Numerical Methods: Finite Difference Approach
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Lecture – 20
Wendroff's method for solving hyperbolic equations

Welcome to the lecture series on Numerical Methods Finite Difference Approach. In the present lecture we will discuss about first order hyperbolic equations using a Wendroff's method followed with some examples.

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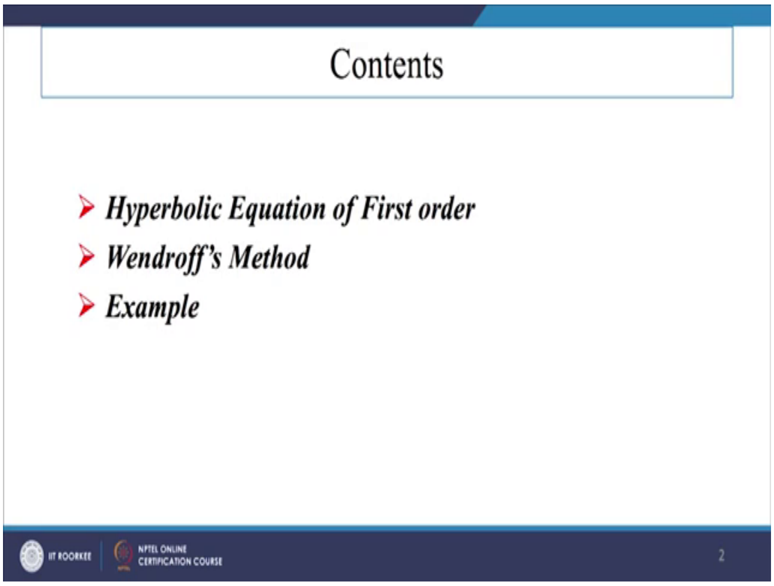

So, if we will just to go for this a hyperbolic equation of first order.

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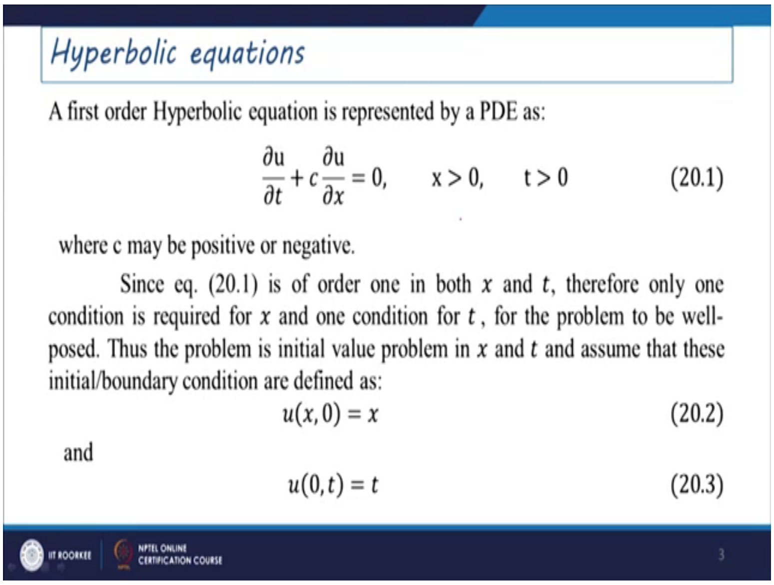

So, especially this a equations are written in the form of like a del u by del t plus c del u by del x this equals to 0 and for a well posedness of this problem we will have like a two conditions it is required, one in initial condition and one boundary condition there itself. And if you will just see here this coefficient c especially that depends on this like physical phenomenon of this problem that how this problem is occurring in a system. Sometimes in the heat transfer problems we are just considering c as the like specific heat capacity, sometimes in the wave equation we can just consider this as the wave parameters.

So, likewise in different scenarios or a different physical behaviour we can just consider these coefficients in different forms. And if you will just consider this problem as a initial value problem, where x and x is a just varying in the space coordinates and t is just varying in the time scale then this initial and boundary conditions are prescribed as like u of x 0 equal to x and this boundary condition it is prescribed as u of 0 t equals to t there.

