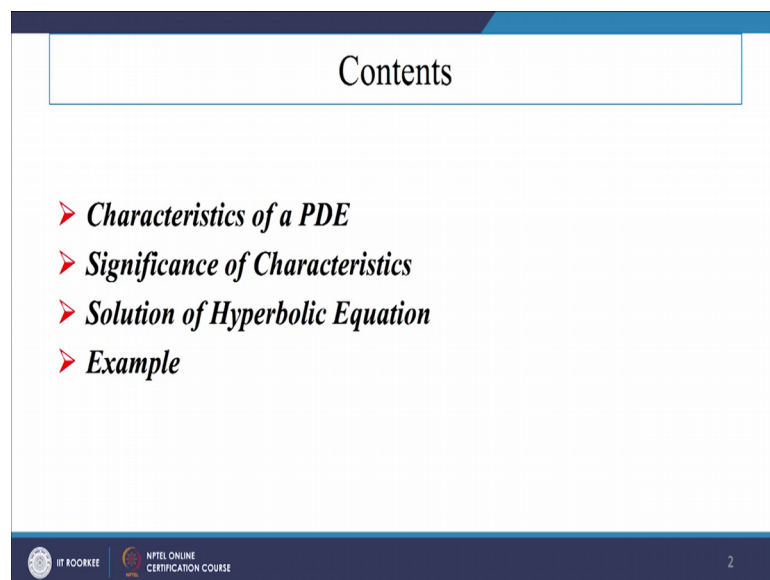


Numerical Methods: Finite Difference Approach
Dr. Ameeya Kumar Nayak
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 18
Characteristics of PDE

Welcome to the lecture series on Numerical Methods: Finite Difference Approach, in this approach we have discussed like elliptic equations, parabolic equations, hyperbolic equations, and their solution techniques.

(Refer Slide Time: 00:34)



Contents
➤ <i>Characteristics of a PDE</i>
➤ <i>Significance of Characteristics</i>
➤ <i>Solution of Hyperbolic Equation</i>
➤ <i>Example</i>

IIT ROORKEE NPTEL ONLINE CERTIFICATION COURSE 2

In the present lecture we will discuss about, this characteristics of a partial differential equation then the significance of characteristics and afterwards we will just go for like solution of a hyperbolic equations with examples.

(Refer Slide Time: 00:45)

Characteristics of a Partial Differential equation

A general form of second-order PDE is defined over a domain D is written as:


$$a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} + e = 0 \quad (18.1)$$


where a, b, c & e are functions of $x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ generally. Let us write the above equation using the following notations:

$$p = \frac{\partial u}{\partial x} = u_x, \quad q = \frac{\partial u}{\partial y} = u_y$$

$$r = \frac{\partial^2 u}{\partial x^2} = \frac{\partial p}{\partial x}, \quad s = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}, \quad t = \frac{\partial^2 u}{\partial y^2} = \frac{\partial q}{\partial y}$$

We get:




NPTEL ONLINE
CERTIFICATION COURSE

3

So, if you will just go for this characteristics of partial differential equation. So, usually we are just writing the second order partial differential equation in the form is $a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} + e = 0$, where especially a, b, c and e are functions of x, y, u, u_x, u_y ; generally it is just expressed in this form.

So, if you will just write these expressions in the notations as p equals to suppose $\frac{\partial u}{\partial x}$ as u_x here and q as $\frac{\partial u}{\partial y}$ which is just defined as u_y here and r is the second derivative term that is especially $\frac{\partial^2 u}{\partial x^2}$ that is nothing, but $\frac{\partial}{\partial x}$ of $\frac{\partial u}{\partial x}$. So, that is why it is just written as $\frac{\partial p}{\partial x}$ here, s which is defined as $\frac{\partial}{\partial x}$ of $\frac{\partial u}{\partial y}$ which is nothing, but $\frac{\partial p}{\partial y}$ or $\frac{\partial q}{\partial x}$. We can just write and if you will just write t terms here. So, t can be written as like $\frac{\partial^2 u}{\partial y^2}$ that is nothing, but $\frac{\partial}{\partial y}$ of $\frac{\partial u}{\partial y}$ that is nothing, but $\frac{\partial q}{\partial y}$ here.

(Refer Slide Time: 02:02)

Characteristics of a PDE (continue...):

$$ar + bs + ct + e = 0 \quad (18.2)$$



Let G be a given curve in D where PDE (18.1) holds. Assuming that u , u_x & u_y are continuous functions in D. On curve G, we have:

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = r dx + s dy \quad (18.3)$$

$$dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy = s dx + t dy \quad (18.4)$$

Since PDE is satisfied at each point along the curve G, therefore by putting the values of r and s from eq. (18.3) and (18.4) in eq. (18.2), we get:

$$a \left(\frac{dp}{dx} - s \frac{dy}{dx} \right) + b s + c \left(\frac{dq}{dy} - s \frac{dx}{dy} \right) + e = 0$$

 IIT ROORKEE
  NPTEL ONLINE CERTIFICATION COURSE
 4

So, if you will just write it in incomplete form then it can be expressed as ar plus bs plus ct plus e equals to 0. Suppose a curve like G be a curve which is defined in a domain D where this partial differential equation holds. So, assuming that suppose u , u_x , u_y are continuous functions in D and along this curve if you will just consider like dp equals to $\frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy$.

