Numerical Methods: Finite Difference Approach
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Lecture - 03
Introduction to PDE

Welcome to the lecture series on numerical methods a finite difference approach. And last class we have discussed about the various solution methods for ordinary differential equations. And in this lecture, we will start about partial differential equations. And in this partial differential equation solution approach, first we will just go for this classification of a partial differential equations. So, whether it is like Parabolic, Elliptic or Hyperbolic. And based on that we can just proceed for the solution methods.

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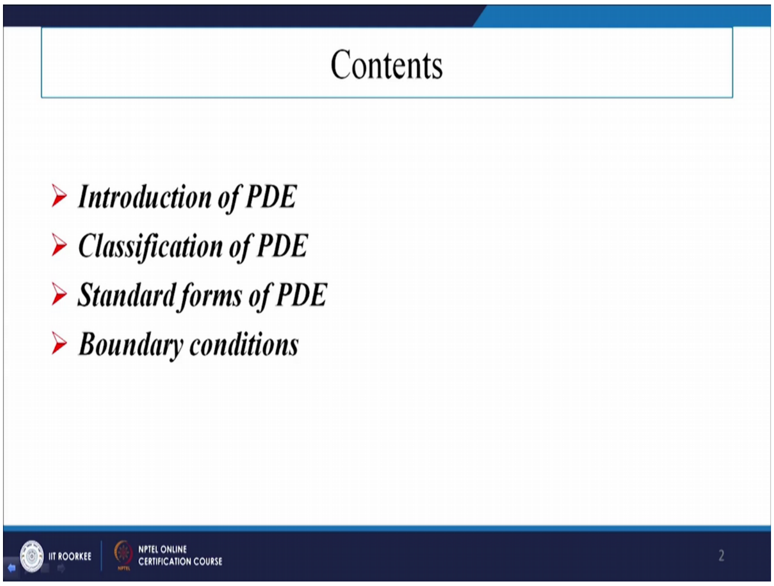

So, especially if you will just go for the contents here, first we will just to go for this introduction of a partial differential equations, then this classification of partial differential equations. Then how we can just represent this partial differential equations in a standard form. And then we will just describe the boundary conditions, through which we can just obtain the solutions of the partial differential equations.

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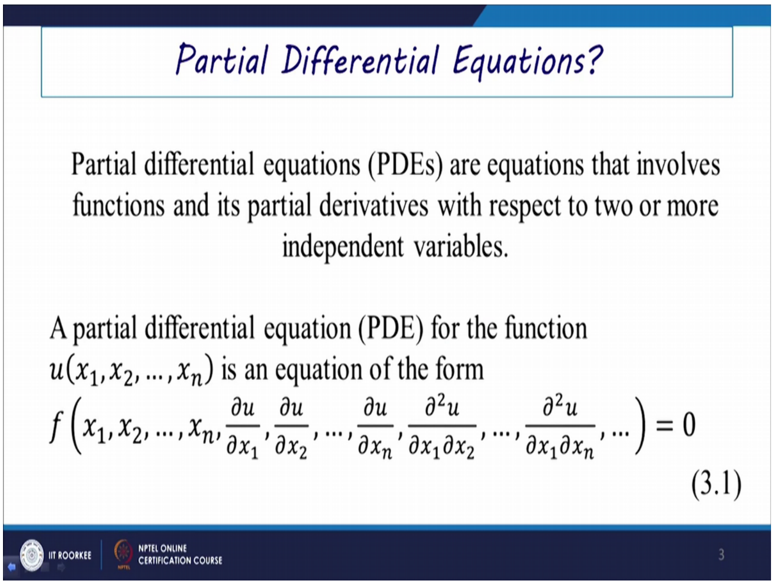

So, partial differential equations are equations so that involves functions and it is partial derivatives with respect to 2 or more independent variables. This means that, in ordinary form especially if you will just see that we used to write dy by dx, this means that y is the dependent variable and x is the independent variable here. But if you will just go for like differential equations with more than one independent variables, then it is called partial differential equations.

So, any partial differential equation it can be represented as a function like u is a function like x 1, x 2 up to xn, then it is partial differential form can be written as either as a del u by del x 1, del u by del x 2 up to del u by del xn, or del square u by del x 1 del x 2 up to del square u by del x 1 del xn, or so many forms in partial sense we can just express.

So, that is why a partial differential equation for the function u, which is a function of like variables x 1 x 2 up to xn is an equation of the form that is f of x 1 x 2 up to xn, del u by del x 1, del u by del x 2 up to del u by del x n. Then, this is especially your first order partial derivatives, and then if you will just go for second order partial derivatives, it can be written in the form of like del square u by del x 1 del x 2, up to del square u by del x n del x 1 del x n. So, afterwards similarly we can just express this partial differential terms of 2 nth order.