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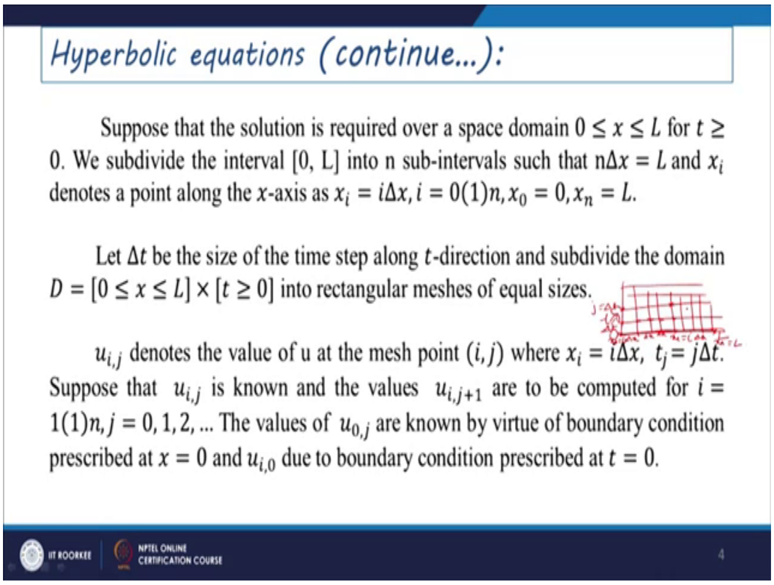

Suppose the solution is required over a space domain that this space domain in the x axis it is just varying from 0 to L here for all t greater than equal to 0 we have to find the solution there. And for this we will just a discretize this a x space coordinates into several small grids which is of like grid length has a del x and if you will just find a particular coordinate as suppose x i here which can be written as i delta x here. And the starting point we are just considering as a x 0 equals to 0 here and the end point it is just considered as x n equals to L here. Hence we can just consider that i is just varying from 0 to n here and for each of this a grid sizes of length del x is placed.

And if you will just consider the timing increments here suppose del t is the small time increment in the first level here and the proceeds with the steps as a del t in each step size and after suppose the jth step we can just say that this time is just moving up to the level j del t there and if we want to find the unknown at the level. Suppose i j plus 1 here then the values of u at i jth levels should be known to us here. Since always we will have a boundary condition which is prescribed along this boundary or in the x surface here and if we will just consider this know values as u 0 j as the boundary conditions here that is prescribed at x equals to 0, which is just prescribed at t equals to 0 initially we can just consider that one as.

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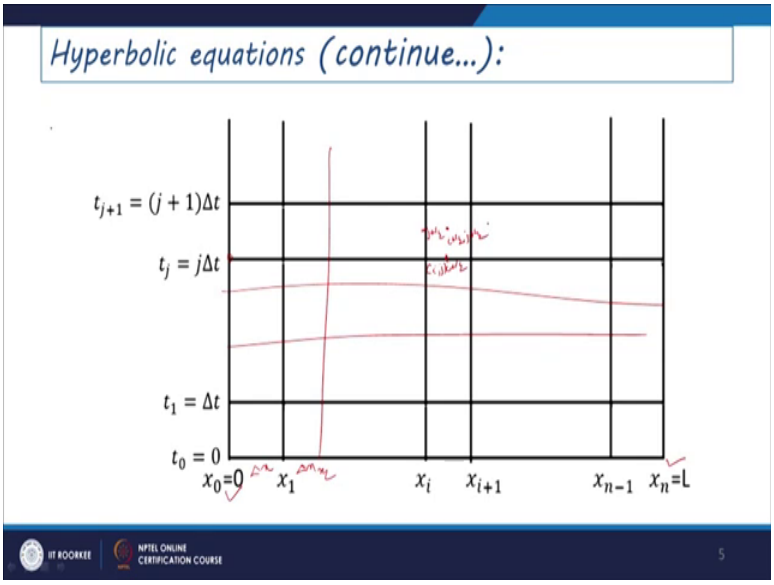

And if you will just proceed then we can just discretize this complete domain as defined in this picture here that is x 0 equals to 0 is the starting point and x n equals to L is the end point here we are just dividing this a total x axis or x space as m coordinates here that each of length of del x here. So, similarly we can just assume this time scale coordinates as a del t here and each of these time increments it is just added this a del t at each of these levels there. So, that is why at t jth level we will have this coordinate as a j del t there.

