Numerical Methods: Finite Difference Approach

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Lecture – 04
Finite differences using Taylor's series expansion

Welcome to the lecture series on numerical methods, a finite difference approach. And last lecture, we have discussed about to classification of partial differential equations and the boundary conditions.

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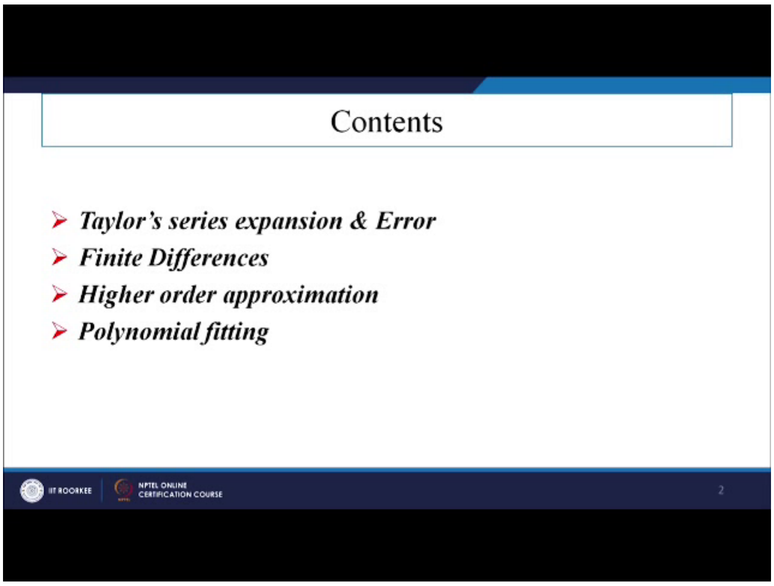

And in this lecture, we will just start about this like a Taylor series expansion and it is associated error, then the finite difference approximations, that how we can just find out this finite difference approximations using Taylor series expansion, then how we can just move to the higher order approximations and in the final form, we can just continue this polynomial fitting.

So, if we will just go for this Taylor series a form here, this means say that if f of x is a function which possesses it is continuous partial derivatives at certain point, this means that if the function is continuous along it is neighborhood point and it has continuous partial derivatives or it has possessing derivatives of all orders up to n plus 1 in an interval I, then we can express this f of x in the Taylor series form.

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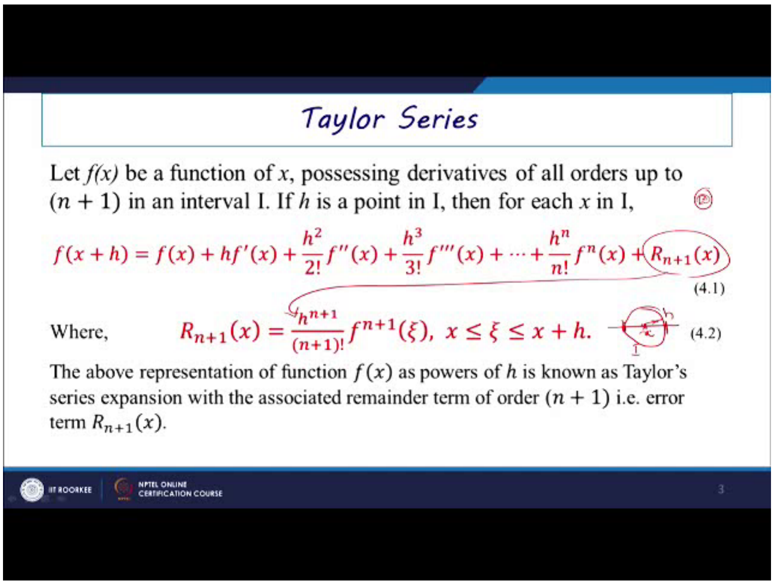

So, especially the statement is like, a let f of x be a function of x possessing derivatives of all orders up to n plus 1 in an interval I. Suppose, if h is a point in I then, for each x in I; that means, if suppose we are just defining a point and this point, we can just express or we can just expand this Taylor series at all of it is neighborhood points, if the function is continuous along all of it is a neighborhood points there.

So, that is why, if h is a point in I, then for each x within that particular interval I, we can just express as f of x plus h equals to f of x plus h f dash x plus h square by factorial 2 f double dash of x plus h cube by factorial 3 f triple dash of x plus it will just continue it up to nth order term, that is h to the power n by n factorial f to the power n of plus the reminder term.

So, especially this reminder term is written in the form of Rn plus 1 x here which is a defined as like nth plus 1th power term here that is h to the power n plus 1 by n plus 1 factorial f to the power n plus 1 zeta where zeta should be lies between x to x plus h here this means that, if we are just finding this Taylor series expansion at a point x, then this function should be continuous at all of it is neighborhood point which is a existing at a distance h from this point x here.

