# Numerical Methods: Finite Difference Approach

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## Lecture – 18
## Characteristics of PDE

Welcome to the lecture series on Numerical Methods: Finite Difference Approach, in this approach we have discussed like elliptic equations, parabolic equations, hyperbolic equations, and their solution techniques.

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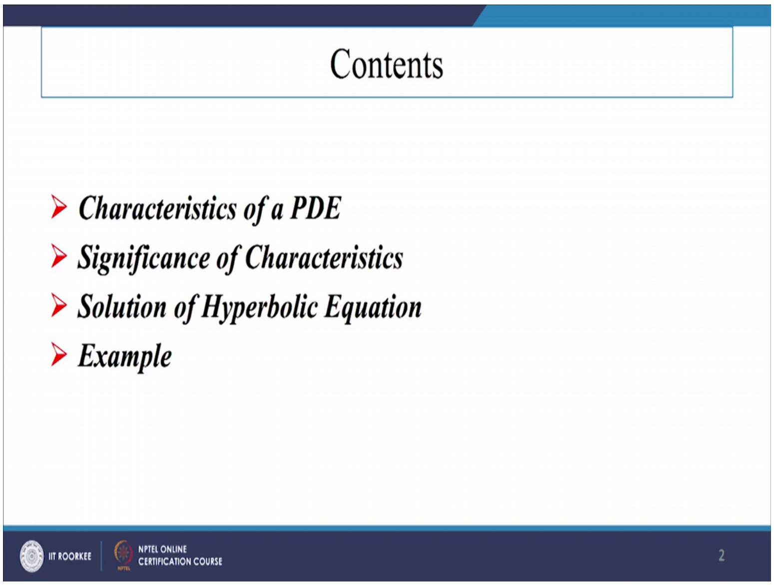


In the present lecture we will discuss about, this characteristics of a partial differential equation then the significance of characteristics and afterwards we will just go for like solution of a hyperbolic equations with examples.

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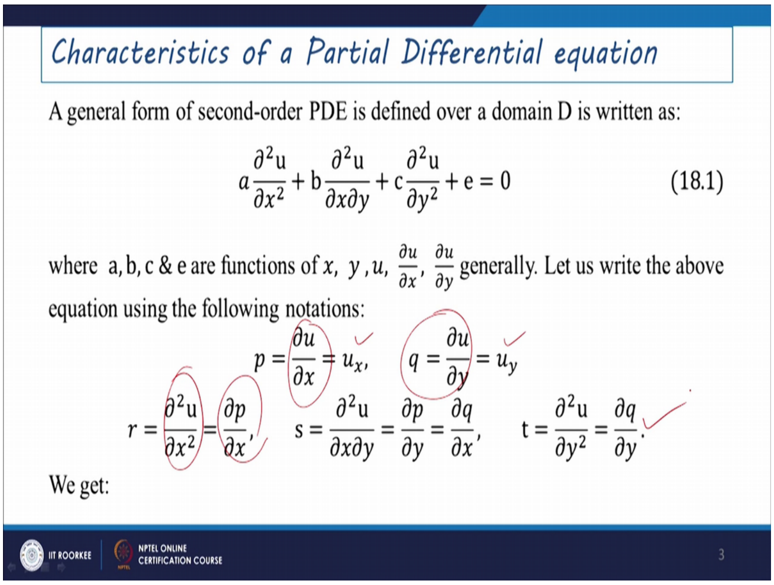


So, if you will just go for this characteristics of partial differential equation. So, usually we are just writing the second order partial differential equation in the form is a del square u by del x square plus b del square u by del x del y plus c del square u by del y square plus e equals to 0, where especially a, b, c and e are functions of x, y, u, u x, u y; generally it is just expressed in this form.

So, if you will just write these expressions in the notations as p equals to suppose del u by del x as u x here and q as del u by del y which is just defined as u y here and r is the second derivative term that is especially del square u by del x square that is nothing, but del by del x of del u by del x. So, that is why it is just written as del p by del x here, s which is defined as del by del x of a del u by del y which is nothing, but del p by del y or del q by del x. We can just write and if you will just write t terms here. So, t can be written as like del square u by del y square that is nothing, but del by del y of del u by del y that is nothing, but del q by del y here.