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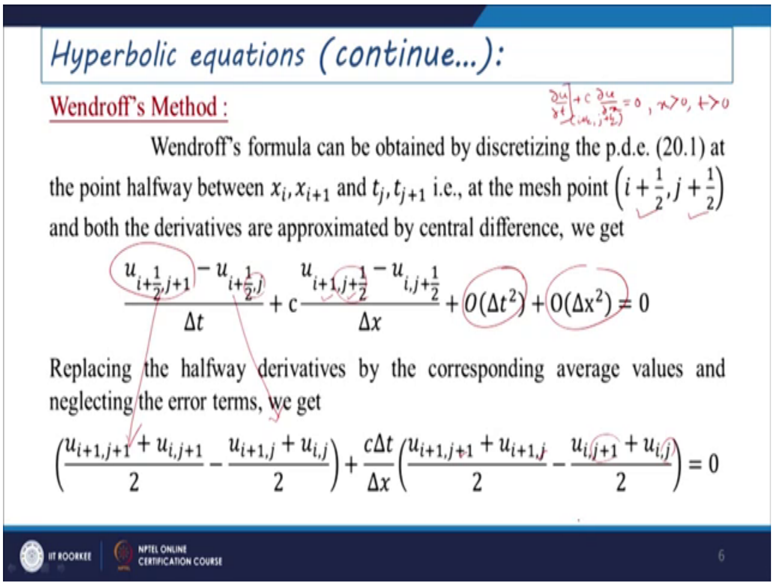

So, if will just go for a Wendroff's method here wendroff formula especially it is just obtained by discretizing this partial differential equation which is just a defined as a del u by del t this plus c del u by del x this as 0 here for x greater than 0 and t greater than 0 here we can just consider this point of discretization at the half way between x i, x i plus 1 and t j t j plus 1 here.

So, if suppose we will just consider i j coordinate is that this point here then we can just consider we are just moving here i plus half here and we are just moving j plus half here. So, especially this is the point where especially this computation we will go ahead. So, that is why we are just considering as the point mesh point as i plus half and j plus half there, and the both the derivatives are approximated by central differences here and if you will just discretize this del u by del t term at the point like i plus half and j plus half using central difference scheme here then we can just write this one as u i plus half this is a half point again it will just add off.

So, that is why it is just taken as j plus 1 there minus u i plus half. So, j plus half minus half it is there. So, that is why it is just taken as a j point there by del t plus c i plus 1 since we are just considering i plus half plus half there. So, that is why this is just coming as i plus 1 there. So, j plus half it is there minus u i plus half minus half it is there. So, that is why it is just taken as i there itself and j plus half is there. So, and if you especially if you will just see here. So, that is nothing, but if you can just see. So, this time merging it is also just going to del t by 2. So, 2 into 2 it is just cancel it out and del x by 2. So, that is why it is to going to it is just cancel it out.

So, if you will just replace this half way derivatives where the corresponding average values again and neglecting the error terms we can just write this one as since we have this point as in the form of u i plus half j plus 1, since i plus half point especially we do not know what is the exact value of u there itself. So, that is why we have to take the average of i plus 1 point and i point there itself. So, that is why this point is just to replaced as u i plus 1 j plus 1 plus u i j plus 1 by 2 and at i plus half especially this is also a unknown point and this is not exactly placed at the grid point. So, that is why to correspond to this grid point here we can just consider this one as i plus 1 here j plus u i j by 2 here.

Similarly if you will just go for this j plus half point and here then we can just write this one as a like j plus 1 and the j average of these two points and similarly if you will just you go for here like i plus half here. So, this can also goes from j plus 1 j here by 2 here. So, i remain fixed, so that is why it is just taken the average there this equals to 0 here.

