Numerical Methods: Finite Difference Approach

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Lecture – 05
Polynomial fitting and one sided approximation

Welcome to the lecture series on numerical methods, a finite difference approach. In this method, we will last class we have discussed this Taylor series expansion, and how we can just obtain this finite difference formulation using a Taylor series expansion.

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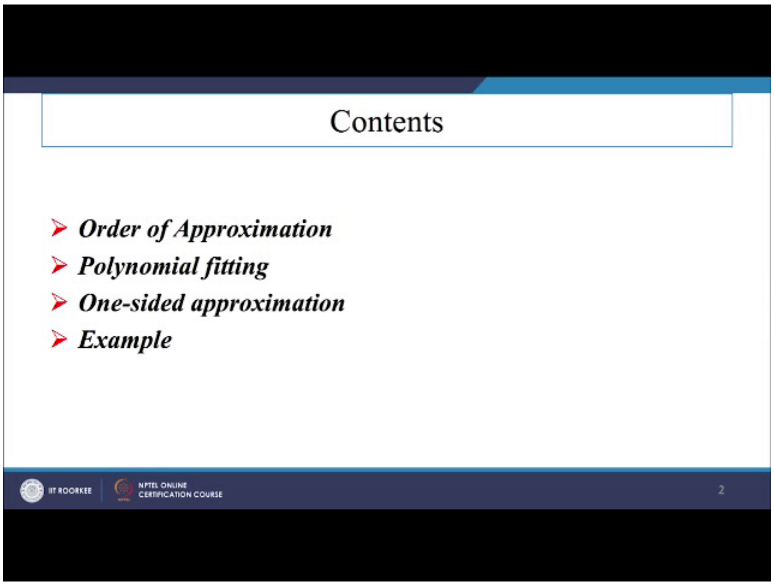

And in this lecture, we will just discuss about this order of approximation, then the polynomial fitting and one-sided approximation means, if suppose a first order differential equation, we are just solving how we can just extend it to like more than 2 points so that we will just consider in this presentation.

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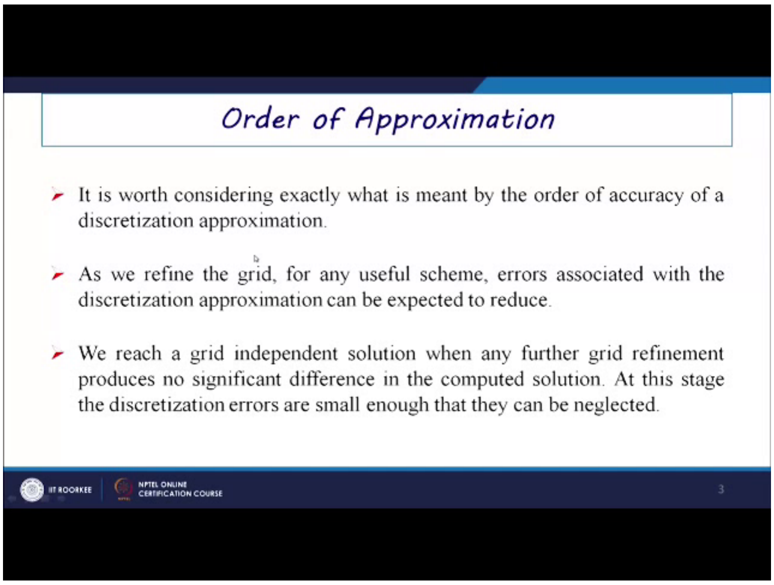

So, if you will just go for the overview of this order of approximation, especially in the last lecture, I have just discussed that if we are just expanding any Taylor series.

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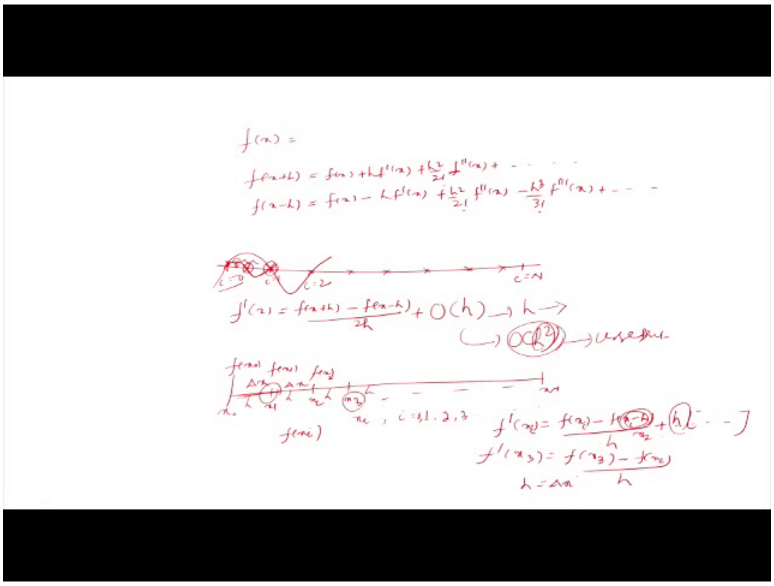