This is as a sense we have just define r as $\frac{\partial p}{\partial x}$ and s as $\frac{\partial p}{\partial y}$. So, you can just write this one as $rdx + sdy$ over here and similarly if you will just defining as dq that is $\frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy$ here which can be defined as $sdx + tdy$ and if we will just express this partial differential equation is satisfied at each point in the curve G which is defined inside this a domain D.

So, if you will just put all these values of r and s from like first equation in the complete form then we can just write this one as since it is just expressed as $a \frac{dp}{dx} - s \frac{dy}{dx} + bs + c \left(\frac{dq}{dy} - s \frac{dx}{dy} \right) + e = 0$. So, that is why r can be written as like $\frac{dp}{dx}$ minus $s \frac{dy}{dx}$ plus bs .

Especially if it is it is like that way it is just present there c into t , especially t can be replaced as like $\frac{dq}{dy} - s \frac{dx}{dy} + e$ equals to 0 here.

(Refer Slide Time: 03:44)

Characteristics of a PDE (continue...):

or

$$\left(a \frac{dp}{dx} + c \frac{dq}{dy} + e \right) - s \left(a \frac{dy}{dx} - b + c \frac{dx}{dy} \right) = 0$$



or

$$\frac{dy}{dx} \left(a \frac{dp}{dx} + c \frac{dq}{dy} + e \right) - s \left\{ a \left(\frac{dy}{dx} \right)^2 - b \frac{dy}{dx} + c \right\} = 0 \quad (18.5)$$

If the curve G is described by the solution of eq.

$$a \left(\frac{dy}{dx} \right)^2 - b \frac{dy}{dx} + c = 0 \quad (18.6)$$

then on curve G, from eq. (18.5), we have



5

And in a compact form if you will just express this equation then it can be written as a $\frac{dp}{dx}$ plus $c \frac{dq}{dy}$ plus e minus s a $\frac{dy}{dx}$ minus b plus $c \frac{dx}{dy}$ this equals to 0 and if you will just multiply dy by dx at both the sides. So, it can be expressed as a $\frac{dy}{dx}$ a $\frac{dp}{dx}$ plus $c \frac{dq}{dy}$ plus e minus s . Since $\frac{dy}{dx}$ is multiplied so, it can be expressed as a $\left(\frac{dy}{dx} \right)^2$ minus $b \frac{dy}{dx}$ plus c .

Since it is just cancel it out since a $\frac{dy}{dx}$ into dx by dy so, that is why it is just convenient 0 there. So, if they curve G which is satisfied by the solution of this equation as a $\left(\frac{dy}{dx} \right)^2$ minus $b \frac{dy}{dx}$ plus c equals to 0. We can just consider this left hand part of this equation as 0 there. So, if you will just put this one as 0 here.

(Refer Slide Time: 04:42)

Characteristics of a PDE (continue...):

$$\frac{dy}{dx} \left(a \frac{dp}{dx} + c \frac{dq}{dy} + e \right) = 0$$

or


$$a \frac{dy}{dx} \cdot \frac{dp}{dx} + c \frac{dq}{dy} + e \frac{dy}{dx} = 0$$


or

$$a \frac{dp}{dx} dy + cdq + edy = 0 \quad (18.7)$$

Eq. (18.6) is quadratic in $\frac{dy}{dx}$ whose roots are given as:

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} \quad (18.8)$$



 NPTEL ONLINE
CERTIFICATION COURSE

6

So, we can just write dy by dx into $a \frac{dp}{dx} + c \frac{dq}{dy} + e$ equals to 0 and ah; obviously, if you will just multiply this term in the equation, then it can be expressed as $a \frac{dy}{dx} \frac{dp}{dx} + c \frac{dq}{dy} + e \frac{dy}{dx}$. Since you are special dy you it will just consider out plus $e \frac{dy}{dx}$ this equals to 0.

So, if you will just see here this can be written as like in a differential form if you will just express this can be expressed as $a \frac{dp}{dx} dy + cdq + edy$ this equals to 0 and if you will just solve this equation that is expressed in 18.6 here, that is $a \frac{dy}{dx}$ whole square minus $b \frac{dy}{dx} + c$ equals to 0..