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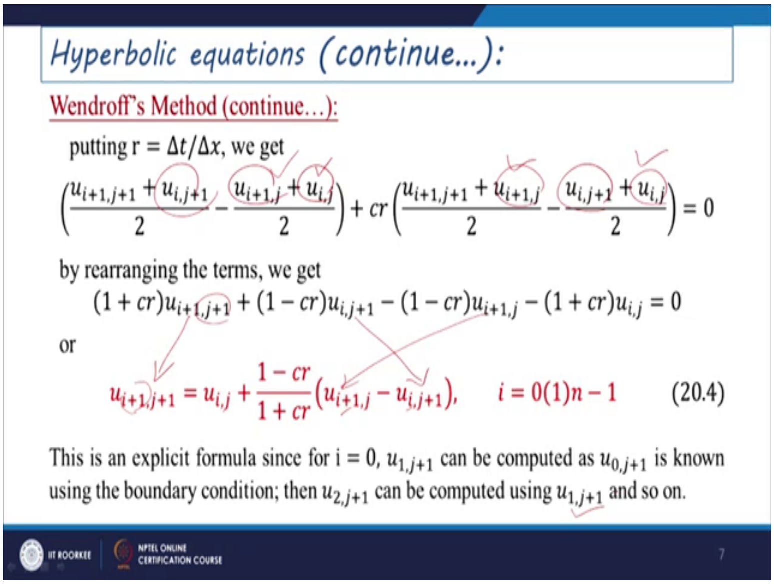

And if you will just put here r equals to del t by del x and rearrange all the terms since we will have this a terms like u i plus 1 j term here u i j term is here. So, u i j term is here u i plus 1 j term it is here and we will have the terms containing u i j plus 1 here we will have this u i j plus 1. So, combining from if you will just arrange all these terms by separating all j plus 1 terms at the beginning and j term at the end of this equation here we can just write this one as 1 plus cr since you if you will just see here u i plus 1 j plus 1. So, half it is also present there, half it is also present there. So, two both the sides we can just cancel it out from the denominator part. That is why we can just write this one as a 1 plus cr i plus 1 j plus 1 1 minus c r if you will just see here u i j plus 1 minus 1 minus c here u i plus 1 j here if you will just see this term and this term here minus 1 plus cr u i j term here u i j term here.

So, if you will just a separate this u i plus 1 j plus 1 time here we can just try this one as a u i j plus 1 minus here by 1 plus here and this is a u i plus 1 j term this is u i j plus 1 term. So, you can just write this term. So, if you will just see here this is a explicit formula since for i equals to 0 if you will just start the calculation here. So, i equals to 1 it will just your take. So, u 1 point it if you will just put here i equals to 0 then we will have a 0 value here then we will have one value here and here also 0 value it will be there, but this u 1 j plus 1 it can be computed as u 0 j plus 1 here which is known using the boundary conditions. If you will just go for like u 2 j plus 1 which can be computed it you will use like u 1 j plus 1 and so on it will just move.

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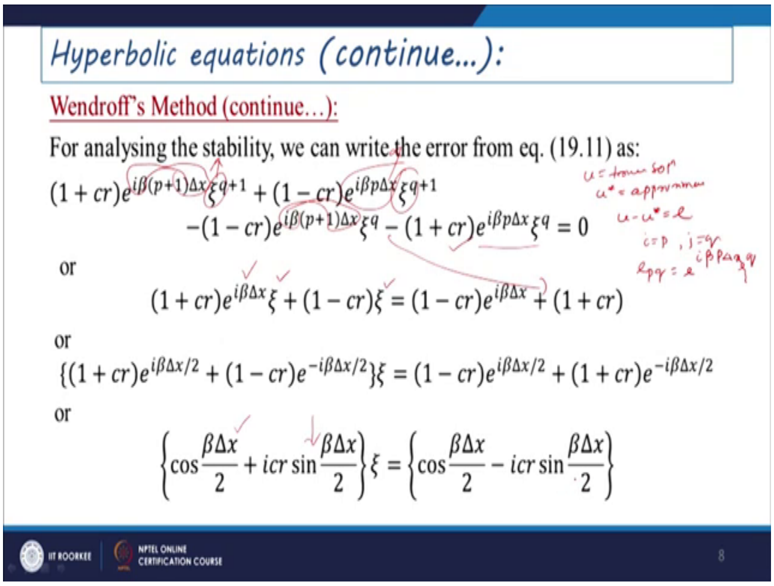