So, this Taylor series can be written in the form of like, if suppose at a point h, you if you want to expand, then we can just write this as f of x plus h; that is nothing but like f of x plus h f dash x plus h square by factorial 2 f double dash of x plus all other terms. And if we want to expand, suppose f of x minus h that can be written in the form of f of x minus h f dash x plus h square by factorial 2 f double dash of x minus h cube by factorial 3 f triple dash of x plus the rest of the terms and whenever we are just adding or subtracting or we are doing some manipulation here, then we are just obtaining they say first order approximations or second order approximations for this finite difference form here.

So, especially if we are just saying that, whenever we are just using this forward difference formulation or backward difference formulation, we are just getting this order of approximation as one there. But, if you are just using like central difference approximation, so then we are just obtaining this order of approximation is 2 here. So, what is the it is overview our, why we are just considering that first order approximation or second order approximation or what is it is benefit? That we will just discuss in this lecture.

So, if you will just see, so first we are just considering here that it is worth considering exactly what is meant by the order of accuracy of a discretization approximation. Since it is very essential to know whether where we are just using like first order approximation where we are just using second order approximations or why we are just using this. So, it depends on this discretization formulation there.

So, if we are just refining the grids like, if you as we refine the grid for any usual scheme errors associated with the discretization approximation can be expected to reduce. So, especially, if you will just see like whenever, we are just using this discretization schemes. So, suppose we are just starting at suppose, I equals to 0. So, proceeding with I equals to one up to I equals to n here. Suppose, then at each of this grid points we are just using this finite difference operators to get this formulation in a difference form here, like if you will just consider, suppose I equals to one here we are just considering for the central difference approximation, we are just considering I equals to 0 and I equals to 2 here and if we will just refine this grid. So, this means that this domain, if you are just considering this I equals to 0 to I equals to one, we can just again discretize this domain into 2 parts there.

So, then when we want to evaluate this value at the middle point of these 2 grid points like I equals to 0 and I equals to one, then we can just find that whenever any fitting of a polynomial or any fitting of function. If you will just consider for this point then, this error of approximation, it will just get reduced. Since directly if you will just consider this function for 2 points. So, the function will come in this form here, but if you will just consider a middle point in between that one. So, again this curve will pass through the middle point. So, that is why, this difference between this axis, if you will just see here. So, that reduces and this causes the difference for this approximation here also.

So, this means that as we refine the grid for any useful scheme, errors associated with the discretization approximation can be expected to reduce and further point is that, we reach a grid independent solution. Since different grid size, if you will just consider like, first if you are just considering h then h by 2 then h by 4 then h by 8, whenever we are just reducing the size of the grid length there over, then at a grid point, we can just find that there will be no change of solution will come; that is specially called the grid independent solutions.

So, when we will just reach for this grid independent solution, this means that for the further grid refinement, we will not find any significant difference in the solution process and at this stage only the discretization errors are small enough that they can be neglected.

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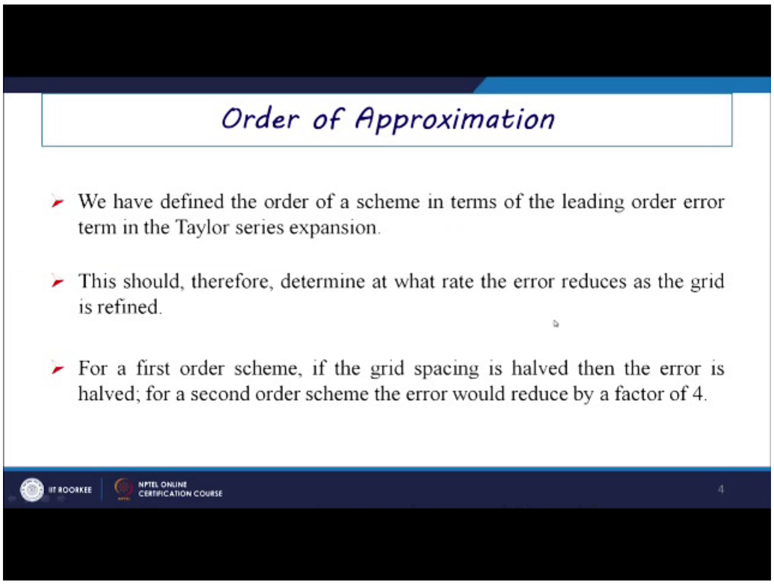

So, we have just defined this order of a scheme in terms of a leading order error term in the Taylor series expansion, if you will just see for this like a central difference approximation, especially we are just writing the central difference approximation as like f dash x is f of x plus h minus f of x minus h divided by 2 h here plus order of certain terms, we are just writing here whether it is order of h or order of h square we do not know.