This is a quadratic equation here and its root can be expressed as $\frac{dy}{dx}$ equals to b plus or minus square root of b square minus $4ac$ by $2a$ here..

(Refer Slide Time: 05:46)

Characteristics of a PDE (continue...):

The solution of (18.8) are called characteristic curves and defines the curve G in $x - y$ plane. If a, b, c are constants or functions of x, y then the equation of curves can be obtained explicitly. Eq. (18.8) gives the direction of characteristic curves at the point (x, y) of G .

If $b^2 - 4ac < 0$ in (18.8), there exist no real characteristics i.e., in case of elliptic equations there will be no real characteristic curves; if $b^2 - 4ac = 0$, then there will be one characteristics which corresponds to parabolic equation; if $b^2 - 4ac > 0$, then eq. (18.8) gives two real characteristics i.e., in case of hyperbolic equations there will be two real characteristic curves.

IIIT ROORKEE | NPTEL ONLINE CERTIFICATION COURSE

7

And if you will just go for the solution, of this a characteristic a curves which is a just defined on this curve G in $x - y$ plane. If a, b, c are constants or functions of x, y then the equation of course, can be explained it in explicit form and if you will just go for this like equation of 18.8 here.

So, which is just expressed as a dy by dx form here. So, that just gives the direction of this characteristic cause at the point x, y of G there. So, first condition if you will just considered here as b square minus $4ac$ suppose less than 0. So, then there does not exist any real characteristics there that is in case of elliptic equations. So, especially this is the condition that is justified in case of elliptic equations..

There will be no real characteristic course and if you will just consider like b square minus $4ac$ goes to 0 then that will just represent the parabolic equation form and if you will just see here this can be justified for a single characteristics there. .

So, we will have like one characteristics there , but if you will just go for b square minus $4ac$ greater than 0 then this equation 18.8 it will just provide two real characteristics that is in case of hyperbolic equations there will be two real characteristic curves there.

(Refer Slide Time: 07:07)

Characteristics of a PDE (continue...):

Significance of Characteristics:-

The importance of characteristic curves defines as the solution of a linear PDE is unique at any point in the region enclosed by an initial curve C and characteristic curve.

In case of parabolic equations, we have $a = 1, b = 0, c = 0$ so the characteristic curve is given from eq. (18.8) as:

$$\frac{dy}{dx} = 0 \quad \text{or} \quad y = \text{constant} \quad (18.9)$$

Thus the region of determinacy of solution in case of parabolic equation is the region enclosed by initial line, $y = y_0$ and the characteristic $y = y_c(\text{const.})$ in the $x - y$ plane.

III ROORKEE NPTEL ONLINE CERTIFICATION COURSE 8

So, if you will just go for this significance of this a characteristic curves there as a solution of like linear partial differential equation, it should be unique at any point and this region is enclosed by an initial curve C and have a characteristic curve there. In case of parabolic equations we will have like a equals to 1, b equals to 0, c equals to 0 there, since we have to show that $b^2 - 4ac$ equals to 0 there.

So, in that case if you will just put all these values then we can just obtain this dy by dx which is defined as like $a \pm \sqrt{b^2 - 4ac}$ by $2a$ that can be taken as a 0 value there and if you will just find the solution of this equation that is much especially nothing, but y equals to a constant there..

And if you will just go for this region of a determine C of solution in case of parabolic equation is the region enclosed by the initial line that is y equals to y_0 and the characteristic curve that is just defined by this constant function y equals to y_c there in the $x - y$ plane.

(Refer Slide Time: 08:14)

Characteristics of a PDE (continue...):

In case of hyperbolic equations



$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2},$$

we have $a = 1, b = 0, c = -1$ so the characteristic curve is given from eq. (18.8) as:

$$\frac{dy}{dx} = \pm 1$$

By changing the variable y to t , the equations of characteristics are given in $x - t$ plane as following:

$$t = x + \text{constant} \quad (18.10)$$
$$t = -x + \text{constant} \quad (18.11)$$

 IIT ROORKEE  NPTEL ONLINE CERTIFICATION COURSE

9

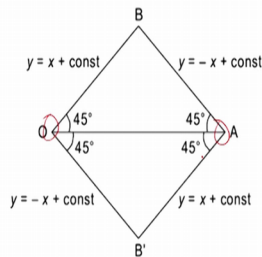
And if you will just go for this hyperbolic equation sense here so, this hyperbolic equation especially if you will just write a generalized hyperbolic equations. So, it can be expressed as the del square u by del x square this equals to del square u by del y square or in generalized form we are just expressing this one as a del square u by del t square this equals to del square u by del x square there, where we are just writing a equals to 1, b equals to 0, c equals to minus 1. So, that the characteristic curve will be given by as dy by dx this equals to plus or minus 1; we will have two solutions here.