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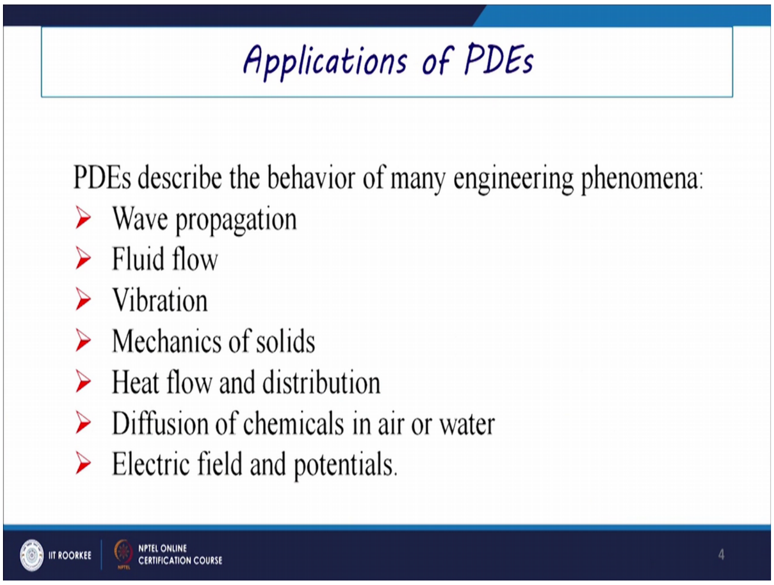

So, this partial differential equation, if you will just see it is occurring in all classes of engineering problems. Specifically, when this wave propagation problem if we will just see, so, one dimensional wave propagation equation it can be represented in the form of like del u by del t equals to c del square u by del x square; there is nothing but a partial differential equation. Then this fluid flow equations like continuity governing equations like momentum equations, energy equation, so, all these equations involving the partial differential equations of first order or it can be of more than one order.

So, then like a vibration problems, if you will just go for this mechanical systems so, there also you can just find this partial differential equations, which expressed in a different forms. So, then this mechanics of solids; that is, solid mechanics problems if you will just discuss like tensor fields, or like stress fields, then usually it can be represented either in the 2-dimensional space or in 3-dimensional space, which can be represented as a tau of x x tau of xy tau of y z. So, all these forms can be represented in the, for partial differential equations. Then heat flow and distribution; that is, especially if you will just go for like one dimensional heat transfer equation, or 2-dimensional heat transfer equation, then we can just find different class of partial differential equation.

Then diffusion of chemicals in the air or water; especially, if you will just using this cigarette smokes, you can just find this diffusion model there. Especially if you will just write that is the equation in the form of del square u by del x square plus del square u by del y square, then we can just find the diffusion equation terms there. Then if you will just discuss here electric field and potentials also, like Maxwell's equation or any class of like electric field equations, that also like expressed as in the form of partial differential equations.

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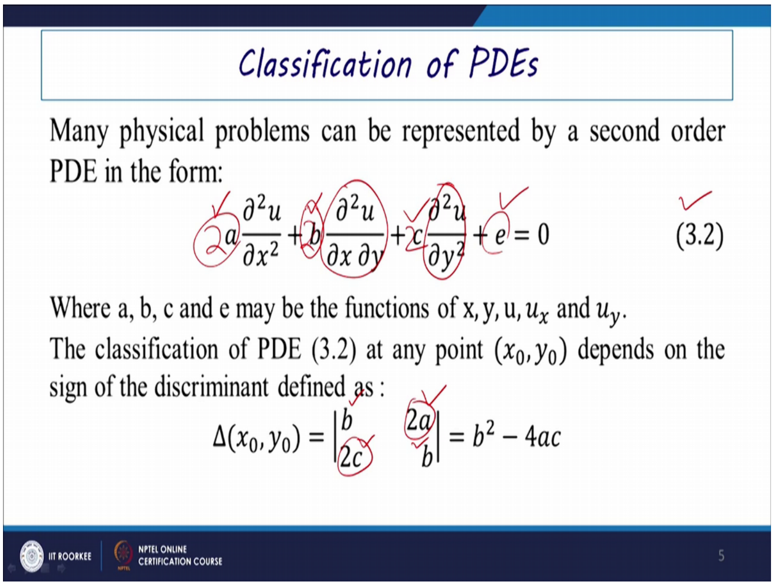

So, based on this if you will now go for this like classification of partial differential equations. So, first we want to express this partial differential terms in a generalized form. So, many physical problems can be represented by a second order partial differential equations. If you will just to see in the textbooks or in a like natural phenomena problem, so, whatever these researchers are using especially that is represented in the form of second order differential equations in partial sense.

If we can just express this generalized form as a del square u by del x square, we can just express this one as in the form of like del square u by del x del y or del square u by del y square. So, plus e means it can be involved either this is a first order partial differential terms, or all of this like dependent variable terms or independent variable terms. So, that is why we have just kept e as a source term here and all other terms is the second order partial differential terms here.

So, that is why I have just written it up here that, is if the partial differential equation in second order form can be expressed as a del square u by del x square plus b del square u by del x del y plus c del square u by del y square plus e equals to 0, where a b c and e may be the functions of either x y u ux and uy.

So, then when we just go for this classification of partial differential equation if you will just consider this equation which is represented as 3.2 here at any certain point suppose x 0 y 0, this depends on the sign of the discriminant defined as delta of x 0 y 0 equals to determinant of this coefficients if you will just see this diagonal coefficient b is written here, and especially this is the coefficients that is just represented as a 2 a here and 2 c here, especially if we can just discuss about quadratic equations.