So, if we are just defining this interval, that is x plus h and x minus h here, then at this point x we can just expand this series at the point x comparing with all of it is neighboring points at that point. So, above expression of this function f of x in the form of like Taylor series here is known as the powers of h and which is specially called the Taylor series expansion with the associated remainder term of order n plus 1 here. And especially this remainder term is called the error term for this Taylor series expansion.

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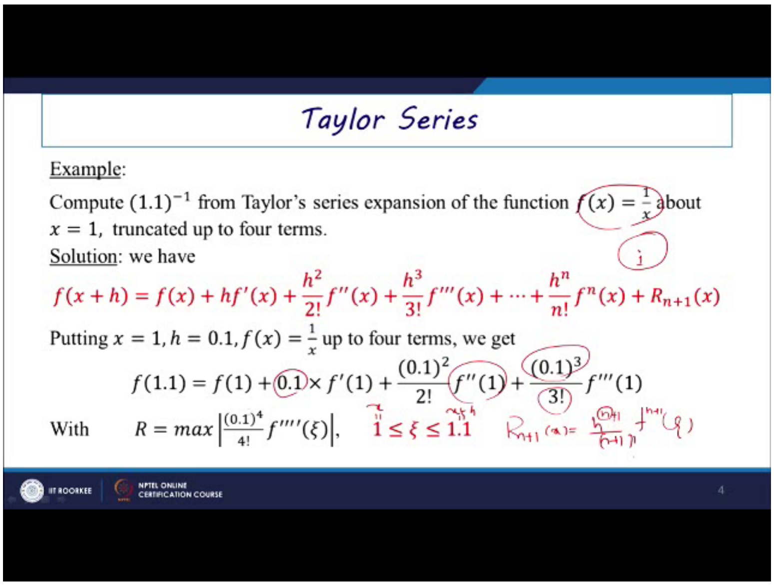

For example, if you will just consider like 1 point 1 whole to the power minus 1, it is asked to find the expansion using Taylor series expansion, then we can just consider that x equals to 1 here and h is point 1 here. And if the question is asked suppose compute one point one whole inverse from Taylor series expansion of the function f of x equals to 1 by x about x equals to one here and we should have to find this result truncated up to 4 terms here. And if it is asked to find this Taylor series expansion of this function f of x equals to 1 by x about x equals to one; about x equals to one means, we can just consider the point one and it is all of it is neighborhood point, where this function should be continuous.

Then, that is why we are just expressing one point one as in the form of 1 plus point one here. So, if you will just put here x equals to one, since it is asked or the question is given to find this Taylor series expansion of this function f of x equal to 1 by x at x equals to one, so, we can just expand this series up to 4 terms that is in the form of like a f of x f of x means, we can just write f of 1 plus h that is nothing but 0 point one here and f dash 1 plus 0 point one whole square by factorial 2 f double dash of one here, then h cube h cube minus 0 point one whole cube by factorial 3 f triple dash of one. And the remainder term that is nothing but you are R n plus 1 term here, which is nothing but we are just writing as h to the power n plus 1 by n plus 1 factorial into f to the power n plus 1 zeta here where zeta should be lies between x to x plus h. So, that is why this is the x range here, x equals to one here and x plus h is nothing but 1 plus 1 here. And we can just express this remainder term that is nothing but maximum of h to the power 4 that is a since it is asked to compute or truncate up to 4 terms, so, that is why we are just considering n equals to 3 here and n plus 1 is nothing but 4 here. So, that is why 0-point one whole to the power 4 by 4 factorial and f to the power 4 zeta here.

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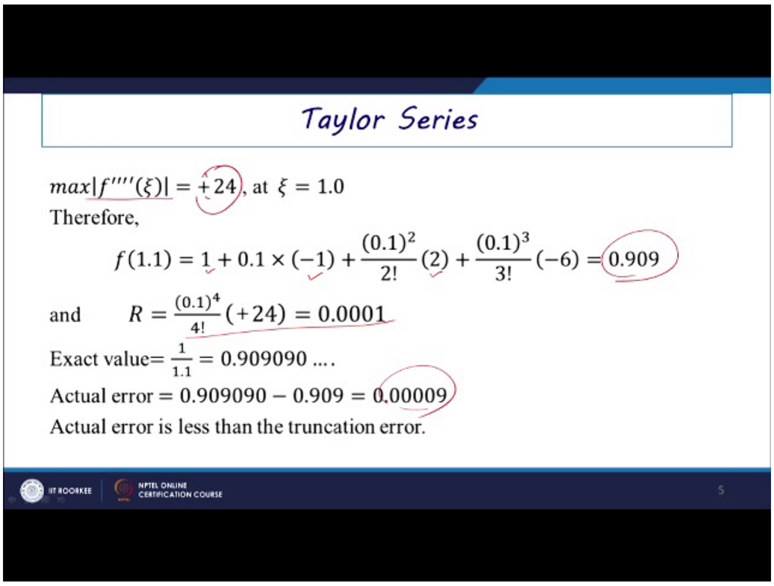

So, if you will just go for this a computation of final value here, then f to the power 4 of zeta for the case of f of x equals to 1 by x this is just a giving the maximum value that is 24 here. It theta equals to one point 0. Therefore, f of one point one it can be expressed as 1 plus like if we are just expressing f of x plus h here, that is nothing but f of x that is nothing but one here, then h f dash x that is nothing but minus 1 here, then h square by 2 factorial and f double dash of x here that is nothing but 2 here. So, if you will just put all these values here. Then we can just obtain the final result as a 0.909 here and for the computation of a error term that is nothing but R is h to the power 4 that is point one whole to the power 4 by 4 factorial into that is the maximum value of f 4th zeta here that is nothing but minus 24 here.