So, if you will just change this variables y in forms of t here in the $x - t$ plane then we can have the solutions like t equals to x plus a constant and another solution that will be minus x plus a constant there. .

(Refer Slide Time: 09:02)

Characteristics of a PDE (continue...):

From the figure, if OA denotes initial curve C on which appropriate boundary conditions are prescribed, then the region of determinacy is given by OBAB'O. There are two characteristics at each of the points O and A.



Region of determinacy of hyperbolic equation

So, if we will just denote it in a figure suppose, if suppose OA denotes the initial curve C on which appropriate boundary conditions are suppose prescribed, then the region of a determinacy is given by like OBAB dash O there. So, there are two characteristics at each of the points O and A which is just given in the form of y equals x plus constant; another one is y equals to minus x plus constant.

So, if you will just determine this two characteristics is one it is just defined as at the point O another which is defined at the point A. So, this can be just a represent in this form here; that is especially the region of determinacy of hyperbolic equations..

(Refer Slide Time: 09:49)

Characteristics of a PDE (continue...):

Method of characteristics for solving hyperbolic equations:-

Referring to the general form of second-order PDE (18.1) and assuming it to be hyperbolic equation then we have two characteristics as:



$$\frac{dy}{dx} = \frac{b + \sqrt{b^2 - 4ac}}{2a} = f \quad \int dy = \int f dx \quad (18.12)$$

and

$$\frac{dy}{dx} = \frac{b - \sqrt{b^2 - 4ac}}{2a} = g \quad (18.13)$$

where f and g are functions of x, y, u, u_x & u_y .

Now consider that the domain is extended between $-\infty < x, y < \infty$ in $x - y$ plane.



 IIT ROORKEE | NPTEL ONLINE CERTIFICATION COURSE

11

And if you will just go for this like characteristics for solving this hyperbolic equations, referring to the general form of the second order partial differential equation which is just defined in the beginning of this slide and assuming, that it should be hyperbolic a inform then the two characteristics are as like dy by dx this is just expressed as like a b plus square root of b square minus $4ac$ by $2a$ one and another one it is just expressed as b minus square root of b squared minus $4ac$ by $2a$ which is just a written in this form.

If you will just signify this first function as f here and second function as g here, where f and g are functions of you can just considered as x, y, u, u_x or u_y there. So, if you will just consider this domain is extended between like a open space like $x - y$ plane. So, which is just a defined from minus infinity to infinity in the $x - y$ plane.

(Refer Slide Time: 10:40)

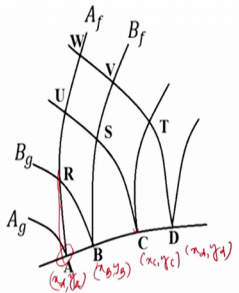
Characteristics of a PDE (continue...):

and the values of u , u_x & u_y is given on the curve .



We will follow the following steps to find $u(x, y)$ at different points in hyperbolic domain:

Step-1 First we choose some points A, B, C, D,...(may or may not be equidistant) on the given curve and denote their positions by $A(x_A, y_A)$, $B(x_B, y_B)$, $C(x_C, y_C)$,...,etc.

Step-2 Then we draw the characteristics through these points A, B, C, D etc. If f and g in eq. (18.12) and (18.13) involves only x and y then equations of characteristics can be drawn as shown in figure; A_f is the f characteristic and A_g is the g characteristic through A and so on.



Characteristic Grid

 IIT ROORKEE
  NPTEL ONLINE CERTIFICATION COURSE

12

Then we can have the values like u , u_x , u_y are defined along this curve and to determine these points. So, if you will just go for like different steps the steps can be like described as follows here.

So, first step we will just choose these points like on the curve here that is characteristic grids we can just form and suppose if you will just to choose these points like A, B, C, D which may not be equidistance if you will just consider in a regular domain on the given curve and denote their position. Suppose A is like x_A, y_A here and B as like suppose x_B, y_B here and C as like x_C, y_C here and D as suppose x_D, y_D here..

And then we can just draw these characteristics through these points A, B, C, D and if f and g ; suppose which is just expressed in the equations like 18.12 here that is one is just taking like B plus another one is just taken as B minus there.

Which just involves so, only x and y then the equations of characteristics can be drawn as shown in the figure as here. That is A_f the f characteristic and A_g is the g characteristics. So, which is just defined at the point A, likewise we will have this a characteristic at the point B like B_f and B_g .

(Refer Slide Time: 12:22)

Characteristics of a PDE (continue...):

However, if f and g involve u and its derivatives then we have to find the location of intersection points namely R, S, T, etc., and compute the values of u there.

Integrating (18.12) from A to R, we get:

$$y_R - y_A = f_A(x_R - x_A) \quad (18.14)$$



Integrating (18.13) from B to R, we get:

$$y_R - y_B = g_B(x_R - x_B) \quad (18.15)$$

By solving the equations (18.14) and (18.15), we get the coordinates $R(x_R, y_R)$ of the grid point R.