And if you will just go for the analysis of stability for this method your scheme it is expressed in the form of like u if it is expressed as the true solution and u star is the like approximate solution. So, especially u minus u star it is just considered as the error term there and each of these coordinates especially we are just writing i equals to p and j as a q there itself and the formula it is just replaced as in the form of e of a p q that is nothing but e to the power i beta p delta x and zeta to the power q there itself.

So, that is why if you will just write these terms here itself then we can just replace this one as like 1 plus cr 1 plus cr u i plus 1 j plus 1. So, that is why it can be written as e to the power i beta p plus 1 delta x zeta to the power p plus 1. Next term 1 minus cr e to the power i beta p delta x is equal to the power q plus 1 since here this is nothing, but i j plus 1 term minus 1 minus cr e to the power i beta b plus 1 delta x zeta to the power q here since you will have this term like i plus 1 j there, minus 1 plus cr e to the power i beta p delta x zeta equal to the q here. So, i j coefficient it is here, so that is why it is just written in this form.

If you will just see here we will have this common term that is in the form of e to the power i beta p delta x into theta to the t power q. So, both the sides if you will just cancel it out we will have like one plus cr e to the power i beta delta x. So, this term it is just there and zeta to the power q it has taken common. So, zeta is a still itself there then we will have 1 minus cr. So, zeta to the power q i beta p delta x it has taken common. So, that is why we will have this term like zeta here, so 1 minus cr, if it is just taken to the right hand side 1 minus cr e to the power i beta delta x term.

Here also same thing, we will have like a only one is a one plus cr is left there. So, that is why it has just taken to the right hand side. So, that is why this sine in sense to plus here. And if you will just go for this further modification of this equation we will have like 1 plus cr e to the power i beta delta x by 2 plus 1 minus cr e to the power i beta delta x by 2 into zeta this equals to 1 minus cr e to the power i beta delta x by 2 plus 1 plus cr e to the power minus i beta delta x by 2 or m is that we can just convert this series to the cos and the sign form. So, that is why we are just multiplying these factors to get it in this trigonometric function from here.

So, if you will just combine this exponential coefficients here. So, the first coefficient it will just to give you like e to the power i beta delta x by 2 and e to the power minus i beta delta x by 2 here, which is a nothing, but it will just give you cross beta delta x by 2. Similarly if you will just take the negation of this one it will just give you icr sine beta delta x by 2 and in the same manner if you will just go for the right hand side we can just open the storm. So, it is cos beta delta by 2 in a size cr sine beta delta x by 2.

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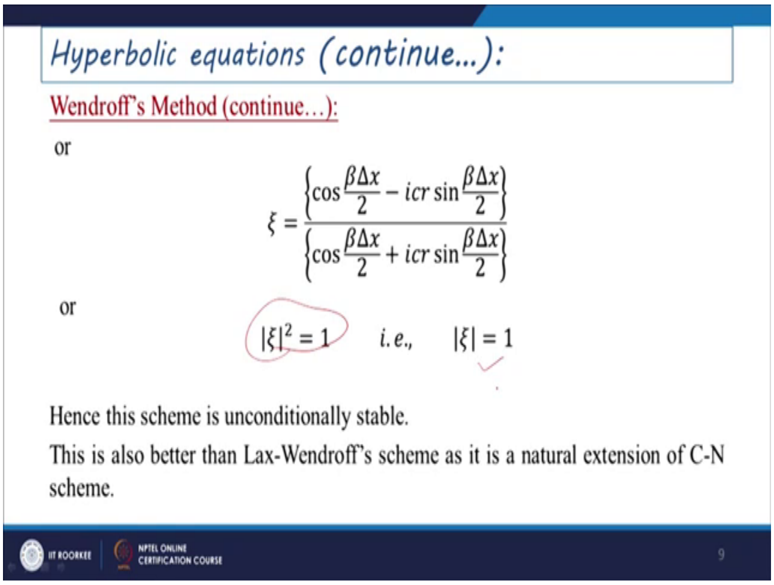