So, usually these coefficients that is written in the form of x square plus 2 bx plus c equals to 0. So, they are itself b always involves this coefficient 2. So, that is why if in a generalized sense if we can just express this partial differential equation as like 2 a 2 b 2 c here. So, then we can just see that this coefficient b can be taken out as in that form, but remaining coefficient for like u with respect to x and u with respect to y it is like 2 a and 2 c here.

So, that is why these coefficients for like del square u by del x square is written as a 2 a here and for del square u by del y square it is just written as a 2 c here, but the coefficient since in the earlier or in the beginning of the sequence or this equations especially it is just written as a coefficient of 2 b. So, that is why their itself also so b is taken as the coefficient that is why we can just write this determinant as b 2 a 2 c and b and it is a determinant can be expressed in the form of b square minus 4 ac here.

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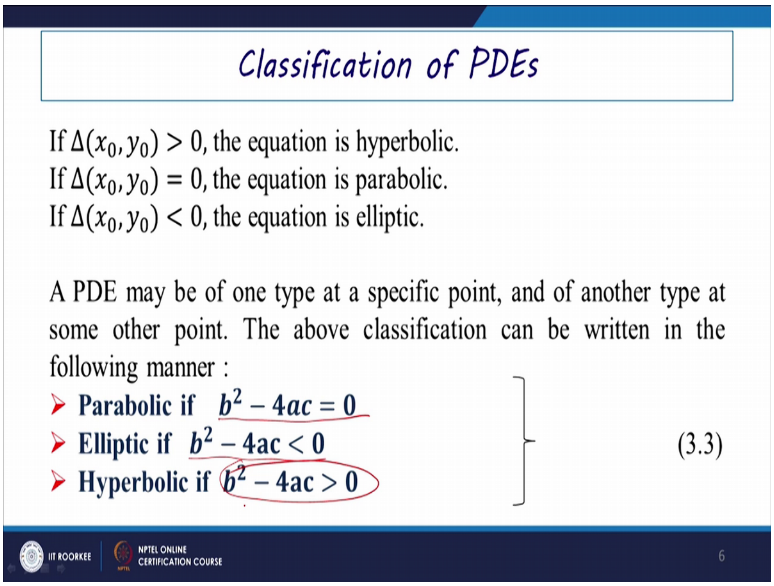

So, when we have just to define these coefficients in a determinant form here. So, we can just define or classify this partial differential equation as if suppose this determinant is greater than 0 the equation is Hyperbolic, and if this determinant equals to 0 this equation is Parabolic here and if this is less than 0 then this equation is Elliptic. Especially in mathematical sense if you will just write all these coefficients as b square minus 4 ac if it is a 0 here, then this system is said to be Parabolic and if b square minus 4 ac is less than 0 then the system is said to be it is in Elliptic form, and if it is greater than 0 then we are just saying it is in Hyperbolic form.

So, that is why a partial differential equation may be of one type at a specific point and of another type at some other points also, this above classification can be represented if you will just consider like another example suppose.

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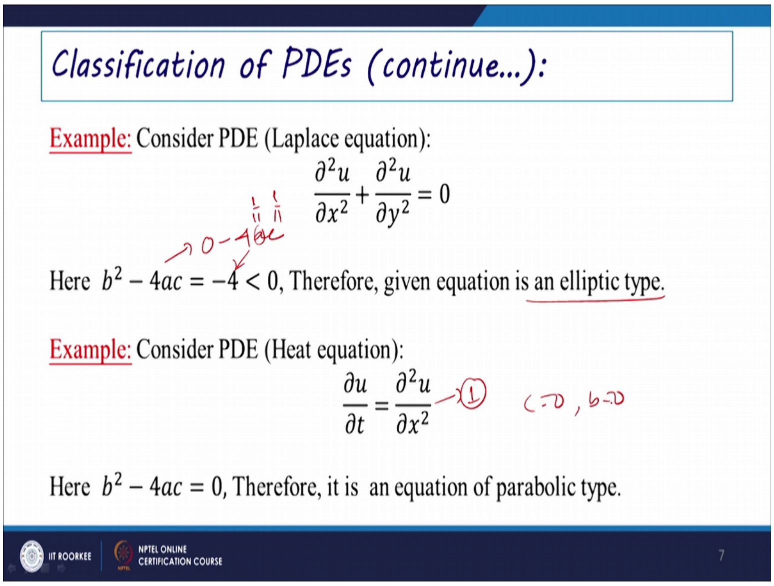

Consider a partial differential equation like del square u by del x square plus del square u by del y square equals to 0. So, if you will just compare this equation with our earlier equation term. Then we can just find these coefficients is like if you will just see here, del square u by del x square plus del square u by del y square this equals to suppose 0 here.