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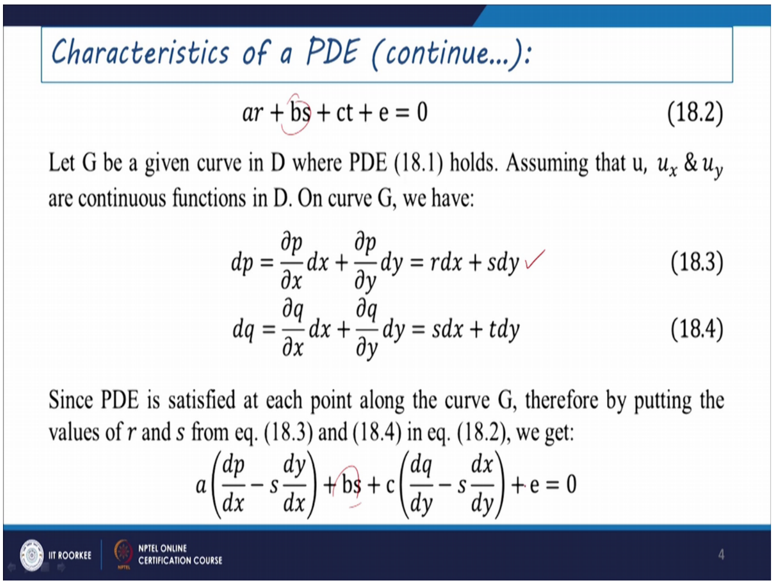


So, if you will just write it in incomplete form then it can be expressed as ar plus bs plus ct plus e equals to 0. Suppose a curve like G be a curve which is defined in a domain D where this partial differential equation holds. So, assuming that suppose u, u x, u y are continuous functions in D and along this curve if you will just consider like d p equals to del p by del x into dx plus del p by del into dy..

This is as a sense we have just define r as del p by del x and s has a del p by del y. So, you can just write this one as rdx plus sdy over here and similarly if you will just defining as dq that is a del q by del x into dx plus del q by del y into dy here which can be defined as sdx plus tdy and if we will just express this partial differential equation is satisfied at each point in the curve G which is defined inside this a domain D.

So, if you will just put all these values of r and s from like first equation in the complete form then we can just write this one as since it is just expressed as a r ar. So, that is why r can be written as like dp by dx minus s dy by dx plus bs..

Especially if it is it is like that way it is just present there c into t, especially t can be replaced as like dq by dy minus s dx by dy plus e equals to 0 here.

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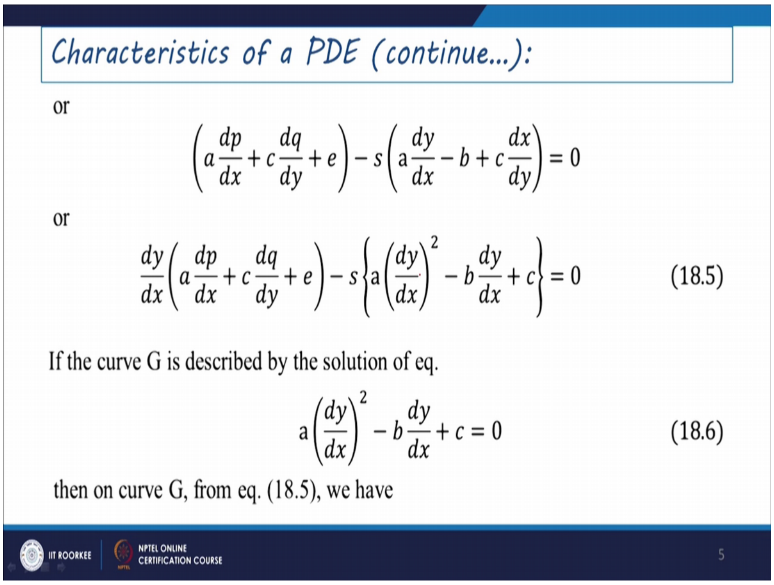