So, first if you are just considering, if this order is suppose h here then we can just find that this error of approximation for like particular h, if you will just see, if suppose this is like h here and sometimes, we are just getting this is order of h square, suppose whenever we are just considering this h square here, this means that h is a very small quantity and whenever we are just considering this term as in square sense, this term will have a very less effect on the solution process.

But if you will just consider only h here, then since it is has a small value. So, that is why it may be contribute something in the solution process that may create the problem. So, that is why this would therefore, determine at what rate the error reduces as the grade is it grid is refined here; this means that whenever we are just doing this grid refinement, we have to consider that how much small scales, we can just go or how much small length we can just consider for a special grid size that this errors will not reduce after certain states that we will just consider here over.

Suppose, for a first order scheme if the grid spacing is halved, then the error is halved. So, if you will just see for a second order scheme if the error would reduce then it will be like order of h square. So, that is why you can just consider as like h square by 4 there. So, this means that error will be reduced by a like a factor of 4 there itself. So, that is why this is the benefit? We can just get if you will just reduce this like a grid length or whenever we are just going for this a higher order or like a neglection of these remainder terms.

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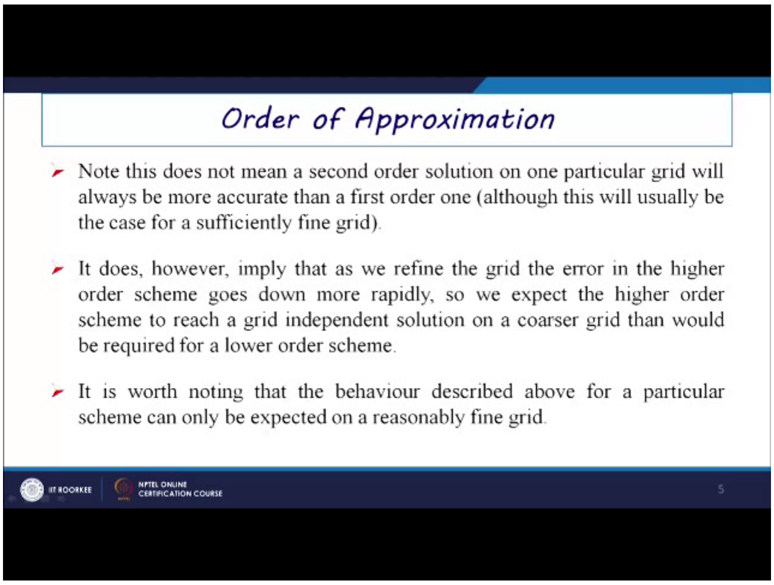

So, note that you should have to keep in in mind that, this does not mean a second order solution on particular grid will always be more accurate than a first order one although this will useful or this you will be usually be the case for a sufficiently fine grid here. If you will just see, so sometimes we are just considering like a first order approximations or accuracy or sometimes we are just considering like second order accuracy for a particular grid.

Since we at that point only, we are just considering like either the 2 points or we are just except extending to like a for forward marching or a backward marching there. So, this does not mean any like a particular grid. It will be effect that the more accuracy of the solution there. So, it does that; however, imply that as we refine the grid the error in the higher order scheme goes down more rapidly.

So, we expect the higher order scheme to raise a grid independent solution on a coarser grid, then would be required for a lower order scheme since whenever, we are just going for like finer a great solution process, then especially the error will be first reduced there, but if you will just go for like larger grid lengths there, then this there will not be converts in a smooth form there. This means, say it maybe takes some time that it can just reach to that level.

So, it is worth when nothing that the behavior described above for a particular scheme can only be expected on a reasonable fine grid. So, from this we can just observe that whenever, we are just going for this a fine grid. So, then we are just finding that it can just give you us this best approximated solutions.