So, if we are just taking this maximum value in a absolute form here, then this absolute should become as plus here. So, that is why we are just writing this absolute sense this total product term is plus here. So, exact value if you will just to go for this computation is 1 by 1 plus 1 here that is nothing but 0.909090 here. An actual error it can be computed as a 0.909090 minus 0.909 this equals to 0.00009 here. Actually, error if you will just see here, it is a less than the truncation error here

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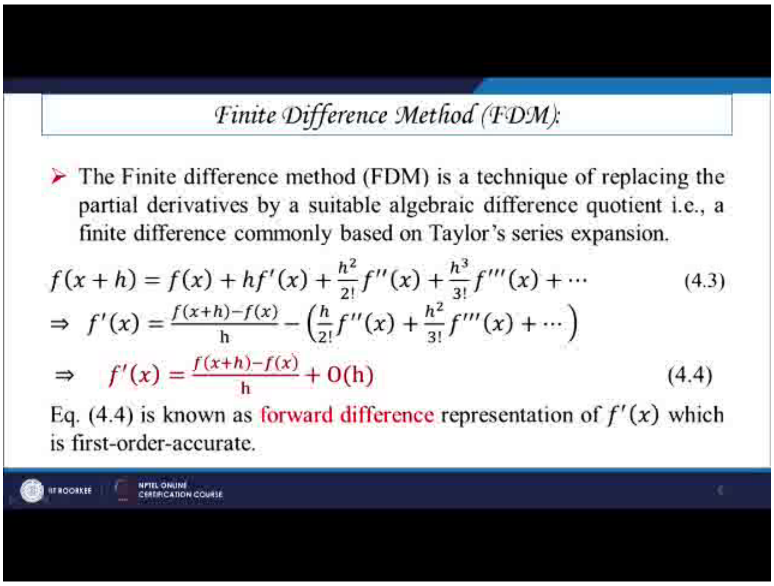

So, if you will just go for this finite difference method here, the finite difference method is a technique of replacing the partial derivatives by a suitable algebraic difference coefficient here, that is a finite difference commonly based on Taylor series expansion especially. So, especially if we are just going for the Taylor series expansion, whenever we will have this function f to be continuous in the neighborhood of x here, then we are just expressing f of x in the Taylor series form corresponding to point h there itself.

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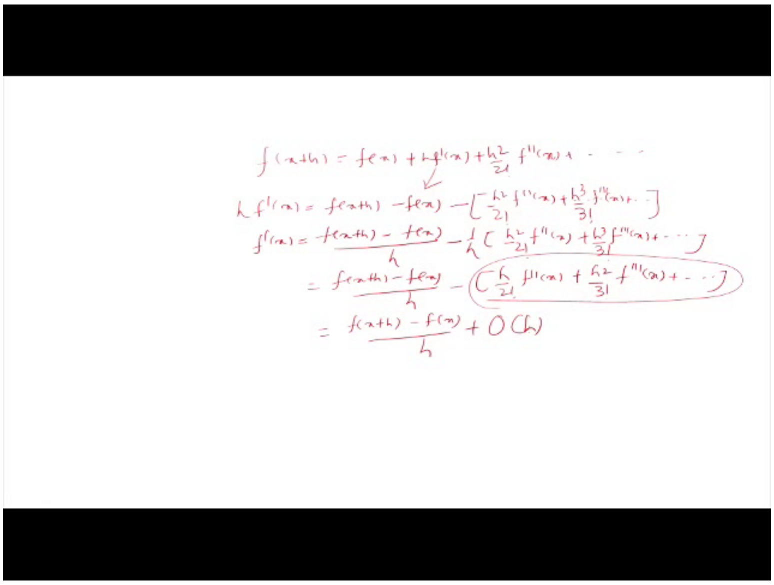

So, that is why we can just express f of x as f of x plus h that is f of x plus h f dash x plus h square by factorial 2 f double dash of x plus remaining terms here and if we want to find, suppose f dash here x here, then we can just write h f dash x as f of x plus h minus f of x minus h square by 2 factorial f double dash of x plus all other terms here; that means, a h cube by factorial 3 f triple dash of x plus all other terms.

So, then we can just express this one as f dash x equals to f of x plus h minus f of x divided by h minus 1 by h, h square by 2 factorial f double dash of x plus h cube by 3 factorial f triple dash of x plus all other terms here. So, if we will just multiply here 1 by h with the remanding terms, we can just get it as f of x plus h minus f of x by h that is nothing but minus of h by 2 factorial f double dash of x plus h square by 3 factorial f triple dash of x plus all other terms here.