And directly we can just write zeta equals 2 cos beta delta x by 2 minus i sine icr sine beta delta x by 2 and cos alpha beta delta x by 2 plus icr sine beta delta x by 2 here take the absolute value of zeta square this equals to 1, that is mod zeta equals to 1.

Hence this scheme is a unconditionally stable. And this is also better than Lax-Wendroff's schemes since it is a natural extension of a Crack-Nicolson scheme which is used in our earlier cases.

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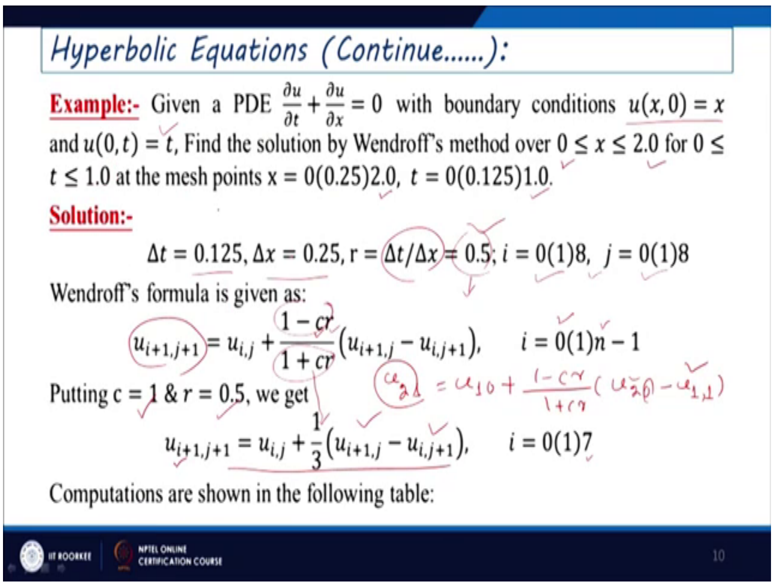

So, the example can be considered as like del u by del t plus del u by del x equals to 0 where c is considered as one here with the boundary conditions u of x 0 equals to x and u of 0 t equals to t here. So, the question is asked to find the solution using Wendroff's method over the x lying between a 0 to 2.0 especially you can just consider a x equals to 0 here and L is equal to 2.0 and your t value it is just lying between like a 0 to 1.0 with this mesh sizes which is just varying as like 0, 0.25, 0.5 in the x direction up to 2.0 and the t is just incremented in the form of a 0, 0.125 up to 1.0.

If you will just see here we have to follow like a from 0 to 8 steps to get the or to reach at x equals to 2.0 and if we will just follows like 8 steps we can just get t equals to 1.0 there. So, that is why we can just write it as like i, it is varying from 0 to 8 incremented with 1 and j is a just varying from 0 to 8 incremented with the value 1 there. And it will just consider like the del t equals to 0.125 and the del x is 0.25 r can be defined as a del t by del x here, which is nothing, but you can just write 0.125 by 0.25, which is nothing, but 0 0.5 here. And this formula if you will just see here that Wendroff's formula which is just a written as u i plus 1 j plus 1 this is nothing but u i j plus 1 minus cr by 1 plus cr into u i plus 1 j minus u i j plus 1 here. And it will just put here c equals to 1 and r equals to 0.5 here. Then we can just get it as a like 1 in the numerator and the 3, in the denominator here and u i plus 1 j minus u i j plus 1 here.