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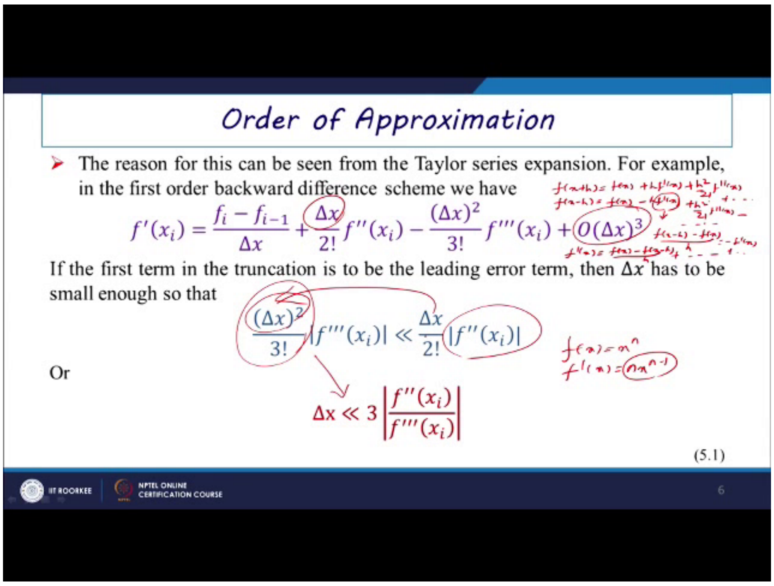

So, then if in a numerical sense, if you will just go for this a first order approximation here that in the Taylor series expansion for example, in the first order backward difference scheme, we are just considering like f dash of x I. Since you we are just considering here, the series is in the form of like f of x plus h that is like f of x plus h f dash x plus h square by factorial 2 f double dash x. So, likewise and if we are just considering like f of x minus h here, this can be written in the form of f of x minus h f dash x plus h square by factorial 2 f double dash of x minus rest of the terms here and if we want to separate, suppose f dash x for this backward difference scheme here.

Then, especially we are just writing that as f of x minus h minus f of x by h; that is nothing but here as a minus f dash x plus all other terms are there. And if you will just change the sign at both the sides, then we can just obtain the solution as in the form of like f dash x this equals to f of x minus f of x minus h divided by h plus rest of the terms they are over and especially for this case, if we are just considering this grid sizes are like if suppose, the starting point is if you are just considering here, that is in the form of like starting is from this to there and then we are just starting x 0 x 1 x 2 x 3 of 2 xn is the last point here. Then especially, we can just say that this difference can be defined at a particular point or we can just say that if x I is the point where I is varying from like 1 2 3.

So, likewise or we can just consider this is 0 1 2 3 likewise here, then we can just say that we can just f of x I can be defined at all the points here that is; that means, we can just write f of x 0 f of x 1 f of x 2 at each of the points there, then at a particular point if we want to find this a finite difference approximation in a backward sense, then we can just write this is like f dash of x we have just defined as f of x minus f of x minus h divided by h plus rest of the terms. We have just to define and at that point if you will just write here like f of f dash of x I here then this is x I then this is x I minus h here, then all of the order terms that will also involving this term h into multiplication of rest of the terms there. Since if you will just a divide here h. So, then the first order h can be come into the picture. So, whenever we are just to have a like second order differentiation here, so h square by h that will just give the h term there over.

So, that is why if you will just go x I minus h means, we are just going back to one of these points there since say this pair size is defined as h here. So, that is why if you will just consider like x I is the point. Suppose x 3 here, then if x I minus h means we can just go to the back of the one point that we can just say as the x 2 here. So, if you want to evaluate suppose f dash of x 3 then we can just write this one as like f of x 3 minus f of x 2 divided by your space length that is nothing but h here and especially in the particular slide we have just considered this distance as the since, we are just moving in the x direction or you can just say that, space length is defined as del x here that is a h equals to del x we have just consider for this expansion here.

So, that is why we are just writing this f dash of x I is nothing but f of x I minus f of x I minus 1 divided by del x plus h by 2 factorial especially, we are just writing that is nothing but del x by 2 factorial here f double dash of x I and minus of del x whole square by 3 factorial f triple dash of x I plus order of a delta x cube terms since a after this term. So, this term will be the coefficient of a this like f forth of a derivative theorem.

So, if the first term in the truncation error is the leading term here, if you will just consider here, so this term is associated with this coefficient that is a del x here over and if we want to find suppose, for this like a coarser grids, if you will just say or finer grids then, we can just find that that del x by 2 is into your absolute of f double dash of x I here. This is always larger than this del x whole square by factorial 3 f triple dash of x I since whenever, any polynomial differences and we are just taking then this polynomial differentiation always reducing to some powers also there.

Suppose, if you will just consider like f of x equals to x to the power n then f dash of x we are just writing n x to the power n minus 1 here this means, that it is just reduces to the power by one also there and if you will just see here this del x square since it is a small term, we are just considering here or del x as a small value. So, it is square of approximation means, always it is less than del x also here and if you will just consider this del x is especially, in the final step here. So, then we can just find del x should be always less than 3 into f double dash of x I by f triple dash of x I here.