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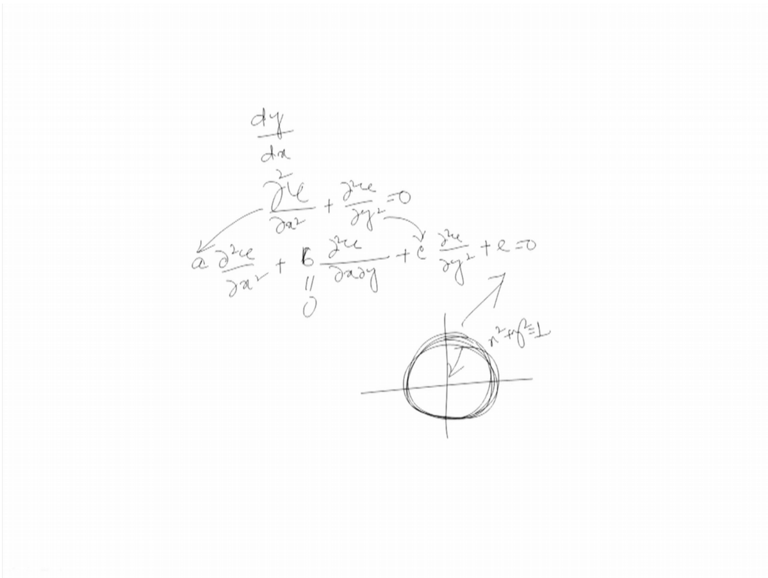

So, earlier our equation we have just defined that I see in the form of a del square u by del x square plus b del square u by del x del y plus c del square u by del y square plus e equals to 0, if you will just compare this coefficients here del square u by del a x square coefficient is a here, del square u by del y square coefficient is this c here, but this b coefficient especially this is just taking the 0 value here.

So, that is we can just represent this one as b is 0 here, a is a like 1 here, c is 1 here. So, that is why this total value that is b square minus 4 ac that is nothing but we can just write 0 minus 4 ac, since a equals to 1 here and c equals to 1 here. So, that is why this total value is coming as minus 4 here and especially minus 4 is less than 0 then this condition, especially if you will just see the previous slide here; that means, that if b square minus 4 ac is less than 0 then the system is a Elliptic in nature.

So, that is why therefore, the equation is an Elliptic equation here. So, for the example if you will just consider like one dimensional heat equation, then this partial differential equation form is a del u by del t equals to del square u by del x square. So, people used to write this one in the form of like del u by del t equals to c del square u by del x square where, c is the like a specific heat constant and here if you will just see the coefficients. So, especially these coefficients this can be written as here del square u by del x square coefficient is a 1 here, and like remaining coefficient that is c equals to 0 here and b equals to 0 here.

So, that is why b square minus 4 ac if you will just see this is just a taking the 0 value. So, that is why this is an Parabolic type of equation and 0 condition means we can just say that the system is Parabolic in nature whenever b square minus 4 ac goes to 0.

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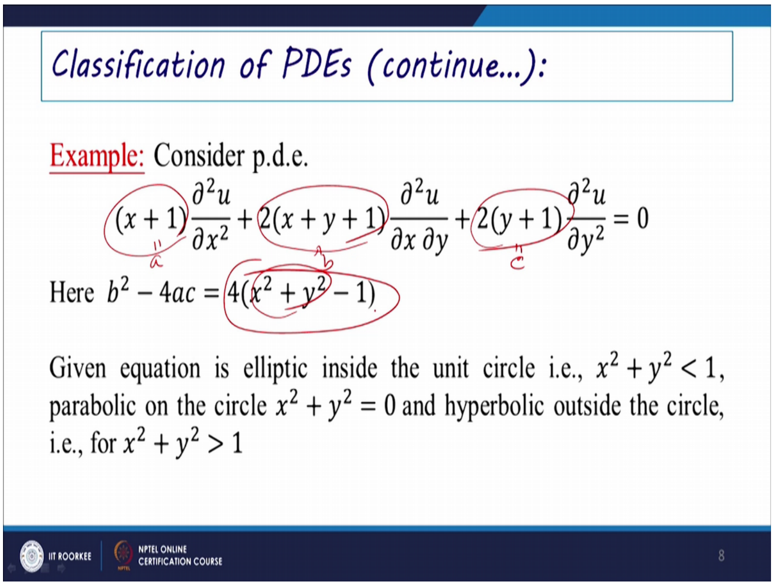

So, then if you will just go for a partial differential equation in a generalized form suppose, generalized form means in the previous slide I have just told that whenever we will have certain domains in certain ranges we can just say that the system is like Parabolic in certain domains, we can just say that the system is Elliptic in nature or so certain domains where this system is like a Hyperbolic in nature.

So, to classify this one we have considered this example as like a this partial differential as x plus 1 into del square u by del x square plus 2 into x plus y plus 1 del square u by del x del y plus 2 into y plus 1 del square u by del y square this equals to 0 here. So, if you will just see this equation here, then you can just find this coefficient of b as 2 into x plus y plus 1 here, and the c coefficient that is as 2 into y plus 1 here, and a coefficient that is nothing but x plus 1 here. So, you can just write a equals to this one, b equals to this one and c equals to this one here. So, that is why b square minus 4 ac if you will just consider here that is nothing but 4 into x square plus y square minus 1 here.

So, there itself we can just justify that 3 conditions can be taken to define the domain, where the system is like Parabolic, where the system is Elliptic, where the system is Hyperbolic here. So, if you will just consider the system that whenever suppose x square plus y square value if it is less than 1, then the whole system will take a like a negative value here, that is the given equation is Elliptic inside the unit circle that is x square plus y square less than 1, and Parabolic on the circle that is x square plus y square equals to 0, and Hyperbolic outside the circle for x square plus y square greater than 1 here.