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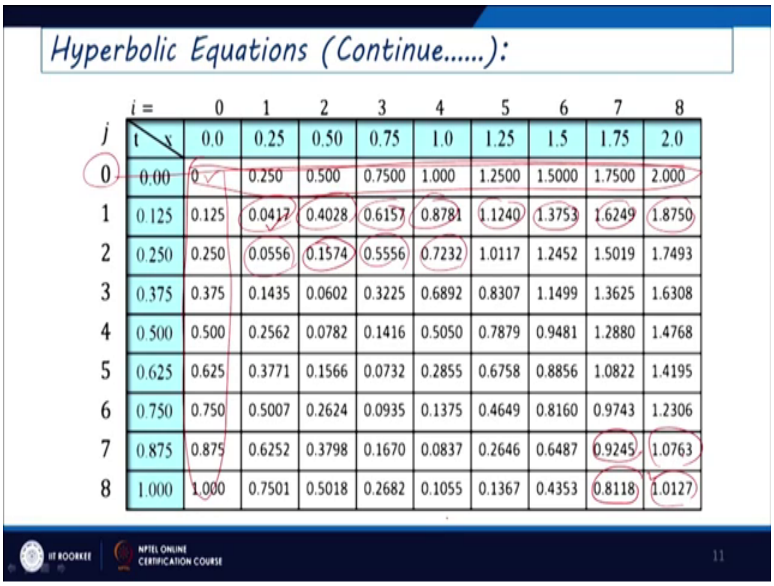

j \ i =		0	1	2	3	4	5	6	7	8
	t \ x	0.0	0.25	0.50	0.75	1.0	1.25	1.5	1.75	2.0
0	0.00	0	0.250	0.500	0.7500	1.000	1.2500	1.5000	1.7500	2.000
1	0.125	0.125	0.0417	0.4028	0.6157	0.8781	1.1240	1.3753	1.6249	1.8750
2	0.250	0.250	0.0556	0.1574	0.5556	0.7232	1.0117	1.2452	1.5019	1.7493
3	0.375	0.375	0.1435	0.0602	0.3225	0.6892	0.8307	1.1499	1.3625	1.6308
4	0.500	0.500	0.2562	0.0782	0.1416	0.5050	0.7879	0.9481	1.2880	1.4768
5	0.625	0.625	0.3771	0.1566	0.0732	0.2855	0.6758	0.8856	1.0822	1.4195
6	0.750	0.750	0.5007	0.2624	0.0935	0.1375	0.4649	0.8160	0.9743	1.2306
7	0.875	0.875	0.6252	0.3798	0.1670	0.0837	0.2646	0.6487	0.9245	1.0763
8	1.000	1.000	0.7501	0.5018	0.2682	0.1055	0.1367	0.4353	0.8118	1.0127

Computations will be performed for a different cells here. So, if you will just see i is varying from like 0 1 2 3 4 5 6 7 8 and the j is varying from 0 1 2 3 4 5 6 7 8 here. Since we are just considering these values as delta equals to 0.125 and a del x as 0.25 where this x is a just going from this region like 0 to 2.0. So, that is why if you will just divide in the form of 0.25 here. So, each of these a grid spacings then it will just vary in the form of like a 0.0, 0.25, 0.50, 0.75, 1.0, 1.25, 1.5, 1.75 and 2.0.

For the time scale if you will just see here. So, time scale is defined starting from 0 to 1.0 with the increment as a 0.125. So, that is why if you will just see this increments in the table here also. So, that is just going as a 0.0 then 0.125, 0.250, 0.375, 0.5, 0.625, 0.75, 0.87 and 1.0. And each of the cells if you will just see here that is the first coordinate if you will just see that is as varying as i equals to 0 and as a equals to 0 here. So, all these initial conditions. So, that is just kept it as a 0, 0.125, 0.250 since at the t equals to 0 we have to define that one. So, that is why if we will just see this problem here over then that is just a defined at a t equals to 0 u of x 0 equals to x here.