Step-3 To find the solution to grid points U, V, the values of u , u_x & u_y will be required at grid points R, S, T. From eq. (18.7), we have

$$a \frac{dy}{dx} dp + cdq + edy = 0 \quad (18.16)$$

 IIT ROORKEE  NPTEL ONLINE CERTIFICATION COURSE 13

If you will just find here f and g involves this u and its derivatives then exactly we can find the location of intersection points namely like if you will just see here that is the points as R, S, T here.

So, to determine these points here we can just integrate this equation like 18.12 from like a A to R here if you will just see point A to R, yes this one. So, we can just integrate this a differential equation there. So, that will just give you like $y_R - y_A$ this equals to f_A into $x_R - x_A$ there..

Since this a equation if you will just see 18.12 here, yes this one dy by dx that is nothing, but defined a as function f here if you will just integrate simply here. So, integration of dy we can just write this one as integration of $f dx$.

Since the point it is just defined at the point a itself there. So, that is why this functional value we can just assume this one as f here there and this integration dx that is just a varying from $x_R - x_A$ there and similarly if you will just integrate a equation like 18.13 from B to R. We can just get it as like $y_R - y_B$ this is equals to g_B into $x_R - x_B$ there.

So, by solving these equations like 18.14 and 18.15 here we can get the coordinates of R that is as x_R, y_R of the grid point are there itself. To find the solution to the grid points U, , the values of u , u_x and u_y will be required at the grid points like R, S, T. .

So, for that we will just solve these equations that is in the form of like a dy by dx dp plus cdq plus edy equals to 0 and if you will just go for this integration of 18.16 equation here.

(Refer Slide Time: 14:23)

Characteristics of a PDE (continue...):

Integrating (18.16) from A to R along the characteristics f , we get:

$$a_A f_A (p_R - p_A) + c_A (q_R - q_A) + e_A (y_R - y_A) = 0 \quad (18.17)$$

Integrating (18.16) from B to R along the characteristics g , we get:

$$a_B f_B (p_R - p_B) + c_B (q_R - q_B) + e_B (y_R - y_B) = 0 \quad (18.18)$$

By solving the equations (18.17) and (18.18), we get the values p_R and q_R .

Step-4 To find the values of u at grid points R i.e., u_R , we use

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$u = p dx + q dy \quad (18.19)$$

So, then we can just find here a dy by dx that is nothing, but it is just defined as the function f there. So, that is why it can be taken as a $A f A$ and this integration range that is just varying dp integration, that is why we have just considered as a p_R minus p_A there. Then if you will just say so, cdq means q range is just varying from R to A there itself. So, $c_A q_R$ minus q_A plus e is the point defined at the point a there itself. So, e_A plus dy is the range. So, y_R minus y_A these equals to 0 there and similarly if you will just integrate this equation 18.16 from B to R along the characteristics curve g . We can just get this one as like a $B f_B p_R$ minus p_B plus c_B and q_R minus q_B plus $e_B y_R$ minus y_B there.

So, same way just point has been just getting it is changed. So, by solving these two equations like 18.17 and 18.18, we get the values of p_R and q_R . To find the values of u at the grid points like R that is u_R , we can just use this formula that is as du equals to $\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$ there, where if you will just find the solution here u is nothing, but a $p dx$ plus $q dy$.

(Refer Slide Time: 15:53)

Characteristics of a PDE (continue...):

Integrating (18.19) either from A to R along the characteristics f (or from B to R along the characteristics g), we get:

$$u_R = u_A + \frac{(p_R + p_A)}{2}(x_R - x_A) + \frac{(q_R + q_A)}{2}(y_R - y_A) \quad (18.20)$$

After getting the approximate values of grid point R (x_R, y_R) and u_R , we will improve these values by modifying the steps 2 to 4 as follows:

Step-5 Since the values of f and g can be computed at the point R also, we modify it by taking their average values, i.e.,

$$y_R - y_A = \frac{(f_R + f_A)}{2}(x_R - x_A) \quad (18.21)$$

and

IIIT ROORKEE | NPTEL ONLINE CERTIFICATION COURSE 15

And; obviously, if you will just integrate this equation from A to R along the characteristics a curve f or from B to R along the characteristics curve G , we can just get u_R equals to u_A plus p_R plus p_A by 2 into x_R minus x_A ; that is the distance from like point A to R. in the x direction and if you will just write for this a q position there. So, it can be written as q_R plus q_A by 2.

So, we are just considering this average of two values there if you will just see the curve there. So, A and B it is just considered and both these curves so, that are just intersecting at the point are there itself. So, that is why we are just considering p_R plus p_A by 2 there and similarly since both these curves they are just intersecting at the point like R and A for q value also..