So, from these equations we can just justify that within certain domain, we can just say that wherever this like the system can just represent a Parabolic nature, or a Hyperbolic nature, or like a Elliptic nature. So, if you will just define the domain like x square plus y square this is a just equals to one here, then we can just say that how it is just be having inside the system here. Suppose if we are just considering inside this circle only or above the circle only then, we are just you saying it is Parabolic in nature, but outside this range it is just showing a different nature and inside this one it is just showing a different nature here.

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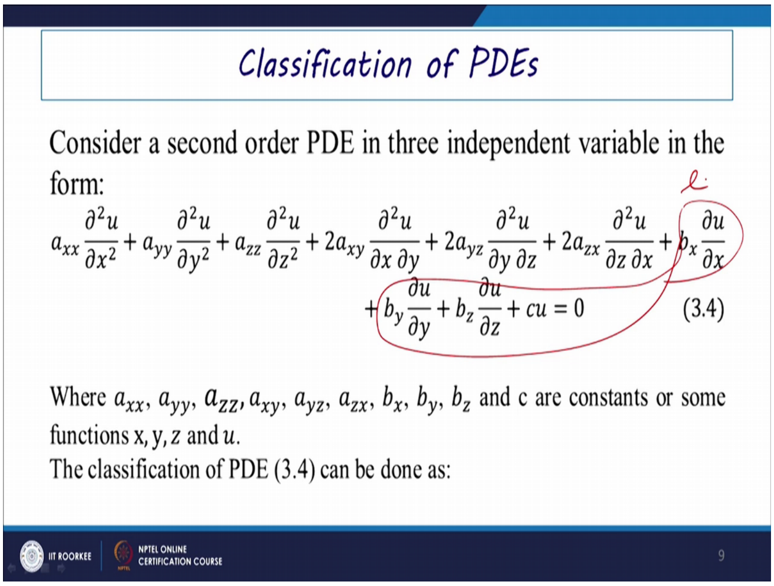

So, then in a generalized form if you will just consider a second order partial differential equation in 3 independent variables suppose, since in the like a 2-variable sense or in a like second order partial differential equation sense, we are just saying that we will have these coefficients like a b and c. So, based on that we are just classifying that whether this b square minus 4 ac is a greater than 0, or it is less than 0, or equals to 0, but if it is more than like 2 variables then we will have like several coefficients in that scale.

So, if we will just go for a partial differential equation in 3 independent variables suppose, that is suppose x y and z and if we can just write this system is like axxs and del square u by del x square plus a y y del square u by del y square plus a z z del square u by del z square plus 2 ax y del square u by del x del y plus 2 a y z del square u by del y del z plus 2 a z x del square u by del z del x plus b x del u by del x plus b y del u by del y plus bz del u by del z plus c u equals to 0. Especially if we want to write this one like in a compact form we can just signify these 3 terms or this last added terms as e times here, since remaining terms if you will just see x axis a yy a zz ax y a yz and azx, which is just consists of all these variables so which is a of second order in partial differential sense.

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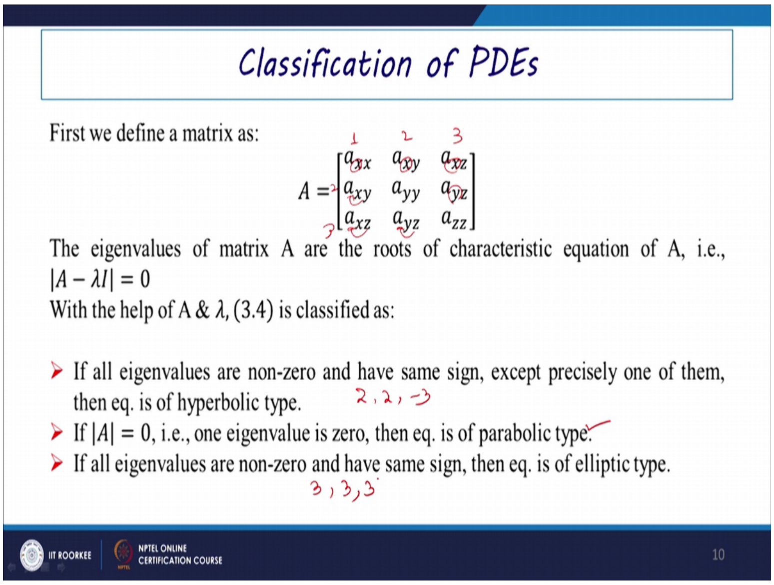

So, if we can just go for this classification of this partial differential equation here. So, then we can just define these coefficients in a matrix form as a equals to like all these x derivative terms in the first row here that is a xx a xy a xz here, for the second one if you will just go we can just define that one in a opposite sense as a of yx a of yy a of yz here, So, then third one we can just write a of zx a of zy a of zz here. This means that each of this independent derivatives if you will just combine like 1 2 3 here then we can just write this one as like 2 3 here.

So, especially the first Eigen value entry we can just write 1 1 then this is 2 2 this is like 3 3 here, and all other differentiations are with respect to like second one it is differentiated with respect to y here, then first one it is differentiated with respect to x here, third one it is just differentiated with respect to z here.

Then if you will just to determine this Eigen values of matrix a are the roots of this characteristic equation of matrix a here, that is nothing but modulus of a minus lambda equals to 0 here, and with the help of this Eigen values we can just classify this partial differential equation whether it is Parabolic in nature, Hyperbolic in nature or Elliptic in nature.