So, if you will just consider or we can just take common of h from this terms here then we can just write this as f of x plus h minus f of x by h plus order of h here. So, that is why it is just written in the form like f dash x equals to f of x plus h minus f of x by h plus order of h here and this formulation of f dash x it is especially called the forward difference representation of f of x here or we can just say, that this is the first order difference form of f of x as in the form of forward sense.

So, if we will just expand this series at the point like a neighborhood point. So, we can just consider either in the plus sense or in the minus sense. So, that is why if you will just consider this point as like.

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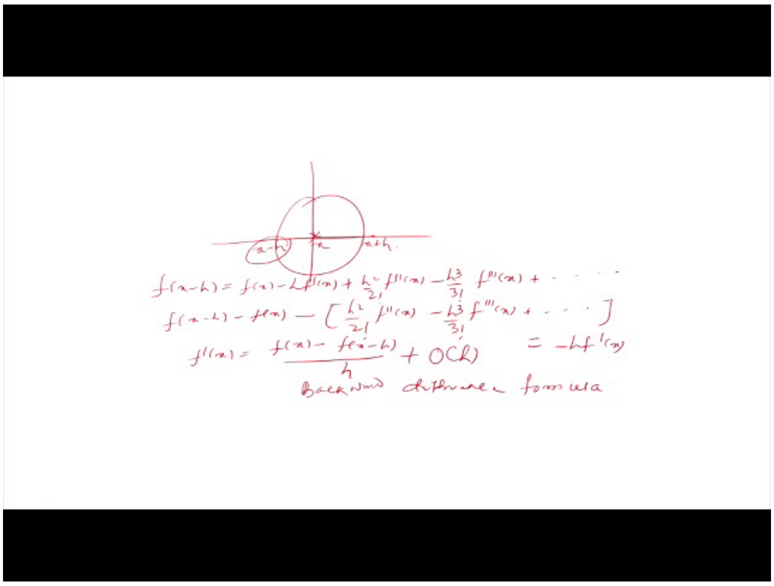

Suppose, here x is the point. So, it is neighborhood point, if you will just consider this line here, either we can just go in the plus sense that is as x plus h here or we can just go in a minus sense here; that is x minus h here, if you will just expand this f of x minus h here, this can be written as like f of x minus h f dash x plus h square by 2 factorial f double dash of x minus h cube by 3 factorial f triple dash of x plus likewise we can just write.

So, then we can just express f of x minus h minus f of x minus h square by 2 factorial f double dash of x minus h cube by 3 factorial f triple dash of x plus all of the remaining terms is like minus h f dash x here. And if you will just change the sign at both the sides, we can just write this one as like f dash x as f of x minus f of x minus h divided by h plus order of the h terms there. So, this is specifically called your backward difference formulation.

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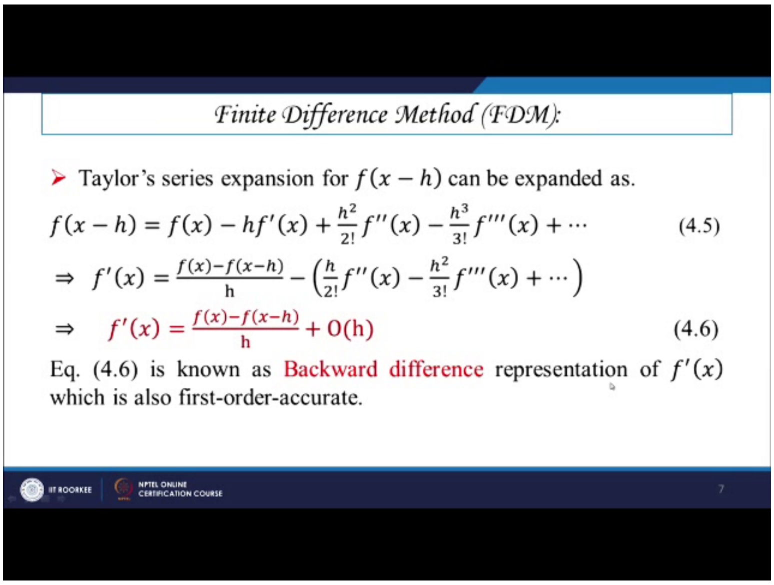

So, backward difference formula we can just find that this representation of f dash x which is also for first order accurate. And in the previous slide, if you will just see this is also this forward difference representation of f dash x which is also first order accurate here.