So, that is why u of x 0 equals to x means at t equals to 0 if you will just see. So, that is just a defined here, that is like 0.25, then 0.5, then 0.75, 1.0, then 1.25, then 1.5, 1.75 and 2.0 here. And if you will just see this boundary conditions especially the boundary values are defined at like u of 0 t here that as t there. So, that is why this boundary conditions that is at x equals to 0 if you will just see. So, that is also defined as a t 0, t 1, t 2, t 3, t 4, t 5, t 6, t 7, t 8. So, that is why from this prescribed values we can just calculate this boundary values and this initial values can be calculated also and it can be placed in that from there itself.

For the first cell calculation that is at i equals to 1 and j equals to 1 if you will just see this formulation here. So, that is especially written as u i plus 1, j plus 1, this is written as u i j plus 1 minus cr by 1 plus cr u i plus 1 j minus u i j plus 1. So, starting value if you will just put here i equals to 0 and j equals to 0 then we can just obtain the values at u 1 1 there. So, if you will just see here u 1 1 value that is as a computed as a 0.0417 which is a nothing, but if you will just see here. So, all computed values if you will just see here, c can be prescribed and r is also given as a del t by del x that is a 0.5 here. So, and u 0 0 that is at i equals to 0 and j equals to 0. So, this value is a known to us.

Aince if you will just see here this is 0 0 cell value for u it is just taken as a 0 value there and then u 1 0 minus u 0 1 there. So, all the values are known to us for this problem here. Since c equals to 1 r equals to 0.5 here then this is separated equation it is just known as that u i plus 1 j plus 1 is the unknown and u i j it is a known value to us and u i plus 1 j minus u i j plus 1 these two values are also known to us wherever if you will just see i us varying from a 0 to n minus 1 since it is cell counting value it is just a final value is going up to 8. So, that is why I will vary to from 0 to 7 here incremented as a 1 waveform here. So, that is why I equals to 0 1 2 3 up to 7 we will just calculate these values.

Since in the final form if you will just put here i equals to 7, so that will just take as u 8 3 value and j plus 1 which it means they can varied from like a first initial value it can be taken as a from the first step calculation then after that all other incremented value it can be calculated by using this equation. So, if you will just go for this a second value here that is in the step of like j equals to 1 and i equal to 2 here which can just take this value as 0.4028 which is nothing, but if you will just see here this means that i equals to 2 here j equals to 1. So, that is why we can just put i equals to 1 here and the z equals to 0 there. So, that is why it will just provide us as the value of like u 21. So, that is why we can just write this values as a u 1 this will take as a 0 here plus 1 minus cr by 1 plus cr and it can be written as like since a i value is a taken as a 1 here, so that is how it can also be written as a 2. Then j value is a taken as a 0 here to 0 minus u 1 then j plus 1 means this is 1 there.

So, this u 1 1 value it can be taken from this like reverse type of calculation from here and then if you will just see here u 2 here is 0 value, 2 0 means especially you can just say that or equals to 2 here and j equals to 0. So, this can be taken from this initial condition and then we can just obtain the value of u 2 1 there.

Similarly if you will just consider this a 2 2 value here then we will just considered I as 1 there and j as 1 there. So, then for this step if you will just see here i equals to 1 and j equals to 1 here then you will have like u 2 2. So, then this can be written as like u 1 1 plus 1 by 3 into u 2 1 minus u 1 2 there. So, u 1 2 and u 2 1 it can be computed from this previous step of calculations and it can be used to obtain this value at u 2 2 level there. So, further calculations it can just carried out like a sequential manner whatever I have just explained like in the earlier cases. So, if you will just proceed in that form then you can just obtain this values as a 3 1 here. So, then 4 1, then 5 1, then 6 1, then 7 1 and finally, 8 1 here and if you will just use all of these previous table uses there are no further steps like for a increment of j in each steps we can just obtain these values in this form here. And in the final step if you will just see here like u 8 8 steps, you will just use this value in the u 7 7 step and this value and this value here to get this final value there.

So, if we will just see here that u i plus 1 j plus 1 that is nothing but it can just take as a combined formulation of the values that at the i jth level and i plus 1 the jth level and i j plus 1th level. And this is just a simple calculation if you will just do then you can just obtain at each iteration values up to our last point.

Thank you for listen this lecture.