If we will just go for like this order of approximation, always we can just find that whenever we are just going for like second order approximation or a third order approximation. In the last lecture, we have just discussed that whenever we are just going like a second order approximation, you can just find that this is higher order term whatever it will be associated with this derivative term here. So, that will be always less than this like second order term there over.

So, this means that whenever, we will just go for this finer grid or like very small grid sizes then we can just find that there will be a no change of solution. We can just find this means that the solution will be always equal or whenever we will just consider this in the last neglected term is the larger compared to the remaining term that we are just neglecting, that will also be a sufficient condition for this order of approximation there.

So, then we will just go for this polynomial fitting especially whenever we are just defining this finite difference approximation. So, especially we are just applying this approximation always in the differential sense or in the integral sense there. So, if we are just using this differential sense, then we will always get a polynomial in the final stage. So, if we are just getting this polynomial how we can just fit this polynomial with this function or with these grid points that will just discuss here.

So, if you will just see this above expressions whatever we have just discussed that is the finite difference formulations based on this Taylor series expansion that are especially involved.

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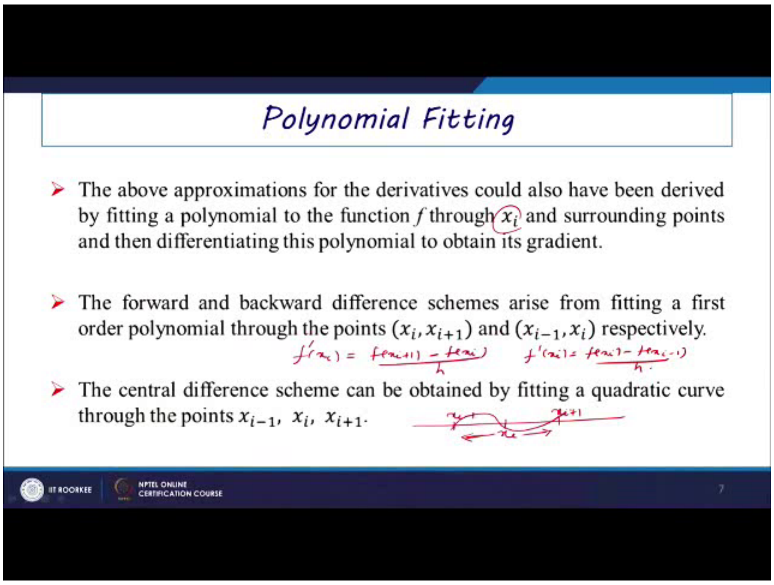

So, this derivative terms which could be derived by fitting a polynomial to the function f through these grid points that is defined in the form of x I here and surrounding points and then, differentiating this polynomial to obtain it is gradient and this forward and backward difference schemes arise from fitting a first order polynomial through the points like x I x I plus 1 x I minus 1 and x I.

Since, if you will just see this a like forward difference formula, especially we are just writing that at the particular point suppose, if you will just see f dash of x I, it can be written in the form of like f of x I plus 1 minus f of x I divided by h there and similarly, we are just defining this backward difference scheme if you will just see here. So, f dash of x I it can be written in the form of f of x I minus f of x I minus 1 divided by h there itself also.

So, that is why, so this function whatever it is associated with the grid points. So, if we can just fit it with a polynomial. It is very easy to get this a differential formulation there itself also. So, for the central difference, it can be obtained by fitting a quadratic curve through these points like x I minus 1 x I and x I plus 1 especially, if you will just see here. So, 3 points we can just consider for this central difference approximation. Since we are just going one forward march and one backward march, so backward march means, we are just considering towards the point x I minus 1 and forward march means we are just considering x I plus 1 here.

So, if we are just combinely just a fitting with a curve here with these 3 points especially, since we are just writing this is a finite difference operator combining these 3 points in a functional form which certain operators there especially that is not just generating a polynomial there itself.