So, that is why we have just considered q_R plus q_A by 2 into y_R minus y_A there. After getting the approximate values of grade point R here x_R to y_R and u_R , we will improve these values by modifying the steps 2 to 4 as follows.

Since the values of f and g as computed like at the points R also, we modify it by taking the average value. So, that is like y_R minus y_A if you will just consider; since earlier we have just considered as f_A there. So, which is just nothing, but the average we have just considered as f_R plus f_A by 2 into x_R minus x_A here.

(Refer Slide Time: 17:19)

Characteristics of a PDE (continue...):



$$y_R - y_B = \frac{(g_R + g_B)}{2}(x_R - x_B) \quad (18.22)$$

By solving the equations (18.21) and (18.22), we get the improved coordinates (x_R, y_R) of the grid point R.

Step-6 Similarly, we can modify step-3 by using the average values of all parameters, we get:

$$\left(\frac{a_A + a_R}{2}\right)\left(\frac{f_A + f_R}{2}\right)(p_R - p_A) + \left(\frac{c_A + c_R}{2}\right)(q_R - q_A) + \left(\frac{e_A + e_R}{2}\right)(y_R - y_A) = 0 \quad (18.23)$$

and

Similarly, if you will just go for like y_R minus y_B so, same point if you will just consider here along the curve g there. So, you can just write that one is a g_R plus g_B by 2. So, that will just take this average of g function at that point so, into x_R minus x_B and if you will just solve this equation say 18.21 and 18.22; we get the improved coordinates x_R, y_R of this grid point R.

Similarly, we can just modify like a step 3 if you will just see here by taking this average. So, we can just write this one as like a_A plus a_R by 2 into f_A plus f_R by 2. Since you both this function so, we have just written in a step 3 here that has like $a_A f_A p_R$ minus p_A plus $c_A q_R$ minus q_A plus $e_A y_R$ minus y_A this equals to 0.

So, each of these points that are just a changed by this taking these averages here, p_R minus p_A itself it is just there then c_A is replaced by c_A plus c_R by 2 and similarly e_A is replaced by e_A plus e_R by 2 here and similarly if you will just replace like a second equation of step 3.

(Refer Slide Time: 18:47)

Characteristics of a PDE (continue...):

and

$$\left(\frac{a_B + a_R}{2}\right)\left(\frac{g_B + g_R}{2}\right)(p_R - p_B) + \left(\frac{c_B + c_R}{2}\right)(q_R - q_B) + \left(\frac{e_B + e_R}{2}\right)(y_R - y_B) = 0 \quad (18.24)$$

By solving the equations (18.23) and (18.24), we get the improved values for p_R and q_R .

Step-7 In the same way, u may be modified by using the latest values of all parameters (x_R, y_R) and p_R & q_R , we get:

$$u_R = u_A + \frac{(p_A + p_R^*)}{2}(x_R^* - x_A) + \frac{(q_A + q_R^*)}{2}(y_R^* - y_A) \quad (18.25)$$

where (*) denotes the improved values obtained in steps 5 and 6. We can iterate the procedure by repeating the steps 5, 6 and 7 to obtain further refinement in estimates.

IIIT ROORKEE | NPTEL ONLINE CERTIFICATION COURSE

17

So, that is nothing, but like a_B f_B p_R minus p_B plus c_B q_R minus q_B plus e_B y_R minus y_P there this equals to 0. So, that is why this e_B is replaced by e_B plus e_R by 2 c_B is replaced by c_B plus c_R by 2 and here we have like this is nothing, but a g_B . So, g_B is replaced by g_B plus g_R by 2 there.

So, if you will just solve these equations so, you can get the improved values of p_R and q_R there. In the same way u can also be modified by using the least values of all these parameters like x_R , y_R and p_R and q_R and if you will just write this one, we can just write in improved words on here..

That is u_R equals to u_A plus p_A plus p_R star, since star just we are just denoting this one as the improved values obtained in step 5 and 6 for this like p_R star and q_R star here and x_R star and y_R star here. We can just iterate this procedure by repeating like steps 5, 6 and 7 to get the further refinement of these values here.

So, based on this see if you will just go for a practical example here.

(Refer Slide Time: 20:13)

Characteristics of a PDE (continue...):

Example:- A function u is a solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + (1-2x) \frac{\partial^2 u}{\partial x \partial y} - 2x \frac{\partial^2 u}{\partial y^2} + x = 0 \quad (1)$$

and initial condition $u = x$ & $\frac{\partial u}{\partial y} = 0, -\infty < x < \infty, y = 0$.

Calculate u at the point R which is the intersection of the characteristics through the points $A(0.2, 0)$ and $B(0.4, 0)$ on the initial curve $y = 0$.