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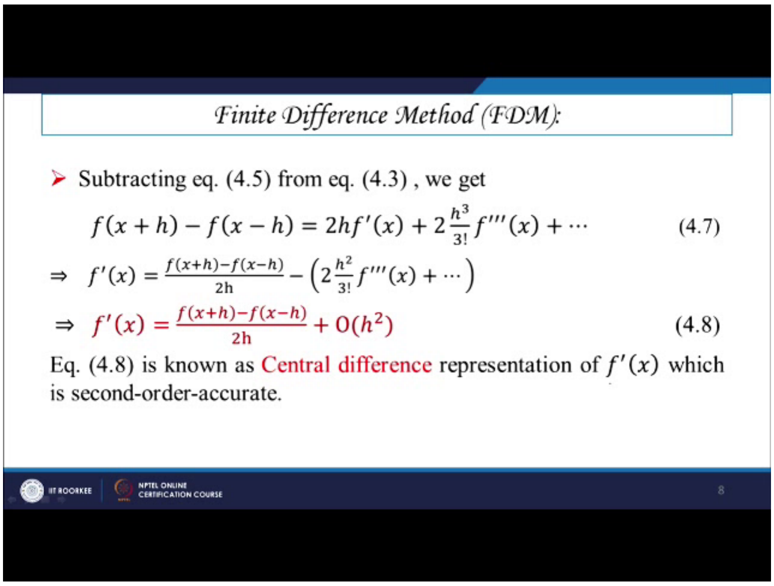

So, if you will just go for this expansion that is in the form of like f of x plus h minus f of x minus h here.

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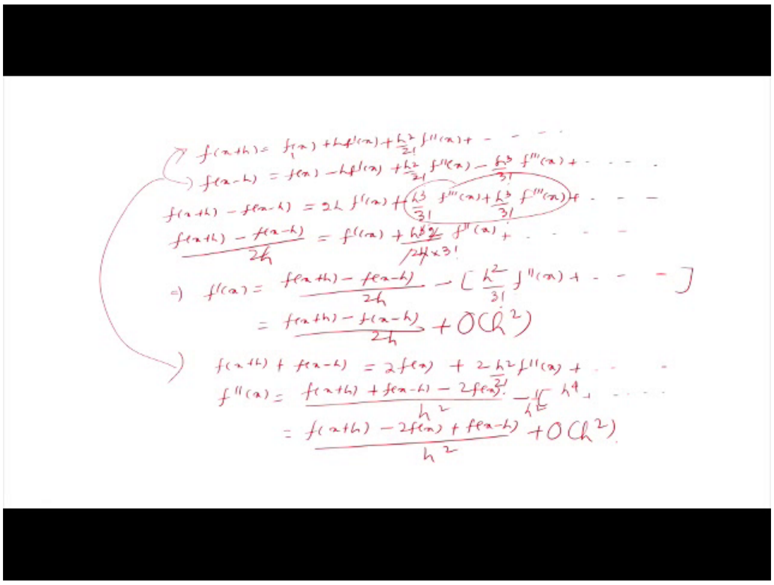

So, which can be represented as, since already we have just got it as like f of x plus h as in the form of like f of x plus h f dash x plus h square by factorial 2 f double dash of x plus all other terms. Similarly, if you will just consider f of x minus h as f of x minus h f dash of x plus h square by 2 factorial f double dash of x minus h cube by 3 factorial f triple dash of x plus all other terms and if you will just have subtract these 2 equations, then we can just find it as f of x plus h minus f of x minus h as faster these 2 terms will cancel it out. So, that is why you can just to get it as 2 h f dash x plus this term is also cancel it out. So, we can just write this one as h cube by 3 factorial f triple dash of x plus h cube by 3 factorial f triple dash of x plus all other terms here.

So, if we can just divide a 2 h, then we can just write this one as f of x plus h f of x minus h by 2 h. It can be represented as f dash x plus h cube by 2 h into 3 factorial f triple dash of x since this is twice repeated here, so we can just consider this one as a twice here plus all other terms here.

So, this implies that we can just write f dash x this equals to f of x plus h minus f of x minus h by 2 h minus this is like the term that is in the form of h square by 3 factorial f double dash of x plus all other terms there. And this can be finally represented as f of x plus h minus f of x minus h by 2 h plus order of h square here. So, in the final form if you will just see this equation is written as f dash x equals to f of x plus h minus f of x minus h by 2 h plus order of h square here.

So, this equation is known as central difference representation of f dash x which is of second order accurate here. So, if you will just see this like a forward difference approximation and backward difference approximation that are like first order accuracy is there, but if you are just taking like difference of f of x plus h minus f of x minus h. So, it is just a giving the second order accuracy here.

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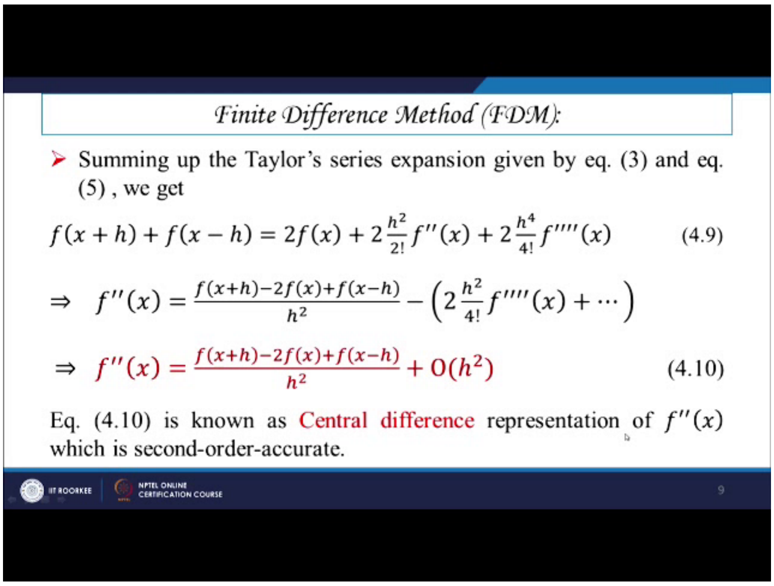