So, then this will just generate since 3 points are involving for this function here. So, that will just generate a polynomial of degree 2 there. So, that is why we are just saying here fitting a quadratic curve through these points x I minus 1 x I and x I plus 1 here then, if you will just go for a linear fit that is for like 2 points formulation here either we can just consider x I x I plus 1 or x I x I minus 1 here. So, for this polynomial,

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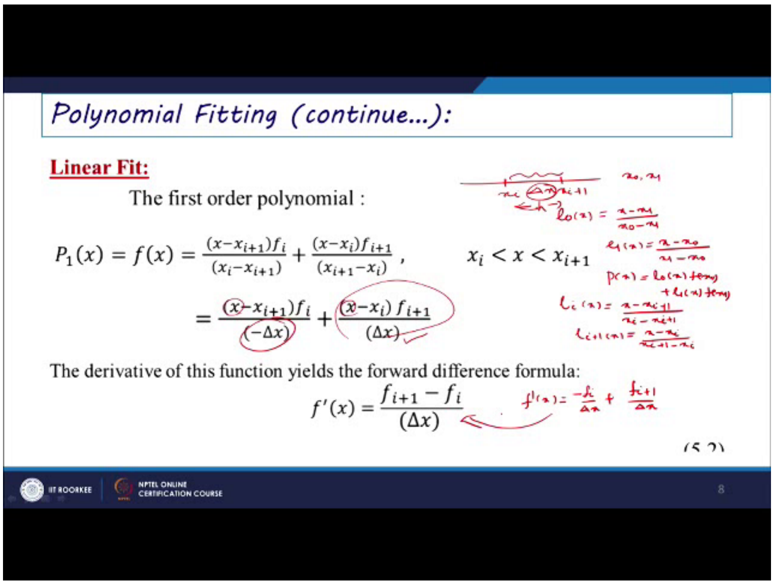

If you will just see here we are just considering this point, suppose like x I and x I plus 1 here and if you will just generate a polynomial function, then this polynomial can be written in the form of like x minus x I plus 1. Especially this one we can just say as a LaGrange polynomial here; that is nothing but like l 0 x if we have like 2 points here x 0 and x one then, especially we are just writing this is x minus x one divided by x 0 minus x one and l one x if you will just write this can be written in the form of like x minus x 0 by x 1 minus x 0 and the total polynomial, especially it is just written in the form of like l 0 x f of x 0 plus l one x f of x one here.

And if you will just a formulate a for these points like x I and x I plus 1 especially, we can just write this as l I of x as x minus x I plus 1 by x I minus x I plus 1 and l I plus 1 of x. This can be written in the form of like x minus x I divided by x I plus 1 minus x I here.

So, obviously, if the complete polynomial can be written. So, this complete polynomial it can be represented in the form of like x minus x I plus 1 divided by x I minus x I plus 1 into f of x I here plus x minus x I f of x I plus 1 by f of x I plus 1 minus x I here and if you will just see that whenever we are just considering this differences here; that is nothing but the del x is we are just considering here or space length has h especially, we are just considering here.

So, that is why we can just write x I minus x I plus 1 here as minus h or minus del x here. So, this complete formulation, we can just write as a x minus x I plus 1 by minus of del x into f of x I here plus x minus x I f of x I plus 1 divided by del x here and the derivative of this function if you directly will just take this difference of this function here. So, this means that we will have associated term x here we will have associate a term x here we can just differentiate directly for this function here.

So, f dash of x especially if you will just write here, so first del x is a constant here we can just take it out, then it can be written in the form of like f of x I divided by like minus alpha del x and this term especially, if it can be written it can be written as a fi plus 1 by del x here. So, the complete differentiation if you will just write here that is f dash x it can be written in the form of like minus of fi by del x plus fi plus 1 by del x here which is just written in this form here that is fi plus 1 minus fi by del x and if you will just go for this a quadratic fit of this polynomial the central difference scheme if you will just see it can involve 3 points here.

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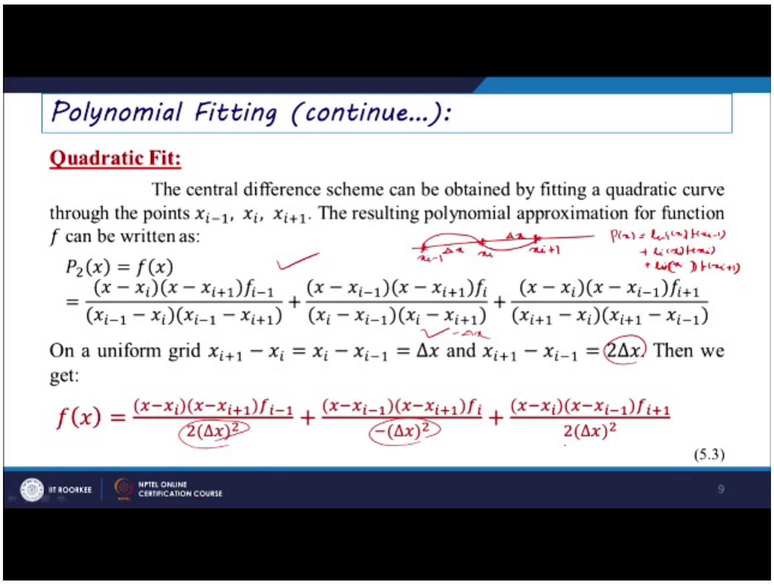

As we have known, that is x I minus 1 x I and x I plus 1 here. So, 3 points in a combined form if you will just consider here that can just generate a polynomial of degree 2. So, especially in a LaGrange form if you will just write.