And in a compact form if you will just express this equation then it can be written as a dp by dx plus c dq by dy plus e minus s a dy by dx minus b plus c dx by dy this equals to 0 and if you will just multiply dy by dx at both the sides. So, it can be expressed as a dy by dx a dp by dx plus c dq by dy plus e minus s. Since dy by dx is multiplied so, it can be expressed as a dy by dx whole square minus b dy by dx plus c.

Since it is just cancel it out since a dy by dx into dx by dy so, that is why it is just convenient 0 there. So, if they curve G which is satisfied by the solution of this equation as a dy by dx whole square minus b dy by dx plus c equals to 0. We can just consider this left hand part of this equation as 0 there. So, if you will just put this one as 0 here.

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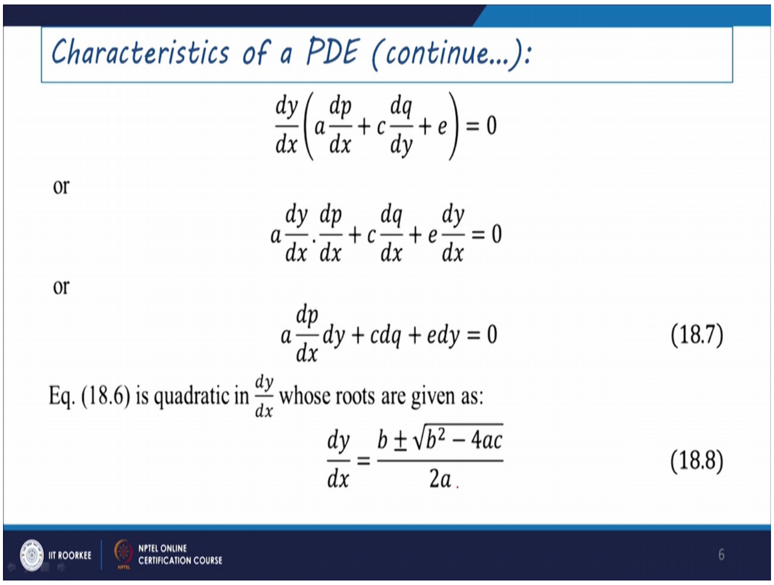


So, we can just write dy by dx into a dp by dx plus c dq by dy plus e equals to 0 and ah; obviously, if you will just multiply this term in the equation, then it can be expressed as a dy by dx into dp by dx plus c dq by dx. Since you are special dy you it will just consider out plus e dy by dx this equals to 0.

So, if you will just see here this can be written as like in a differential form if you will just express this can be expressed as a dp by dx into dy plus cdq plus edy this equals to 0 and if you will just solve this equation that is expressed in 18.6 here, that is a dy by dx whole square minus b dy by dx plus c equals to 0..

This is a quadratic equation here and its root can be expressed as dy dx equals to b plus or minus square root of b square minus 4ac by 2a here..

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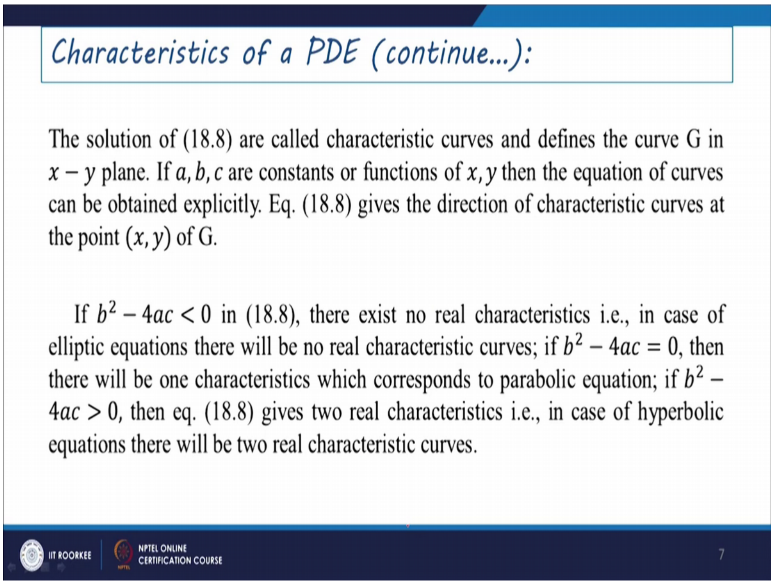