So, the first condition is that if all Eigen values are non-zero, and have same sign except precisely one of them then equation is Hyperbolic in type, this means that suppose we will have these roots like a 2 2 minus 3, then 2 consecutive it is just a repeated and 1 is different and if we are just getting the roots are like that then we can just say that it is Hyperbolic type.

So, that is why this condition is written that the Eigen values are non-zero and have same sign except precisely one of them. So, that is why we can just write this one as suppose 2 2 minus 3, then we can just say that this partial differential equation is like Hyperbolic in nature. Then we know that if this determinant equals to 0 at least one of the Eigen values is 0, then in that sense we can just say that the system is Parabolic in nature. But if all the Eigen values are non-zero and have same sign, then the equation is Elliptic type. So, especially if you can just write this one is like 3 3 3 consecutive roots, then we can just say that the system is a Elliptic in nature.

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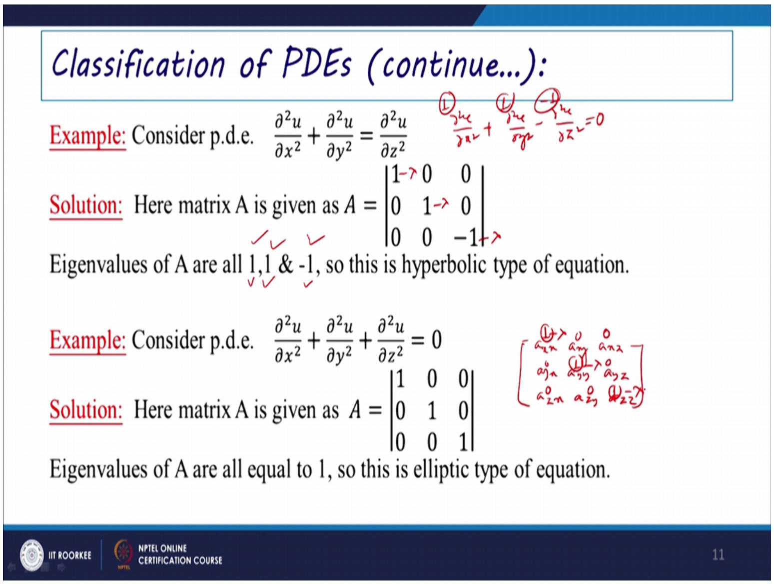

So, to define this problem so we have just consider certain examples here consider a partial differential equation like del square u by del x square plus del square u by del y square this equals to minus sorry del square u by del z square here. So, if you will just take it into the left-hand side here we can just express this one this equation as a del square u by del x square plus del square u by del y square minus del square u by del z square this equals to 0 here. So, these coefficients are like 1 1 and minus 1 here.

So, if we will just take these coefficients in a matrix form here, then this matrix can be written as like u xx coefficient is 1, u yy coefficient is 1 and uz z coefficient is minus 1 here. So, if you will just go for this Eigen values then we can just write this one as like a minus lambda that is nothing but 1 minus lambda here, then 1 minus lambda here minus 1 minus lambda here. So, the Eigen values of a are all like 1 1 minus 1, if all the entries in the diagonal are 1 0 and remaining elements are 0 then the diagonal entries are nothing but the Eigen values. So, that is why this Eigen values of a are all like 1 1 and minus 1. So, if you will just see here, so 2 roots they have the same sign and one has the different sign. So, that is why this equation is Hyperbolic in nature.

So, second example if you will just consider like a partial differential equation of the form del square u by del x square plus del square u by del y square plus del square u by del z square this equals to 0. Suppose, then we can just define the system as like the matrix as like a xx a xy a xz then a yx ayy a yz then a zx azy azz and if you will just place this coefficients here that can be written as 1 here, this is 1 here, this is 1 here remaining elements are 0 here, and if you will just to determine these Eigen values so that can be written as 1 minus lambda 1 minus lambda and 1 minus lambda here, and the Eigen values all are equals to one here. Hence, we can just say that this system is like or this partial differential equation represents a Elliptic nature.

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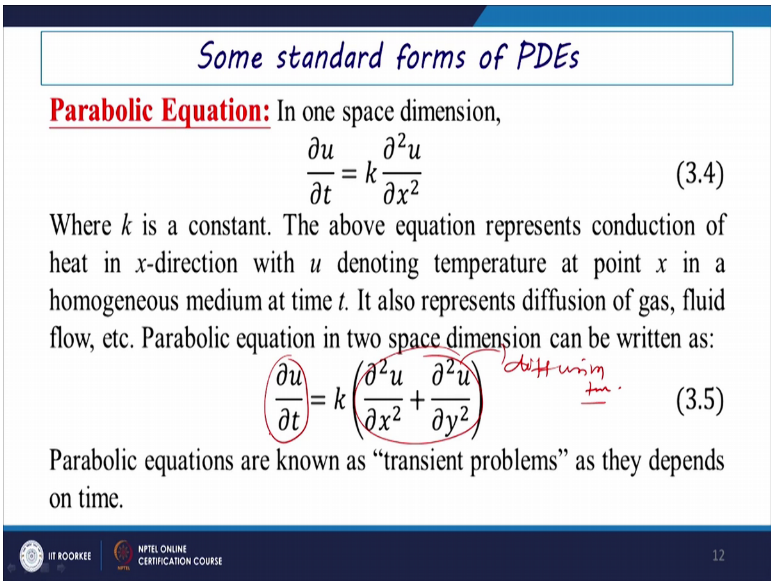

Then we will just go for some standard forms of partial differential equations here, in a one space dimension if you will just write this is a heat transfer equation we can just write this one as del u by del t equals to k del square u by del x square here where, k is a constant. The above equation represents conduction of heat in x direction here with u denoting the temperature at point x in a homogeneous medium at time t and it is also represents diffusion of like fluid flow etc, and if we want to express this Parabolic equation in a 2-dimensional form, then we can just express as a del u by del t equals to k into del square u by del x square plus del square u by del y square.