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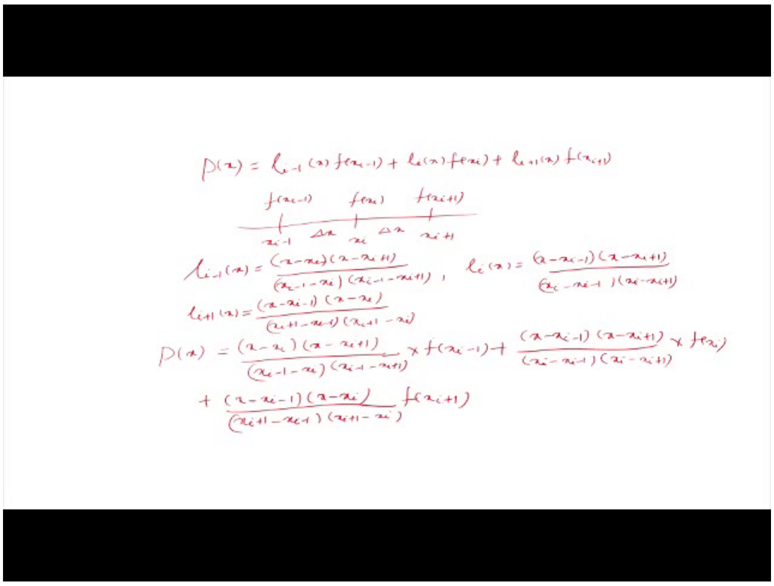

This polynomial as l I minus 1 of x f of x I minus 1 plus li of x f of x I plus l I plus 1 x f of x I plus 1 here and since, it involves 3 points especially if you will just see with each of distances, so del x here. So, we can just write these points as in the form of x I minus 1 x I x I plus 1 and the functional values associated with these points are in the form of fi f of x I minus 1 f of x I and f of x I plus 1 which can be written as like l I minus 1 of x here as x minus x I x minus x I plus 1 divided by x I minus 1 minus x I x I minus 1 minus x I plus 1.

Similarly, you can just write this li of x as x minus x I minus 1 x minus x I plus 1 divided by x I minus x I minus 1 x I minus x I plus 1 here. Similarly, we can just write a lie plus 1 of x as x minus x I minus 1 x minus x I by x I plus 1 minus x I minus 1 into x I plus 1 minus x I here.

So, if you will just combine this a 3 points in a polynomial form here, that we can just write this one as like, if you will just see here, x minus x I x minus x I plus 1 divided by x I minus 1 minus x I x I minus 1 minus x I plus 1 into your f of x I minus 1 plus x minus x I minus 1 x minus x I plus 1 divided by x I minus x I minus 1 x I minus x I plus 1 into f of x I plus x minus x I minus 1 x minus x I divided by x I plus 1 minus x I minus 1 into x I plus 1 minus x I into f of x I plus 1.

So, if you will just see here, so this is your complete formulation of this polynomial. They are over and on a uniform grid since, we are just considering this is a 3 grid points are placed at equal distances, we are just considering this as a del x here this is a del x here. So, that is why we can just write x I plus 1 minus x I this equals 2 x I minus x I minus 1; this is nothing but del x here and if you will just take this a combination of these 2 distances like a x I plus 1 minus x I minus 1 this can be written as 2 del x here.

So, that is why if you will just write in a complete form here x I minus 1 minus x I it can be given you like minus del x here and it can be x I minus 1 minus x I plus 1 here. So, that will just give you like minus 2 delta x here. So, combinely it will just give you 2 delta x square here. Similarly, if you will just find here this will just give you minus delta x square since x I minus x I minus 1 this is your delta x, but x I minus x I plus 1 this will just give you minus delta x here. So, that is why this is just giving you minus delta x whole square here.

So, if you will just take the derivative for this function here.