And if you will just go for the solution, of this a characteristic a curves which is a just defined on this curve G in x - y plane. If a, b, c are constants or functions of x, y then the equation of course, can be explained it in explicit form and if you will just go for this like equation of 18.8 here.

So, which is just expressed as a dy by dx form here. So, that just gives the direction of this characteristic cause at the point x, y of G there. So, first condition if you will just considered here as b square minus 4ac suppose less than 0. So, then there does not exist any real characteristics there that is in case of elliptic equations. So, especially this is the condition that is justified in case of elliptic equations..

There will be no real characteristic course and if you will just consider like b square minus 4ac goes to 0 then that will just represent the parabolic equation form and if you will just see here this can be justified for a single characteristics there. .

So, we will have like one characteristics there , but if you will just go for b square minus 4ac greater than 0 then this equation 18.8 it will just provide two real characteristics that is in case of hyperbolic equations there will be two real characteristic curves there.

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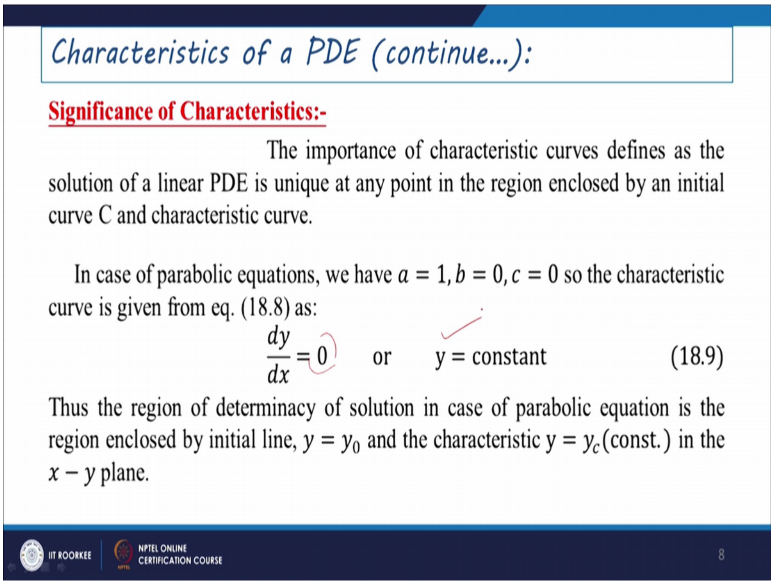


So, if you will just go for this significance of this a characteristic curves there as a solution of like linear partial differential equation, it should be unique at any point and this region is enclosed by an initial curve C and have a characteristic curve there. In case of parabolic equations we will have like a equals to 1, b equals to 0, c equals to 0 there, since we have to show that b square minus 4ac equals to 0 there.

So, in that case if you will just put all these values then we can just obtain this dy by dx which is defined as like a minus b plus or minus square root of b squared minus 4ac by 2a that can be taken as a 0 value there and if you will just find the solution of this equation that is much especially nothing, but y equals to a constant there..

And if you will just go for this region of a determine C of solution in case of parabolic equation is the region enclosed by the initial line that is y equals to y 0 and the characteristic curve that is just defined by this constant function y equals to y c there in the x - y plane.