So, then if you will just go for this second order differentiation formula or the second order approximated formula here, so if you will just a summing of the Taylor series expansion given by equation 3 and equation 5. Suppose, if you will just see here f of x plus h is given and f of x minus h it is just to given here, then if you will just combine these 2 here, then we can just write f of x plus h plus f of x minus h this can be written as 2 f of x plus 2 h square by 2 factorial f double dash of x plus all other terms.

So, if you will just to see here, f double dash of x, it can be written as f of x plus h plus f of x minus h minus 2 f of x here by h square minus your terms like the third order term it can we cancel it out. So, h 4 and all other terms it will be there. So, 1 by h square it will be there. So, that is why this final formulation we can just represented as f of x plus h minus 2 f of x plus f of x minus h by h square plus order of h square there.

So, this formulation is known as central difference representation of f double dash x which is of second order accurate. If you will just see this previous formula also, this is also just providing us the central difference representation of f dash x, which is also of second order accurate. So, the question arises that what is the difference between this first order accuracy and the second order accuracy. So, first order accuracy what about this error we will just get it is larger compared to the second order accuracy error.

So, that is why it is preferable to use this like central difference scheme instead of using this a first order accuracy schemes there.

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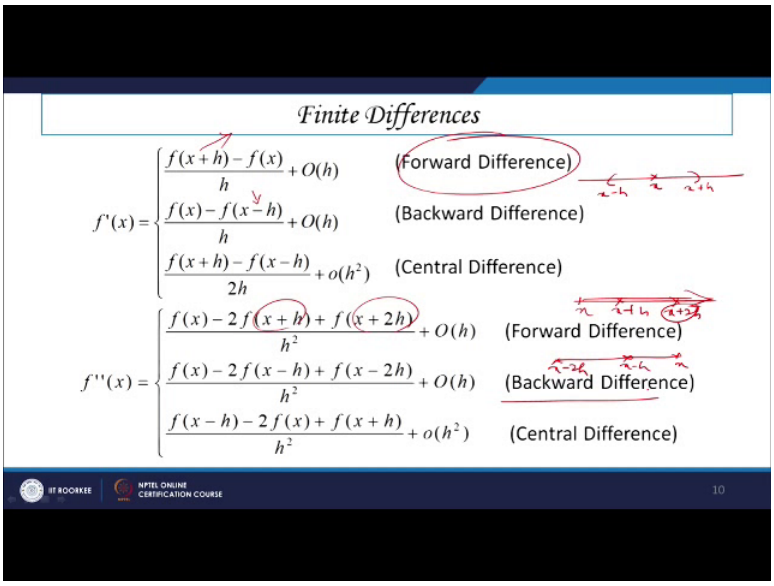

So, especially if you will just express these finite differences, then we can just combine form we can write f dash x as f of x plus h minus f of x by h plus order of h first formula we have just defined; that is as a forward difference formula. And if you will just take a one step backward, suppose this is especially called forward marching. Since if you will just define a line here, so if we are just considering x as the point here. So, one step marching forward we are just getting this point as x plus h here one step backward marching we are just obtaining as x minus h here.

So, that is why this is called forward marching formula. Since we are just moving one step forward here and this is called one step backward marching formula. Since one step backward we are just moving there, so, that is why this is called backward marching formula, sometimes this is called forward marching formula here. And in the central difference scheme, if you will just see we are just considering this is just a combination of like a forward marching and a backward marching.

So, in a physical problem, if you will just see suppose a water is flowing. So, if the water is flowing, it will always have an impact backward impact, it will have a forward impact. If we want to study the properties at the middle point, so we have to consider the properties at the back-step level also at the forward step level. So, that is why to get the more accurate results, we have to consider this for central difference approximations. Central difference approximation considers both this forward step and backward difference approximations. If you individually we will just consider forward difference formula means it can just take this like, future properties and if you will just consider only the backward difference formula, it can just consider the past property of the matter.

As a combination form, if we want to find the formula, we always prefer to use this central difference approximations and if you will just go for like a second order approximation here. So, this second order formula if you will just consider; so forward marching if you will just go. So, then we can just consider this one as f of x minus 2 f of x plus h plus f of x plus 2 h here and divided by h square that we can just derive in a forward form here now. If you will just see this formula that is written as like.