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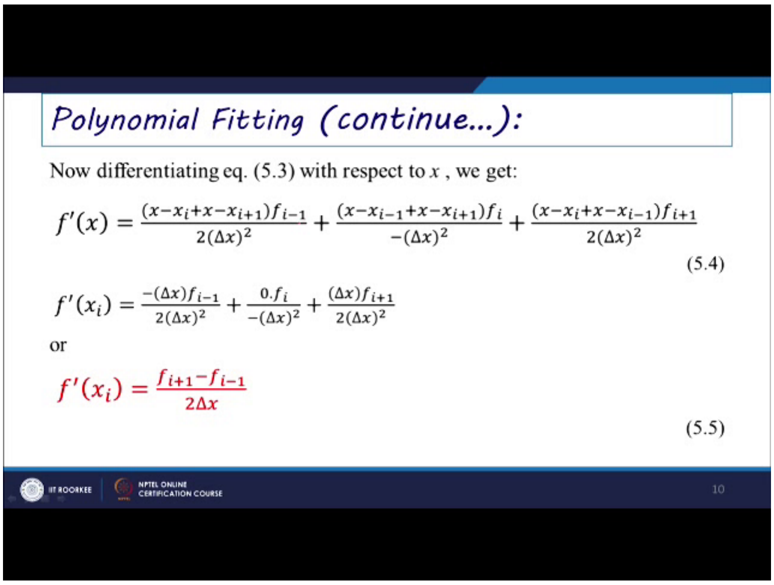

Then this derivative f dash of x can be written in the form of like x minus x I plus x minus x I plus 1 into f of x I minus 1 by 2 del x whole square plus x minus x I minus 1 plus x minus x I plus 1 divided by minus of delta x whole square into f of x I plus x minus x I plus x minus x I minus 1 into f of x I plus 1 by 2 del x whole square and then, we can just write this f dash of x I as in the form of if you will just see here. So, x 2 x it is there and if you will just see here, this is nothing but minus of x I minus x I plus 1 here into fi of minus 1 here and if you will just see this is nothing but 2 del x square here and this is if you will just see all the terms it can just cancel it out here since a minus sign it is if it is just a present it over there. So, combinely if you will just take all these terms.

So, then we can just write this f dash of x I that is like if you will just replace here x I minus x I that it will just give you 0 here. So, x I minus x I plus 1. So, especially this will just to give you here minus del x then, if you will just see x I minus x I minus 1 this will just give you delta x here and x I minus x I plus 1 this will just give you minus delta x. So, that is why this is just cancelling each other this is a giving you 0 here and if you will just see here x I minus x I this is just giving 0 x I minus x I minus 1 this is just giving you delta x. So, that is why this is delta x here into fi plus 1.

So, finally, we are just getting this f dash of x I this is equals to x f I plus 1 minus f I minus 1 divided by 2 delta x. So, if you will just go for like a second order differentiation here. So, that the previous differentiation, once more differentiation if you will just consider here; that is f double dash of x for this equation here.

Then we can just find that since 2 x it is just present it over there. So, that we can just write as a 2 by 2 it can be just cancel it out.

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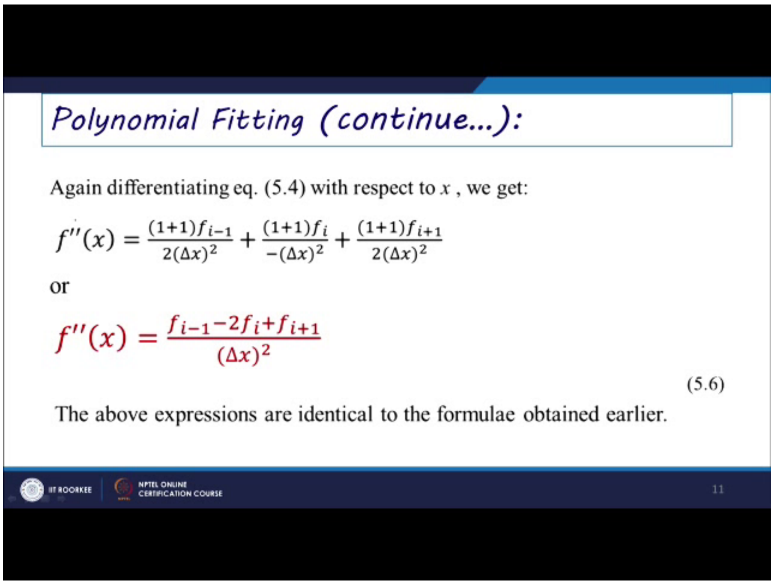

So, that is why it can be it can be written in the form of fi minus 1 by del x square and if you will just see this like this term also here, that is nothing but 2 x here. So, that is why it can be written also like a 2 into fi this will also be written as a 2 into fi plus 1. So, all these terms this differentiation if you will just consider that can be written in the form of like fi minus 1 minus 2 fi plus fi plus 1 by del x whole square and this expression if you will just see this is identical to the formula that whatever we have just obtained in Taylor series expansion also.

Thank you for listening this lecture.