So, especially in the beginning of the lecture whenever I was just speaking about to this application there itself I have just explained the second order partial difference is terms in the form of like del square u by del x square plus del square u by del y square this is called especially the diffusion terms, here and this del u by del t this represents a time dependent variation of u there itself. So, that is sometimes called the transient problems since it depends on time here.

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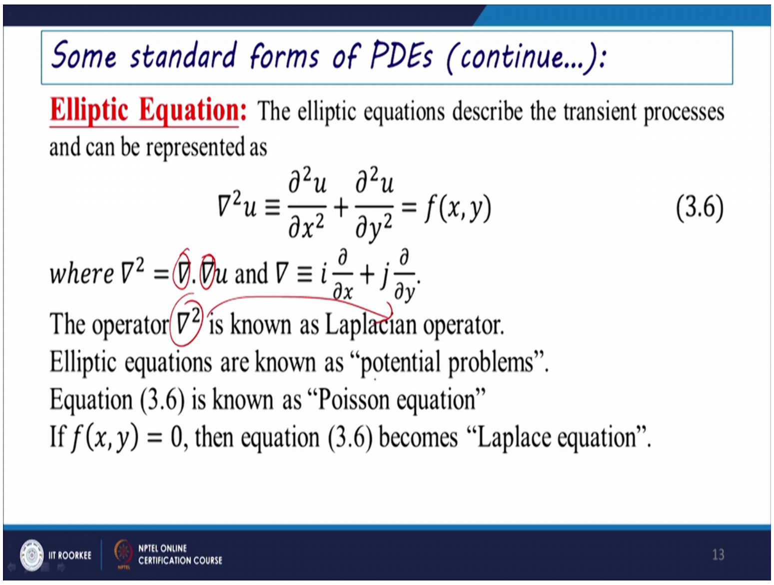

So, if you will just go for like Elliptic equation in a practical problem sense then this Elliptic equation, so describes the transient process which can be expressed as a del nabla square u equals to del square u by del x square plus del square u by del y square this equals to f of x y here, where del square or sorry this nabla square it can be represented as nabla dot nabla here. So, this nabla can be expressed as a like a I del y del x plus j del by del u here and this operator nabla square is known as the Laplacian operator here.

So, we can just say that this equation is a Laplace equation whenever f of x y equals to 0 here, when f of x y is a non-zero then it is called to be the Poisson equation. So, especially we are just saying here that Elliptic equations are known as potential problems. Since, this potential equations either it can be expressed as like Laplace equation or Poisson equation sense.

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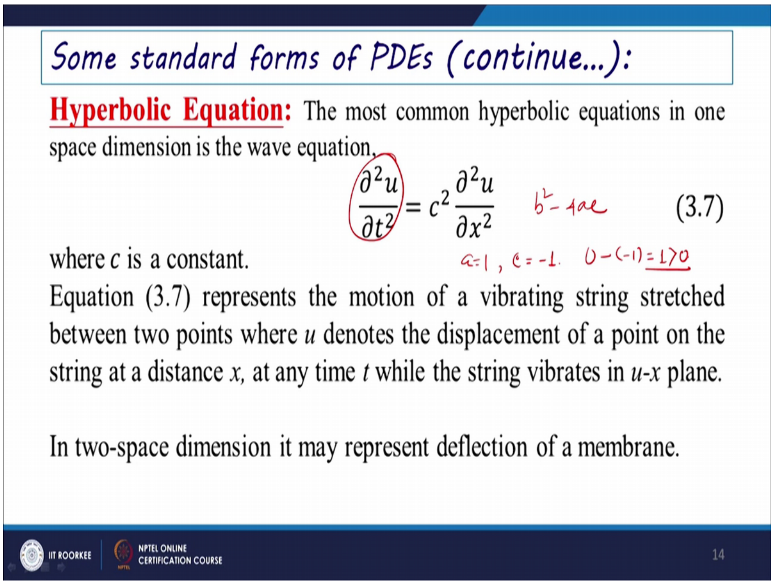

So, then if you will just go for like a Hyperbolic equation here the most common Hyperbolic equation in one space dimension is the wave equation here, if you will just see here this wave equation is represented as a del square u by del t square this equals to c square del square u by del x square here. So, 2 coefficients involved here. So, that is why we can just express this one in the form of like b square minus 4 ac here, and especially we have just to define this one that if b square minus 4 ac if it is like a greater than one this is a Hyperbolic equation.