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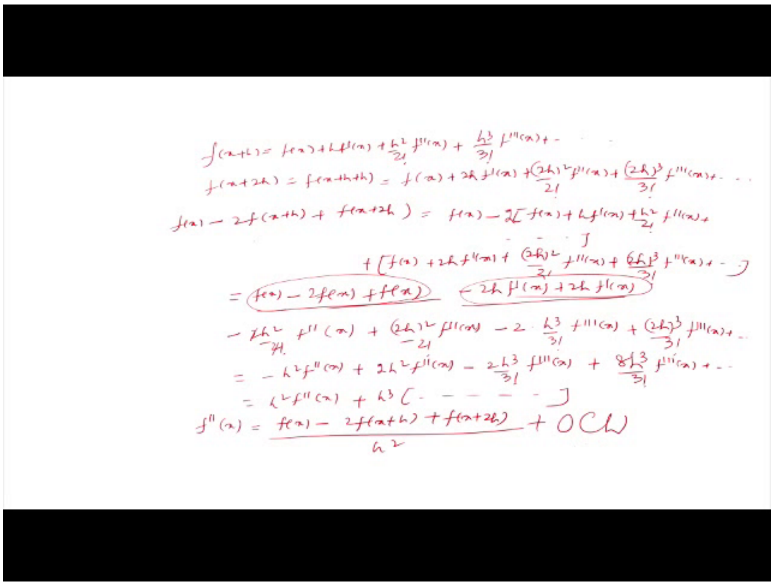

F of x plus h is f of x plus h f dash x plus h square by factorial 2 f double dash of x plus h cube by 3 factorial f triple dash of x. So, likewise we can just write and similarly, if you will just write f of x plus 2 h here, then we can just write this as f of x plus h plus h this means that we can just write this one as f of x plus 2 h f dash x plus 2 h whole square by factorial 2 f double dash of x plus 2 h whole cube by factorial 3 f triple dash of x, so all other formula.

So, if you will just consider like f of x minus 2 f of x plus h plus f of x plus 2 h here then we can just get it as f of x minus f of x plus h f dash of x plus h square by 2 factorial f double dash of x plus all other terms minus sorry, this is plus f of x plus 2 h f dash x plus 2 h whole square by factorial 2 f double dash of x plus 2 h whole cube by factorial 3 f triple dash of x. So, likewise we can just write and if you will just combine all these terms here, then we can just see that f of x. So, this is minus f of x this is plus f of x. So, it can cancel it out.

So, this is nothing but 2 we are just multiplying here. So, that is why, this is the f of x minus 2 f of x plus f of x. So, f of x it will just cancel it out here then minus 2 h f dash x plus this one 2 h f dash x. So, these 2 also cancel it out here. So, then if you will just see here that minus 2 h square by 2 factorial f double dash of x here plus 2 h whole square by factorial 2 f double dash of x here. So, this term only just give you the second order differentiation here. So, immediate next term, if you will just see that will just give you like minus 2 h cube by 3 factorial f triple dash of x then plus 2 h whole cube by 3 factorial f triple dash of x here.

So, if you will just consider this terms, since this cancel it out. So, this can be written as like 2 2 cancel it out. So, we will have like minus h square f double dash of x plus this is a written as like 2. So, cancel it out. So, 2 h square f double dash of x here. So, then minus 2 h cube by 3 factorial f triple dash of x plus this is like 8 h cube by 3 factorial f triple dash of x plus all other terms.

So, finally, we can just get it as h square f double dash of x plus all h cube terms there and if we will just divide h square in the left-hand side and h square in the right-hand side, we can just obtain this one as f double dash of x this equals to f of x minus 2 f of x plus h plus f of x plus 2 h divided by h square plus order of h term here.

So, in this way if you will just go for this computation, then we can just obtain this forward difference formula is like f of x minus 2 f of x plus h plus f of x plus 2 h. If you will just see, always we are just starting at the point x here and first step we are just moving to x plus h and in the second step we are just moving to x plus 2 h here. So, that is why it is called a forward marching formula. Since always we are just move forward here and if you will just go for this like backward difference formula, we have to consider in the similar manner that we will just start at the point x, then we will just go to x minus h, then we have to go to x minus 2 h here. And if you will just combine this one in the form like a f of x minus 2 f of x minus h plus f of x minus 2 h by h square plus order of h, we can just obtain this backward difference formula there.

Since the similar way, whatever we have just a done here, then if you will just do the same thing here.