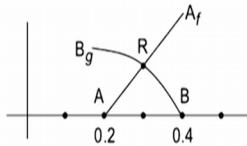
Solution:-

Here,

$a = 1, b = 1 - 2x, c = -2x, e = x$;

$u = x, u_x = 1, u_y = 0$.

The characteristic curve is given by:



18

So, we can just visualize that how clearly zoster this a characteristic and the point of intersection we can just get it out. So, suppose this example we are just considering as like a function u is a solution of this equation $\frac{\partial^2 u}{\partial x^2} + 1 - 2x \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + x = 0$ suppose, with the initial condition u equals to x and $\frac{\partial u}{\partial y}$ equals to 0 where your x plane is lying between like x line is lying between minus infinity to infinity and y equals to zero.

So, the question is ask the to calculate u at the point R which is the intersection of the characteristics through the points A at the point $0.2, 0$ and B which is just defined as $0.4, 0$ on the initial curve y equals to 0 . If you will just consider this a coefficients of this characteristic equation here, this coefficients can be written as like a equals to 1 from this coefficient here and b as $1 - 2x$. Since this is nothing, but a r so, b s. So, c is nothing, but minus $2x$ here and e is nothing, but x here.

. So, we can just define u as the function of x here since the initial condition it is just given as x here and if you will just take the first partial derivative. So, u_x can be written as 1 and u_y equals to 0 so, the characteristic or we feel just right here.

(Refer Slide Time: 21:43)

Characteristics of a PDE (continue...):

$$\left(\frac{dy}{dx}\right)^2 - (1-2x)\frac{dy}{dx} - 2x = 0$$

$$\frac{dy}{dx} = \frac{(1-2x) \pm \sqrt{(1-2x)^2 + 8x}}{2}$$

$$\frac{dy}{dx} = \frac{(1-2x) \pm (1+2x)}{2} = 1 \text{ or } -2x$$

Let

$$\frac{dy}{dx} = 1 = f \quad (1)$$

and

$$\frac{dy}{dx} = -2x = g \quad (2)$$

III ROORKEE NPTEL ONLINE CERTIFICATION COURSE 19

So, that can be written as a dy by dx whole square minus 1, minus 2x into dy by dx minus 2x equals to 0. So, if you will just find the roots so, that can be written in the form of dy by dx equals to 1 minus 2x plus or minus square root of 1 minus 2x whole square plus 8x by 2 and which can just provide the value 1 or minus 2x and if you will just consider like dy by dx equals to 1 which has f here and dy by dx equals to minus 2x as g here.

(Refer Slide Time: 22:13)

Characteristics of a PDE (continue...):

f-characteristic at A(0.2,0) is given as

$$y = x - 0.2 \quad (3)$$

g-characteristic at B(0.4,0) is given as

$$y = -x^2 - 0.16 \quad (4)$$

Differential equation satisfied along the characteristics

$$\frac{dy}{dx} dp - 2x dq + x dy = 0 \quad (5)$$

Step-1 Points A(0.2,0), B(0.4,0) on the initial curve is already specified.

Step-2 Points of intersection of *f*-characteristic through A and *g*-characteristic through B is obtained by solving (3) and (4) which is given by $R(x_R, y_R)$ where $x_R = 0.281, y_R = 0.081$.

Step-3 Equation (5) along *f*-characteristic after using (1) is given as:

III ROORKEE NPTEL ONLINE CERTIFICATION COURSE 20

Then the solution so, we can just get it as like on the f characteristics at 0.20 as y equals 2x minus 0.2.

Since we are just considering here as if you will just see this integration dy along this x here. So, this range is varying if you will just see the y R minus y A this is nothing, but f A into x R minus x A there. So, that is why it is just a taken the values as y equals to; since 0 is the point here y minus y 0 especially it can consider as x minus x 0.

So, x minus 0.2 here similarly if you will just find the solution for second equation here that is a minus 2x into dx so, its integration will be minus x square here. So, if the point is varying like from like 0.4 to 0 here so, y equals to minus x square minus this 0.4 whole square that is nothing, but 0.16 here.

Since this differential equation is satisfied along this characteristic so, you can just write this one as a dy by dx into dp minus 2xdq plus xdy equals to 0. So, if you will just follow these steps like points A as 0.2, 0 and B as 0.4, 0 on the initial curve is already specified and points of intersection of f characteristic through point A and g characteristic through B is obtained by solving like equation 3 and 4 here which is given by like the R coordinate as x R, y R where x R equals to 0.281 and y R as a 0.081 here and if you just follow here equation 5 here along f characteristic after using equation 1.