So, that is why if you will just see here that if you will just take this coefficient to the left-hand side we can just express this one as del square u by del t square minus c square del square u by del x square this equals to 0. So, the coefficient it will take as like a equals to 1 and c equals to minus 1 here. So, that is why b square minus 4 ac if you will just take this one as a 0 minus of minus 1 this is nothing but 1 greater than 0.

So, that is why this is a Hyperbolic in nature and this class of problems are appearing, when this we will just consider the motion of a vibrating string stretched between 2 points where u denotes the displacement of a point on the string at a distance x at any time t while the string vibrates in extra you explained here, in a 2-space dimension it may represent deflection of a membrane sometimes.

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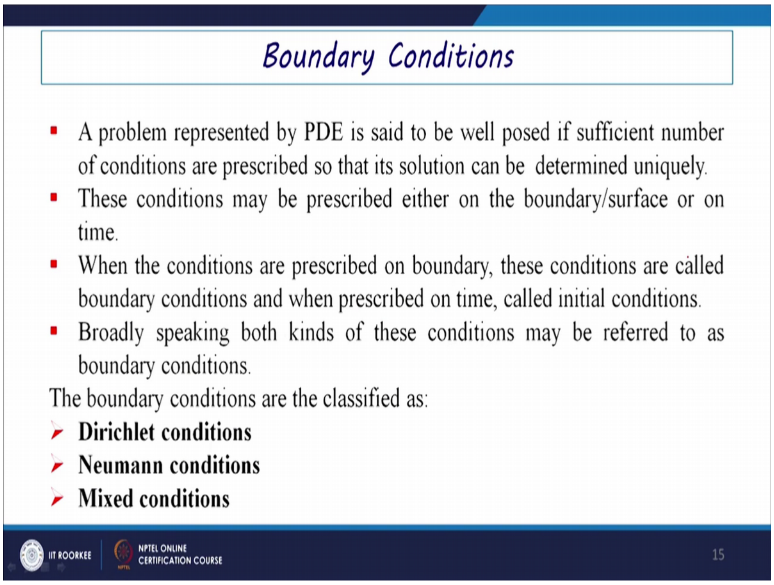

So, if you will just go for this boundary conditions like whenever we will just go for any solution of like ordinary differential equation, or partial differential equation it always requires some of these conditions to find the solutions either it is a called your initial condition or it is called boundary conditions.

So, especially we can just say that the problem represented by a partial differential equation is said to be well posed if sufficient number of conditions are prescribed, so that it is solution can be determined uniquely, and if these conditions are prescribed either on the boundary or along the surface it is called boundary conditions. Sometimes if it is prescribed suppose at initial level suppose at t equals to 0 suppose the conditions are given this is called the initial value problems.

So, we can just say that when the conditions are prescribed on the boundary these conditions are called boundary conditions, when prescribed on time this is called initial conditions. So, broadly speaking both kinds of these conditions may be referred to as one of the boundary conditions like the Dirichlet condition, Neumann conditions and mixed derivative conditions. Especially whenever we are just defining any real class of problems in partial sense, then one of this boundary condition is applicable in any real system.

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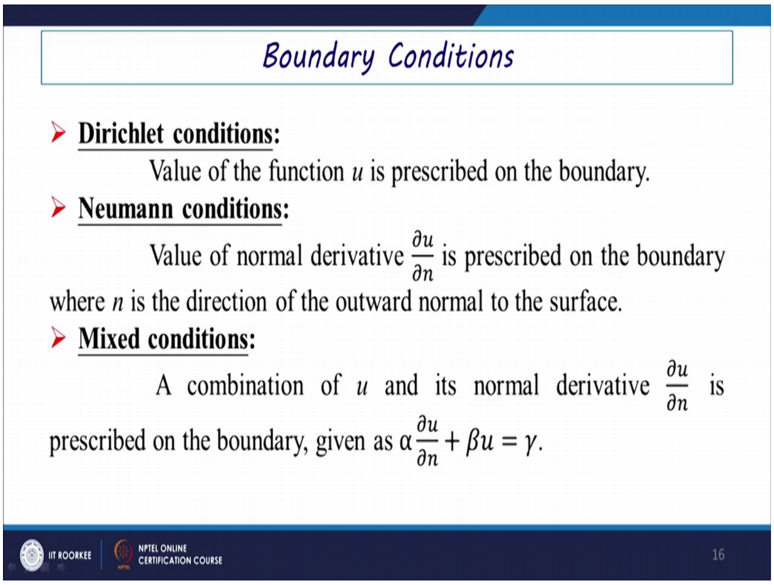

So, for this Dirichlet condition especially we can just consider that u is given directly or u can be prescribed at it is value at all of this boundaries that is specifically called the Dirichlet condition, if it is normal derivative is 0 then that is specifically called a Neumann boundary condition, and if we can just consider this combining of both this Dirichlet condition and a Neumann condition that is called a mixed condition here. This is just a combination of like a fixed value of u and normal derivative in multiplied with some like constant factors.

Then we can just prescribed it on the boundary as alpha del u by del n plus beta u equals to gamma, since our Dirichlet condition means we are just specifying the value u along the boundary, and a Neumann condition means we are just taking the normal derivative along the boundary, and if we are just combining both these 2 here then we can just multiply one factor with this Dirichlet condition and one constant factor with a Neumann conditions and both combinely just forming this mixed condition here.

Thank you for listen this lecture.