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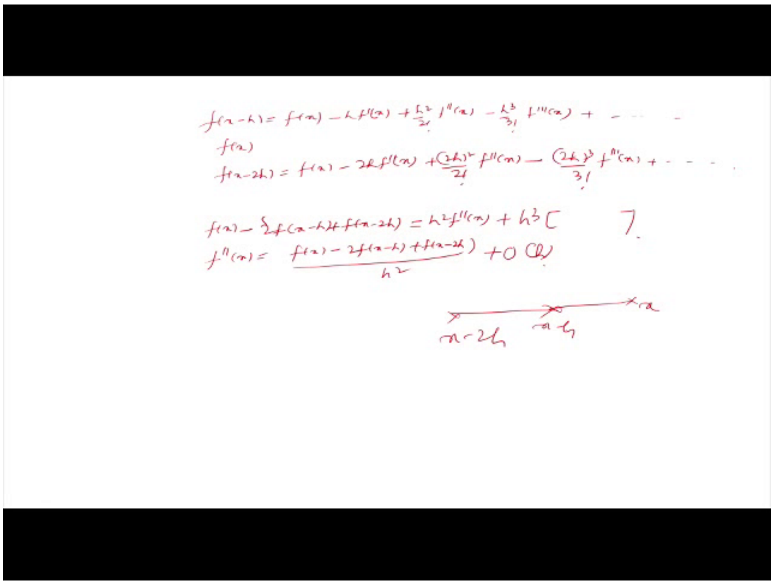

That is like f of x minus h you can just write as f of x minus h f dash of x plus h square by 2 factorial f double dash of x minus h cube by 3 factorial f triple dash of x and f of x is there and like f of x minus 2 h, that is like f of x minus 2 h f dash of x plus 2 h whole square by factorial 2 f double dash of x minus 2 h whole cube by 3 factorial f triple dash of x plus all other terms. And if we can just combine this term, as in the form like f of x minus 2 f of x minus h plus f of x minus 2 h. So, we can just obtain this one as like h square f double dash of x plus h cube and all other terms there.

So, that is why we can just write f double dash of x as f of x minus 2 f of x minus h plus f of x minus 2 h divided by h square plus order of h here and always we are just so moving backward. Since x is the starting point then x minus h then, x minus 2 h we are just considering so, 2 steps backward we are just moving to get this formula. So, that is why it is called backward difference formula for second order derivatives.

So, especially we have just derived this formula for central difference approximations, which is just written as like f of x minus h minus 2 f of x plus f of x plus h by h square plus order of h square. Since this third formula, especially it is just a giving also this second order approximation. So, that is why this can be considered as the best approximated formula, whenever we are just using for the computation.

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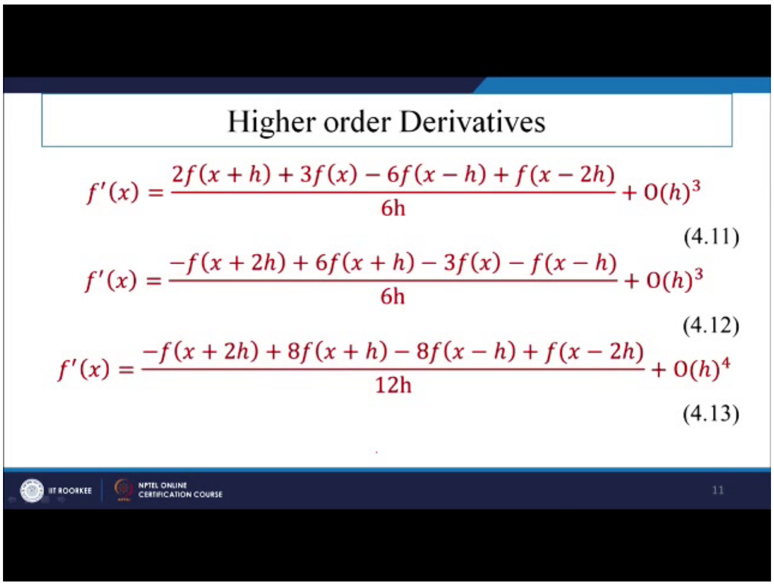

So, if you will just go for like higher order derivatives then, we can just use like f dash x is 2 f of x plus h plus 3 f of x minus 6 f of x minus h plus f of x minus 2 h by 6 h. Sometimes, if it requires like some backward approximations, some forward approximations and if we want to combine all these steps then, if we want to evaluate this f dash x by considering all the data, which is just giving the higher order accuracy compared to the earlier methods then, we can just consider that methods

So, in the second approach, if you will just see f dash x, it is just written in the form like minus f of x plus 2 h plus 6 f of x plus h minus 3 f of x minus f of x minus h divided by 6 h here. This is also giving you this order of approximation as of degree 3 here. Similarly, if you will just consider like f dash x as minus f of x plus 2 h plus 8 f of x plus h minus at 8 f of x minus h plus f of x minus 2 h by 12 h, this is just giving the order of accuracy is 4 here.

Obviously, if you will just see here, so all other terms we can just added it off in a combined form. If we can just consider like some of this a previous step calculations and some of this like next step calculations and it can just to give you the more accuracy compared to the earlier one. Since, in every step if you will just see we are just a marching one step here forward, but 2 steps backward here.

So, that is why, these 2 properties already it is a known to us. Whenever we are just going for the computation of this property at x point here, considering this like future property that at x plus h point there. So, that is why and sometimes also, if we can just consider like a 2 points at the backward 2 points at the forward then, we are just getting this last formulation here, this is the most accurate formula. So, that is why this is just giving the order of accuracy as the 4th order here.

Thank you for listening this lecture.