(Refer Slide Time: 23:56)

Characteristics of a PDE (continue...):

$$dp - 2xdq + xdy = 0 \quad (6)$$

Integrating (6) from A to R, we get

$$(p_R - p_A) - \frac{2(x_R + x_A)}{2}(q_R - q_A) + \frac{x_R + x_A}{2}(y_R - y_A) = 0$$

or

$$(p_R - 1) - \frac{2(0.281 + 0.2)}{2}(q_R - 0) + \frac{0.281 + 0.2}{2}(0.081 - 0) = 0$$


or


$$p_R - 0.481q_R = 0.9805 \quad (7)$$

Equation (5) along g-characteristic after using (2) is given as:

$$-2xdp - 2xdq + xdy = 0$$

or



 NPTEL ONLINE
CERTIFICATION COURSE

21

We can just get it as like dp minus $2xdq$ plus xdy equals to 0 and if you will just integrate this equation from A to R, we can just get this one as like dp that is nothing, but p_R minus p_A minus 2 into x_R plus x_A by 2 we are just taking the average here and q_R minus q_A plus x_R plus x_A by 2; x is the average we are just considering and a dy integration it is just varying from A to R. So, that is why it is just considered as y_R minus y_A equals to 0 here and if you will just put all these points here.

So, we can just write this one as like this point as a 1 and 0 especially we are just considering if you will just see. So, that is why it is just written as like p_R minus 1 minus 2 into your average points so, we have already computed 0.281 and 0.2 it is just a given there. So, by 2 into q_R minus 0 plus 0.281 plus 0.2 by 2 into 0.081 minus 0 this equals to 0; where if you will just solve this equation. So, this is just a taking this from p_R minus 0.481 q_R this equals to 0.9850 here.

So, if you will just go for this equation 5 along g characteristics after using equation 2 it can be represented as like minus $2xdp$ minus $2xdq$ plus xdy equals to 0.

(Refer Slide Time: 25:21)

Characteristics of a PDE (continue...):

$$2dp + 2dq - dy = 0 \quad (8)$$

Integrating (8) from B to R, we get

$$(p_R - p_B) + 2(q_R - q_B) - (y_R - y_B) = 0$$

or

$$(p_R - 1) + 2(q_R - 0) - (0.081 - 0) = 0$$

or


$$p_R + q_R = 1.0405 \quad (9)$$


Solving eq. (7) and (9), we get

$$p_R = 1.0, \quad q_R = 0.0405$$

Step-4 Integrating $du = pds + qdy$ from A to R along f -characteristic

$$u_R = u_A + \frac{(p_R + p_A)}{2}(x_R - x_A) + \frac{(q_R + q_A)}{2}(y_R - y_A)$$




NPTEL ONLINE
CERTIFICATION COURSE

22

Which is nothing, but $2dp$ plus $2dq$ minus dy equals to 0, if you will just integrate this equation from B to R we can just get this equation as in the form of p_R minus p_B plus 2 into q_R minus q_B here. So, minus y_R minus y_B this equals to 0 or p_R minus 1; since a p_B is considered as 1 there plus 2 into q_R minus 0 minus 0.081 minus 0 this equals to 0. So, p_R plus q_R this equals to 1.0405 here.

So, if you will just solve like equation 7 and 9 so, this is a two equations involving p_R and q_R here. We can just get the solution as p_R equals to 1.0 and q_R as 0.0405 here. Again if you will just integrate this equation du equals to $pds + qdy$ from A to R along f characteristics, we can just get this equation as u_R equals to u_A plus p_R plus p_A by 2 into x_R minus x_A plus q_R plus q_A by 2 into y_R minus y_A here..

(Refer Slide Time: 26:32)

Characteristics of a PDE (continue...):

Step-4 Integrating $du = pds + qdy$ from A to R along f -characteristic

$$u_R = u_A + \frac{(p_R + p_A)}{2}(x_R - x_A) + \frac{(q_R + q_A)}{2}(y_R - y_A)$$

$$u_R = 0.2 + \frac{1+1}{2}(0.281 - 0.2) + \frac{0.0405 + 0}{2}(0.081 - 0)$$

$$= 0.2826$$

Note:- If integrating is done along g -characteristic from B to R, then

$$u_R = 0.4 + \frac{1+1}{2}(0.281 - 0.4) + \frac{0.0405 + 0}{2}(0.081 - 0)$$

$$= 0.2826$$

Steps 5, 6, 7 are not required as all parameters are independent of u , u_x & u_y .

23

And again if you will just integrate du equals to this one from like $pds + qdy$ from A to R along f characteristics, we can just get it as u_R equals to u_A plus p_R plus p_A by 2 into x_R minus x_A plus q_R plus q_A by 2 into y_R minus y_A and if you will just put all these points.

So, you can just get u_R as 0.2826 and one note we are just providing here if a integration is done along g characteristics from B to R then we can just get u_R as 0.2826 here. So, steps like 5, 6, 7 are not required as all parameters are independence of u , u_x , and u_y here

Thank you for listen this lecture.