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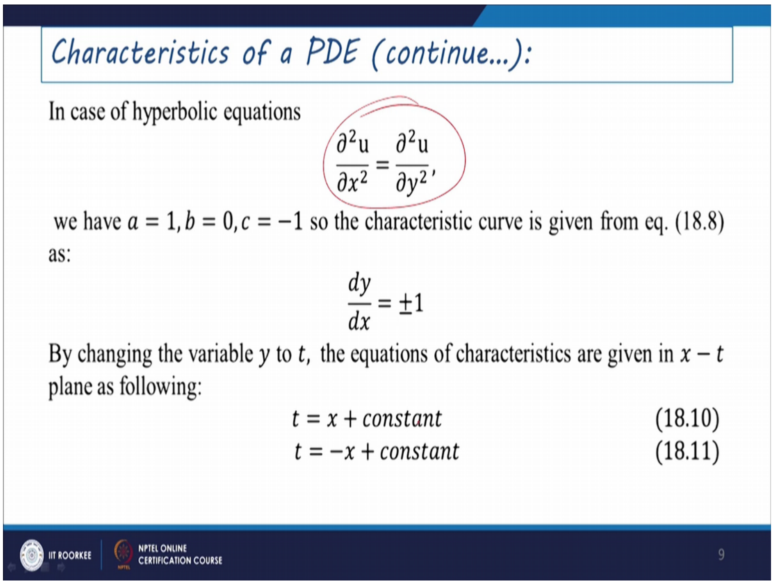


And if you will just go for this hyperbolic equation sense here so, this hyperbolic equation especially if you will just write a generalized hyperbolic equations. So, it can be expressed as the del square u by del x square this equals to del square u by del y square or in generalized form we are just expressing this one as a del square u by del t square this equals to del square u by del x square there, where we are just writing a equals to 1, b equals to 0, c equals to minus 1. So, that the characteristic curve will be given by as dy by dx this equals to plus or minus 1; we will have two solutions here.

So, if you will just change this variables y in forms of t here in the x - t plane then we can have the solutions like t equals to x plus a constant and another solution that will be minus x plus a constant there. .

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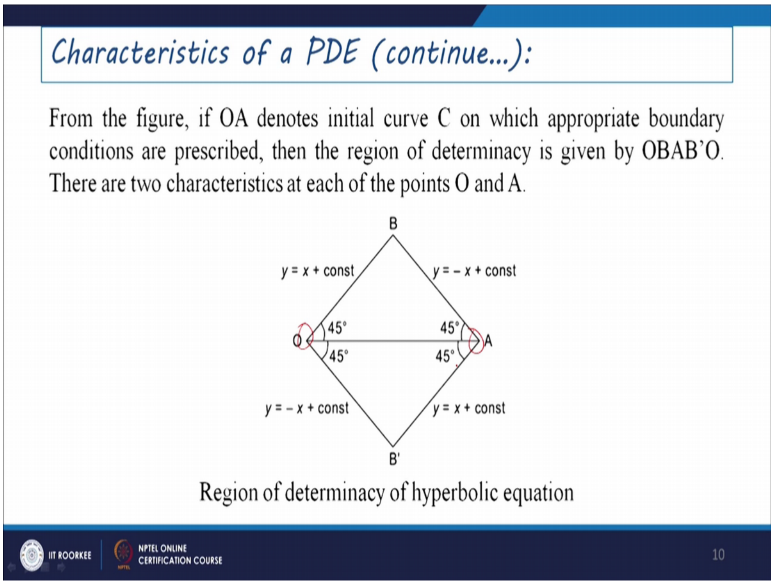


So, if we will just denote it in a figure suppose, if suppose OA denotes the initial curve C on which appropriate boundary conditions are suppose prescribed, then the region of a determinacy is given by like OBAB dash O there. So, there are two characteristics at each of the points O and A which is just given in the form of y equals x plus constant; another one is y equals to minus x plus constant.

So, if you will just determine this two characteristics is one it is just defined as at the point O another which is defined at the point A. So, this can be just a represent in this form here; that is especially the region of determinacy of hyperbolic equations..

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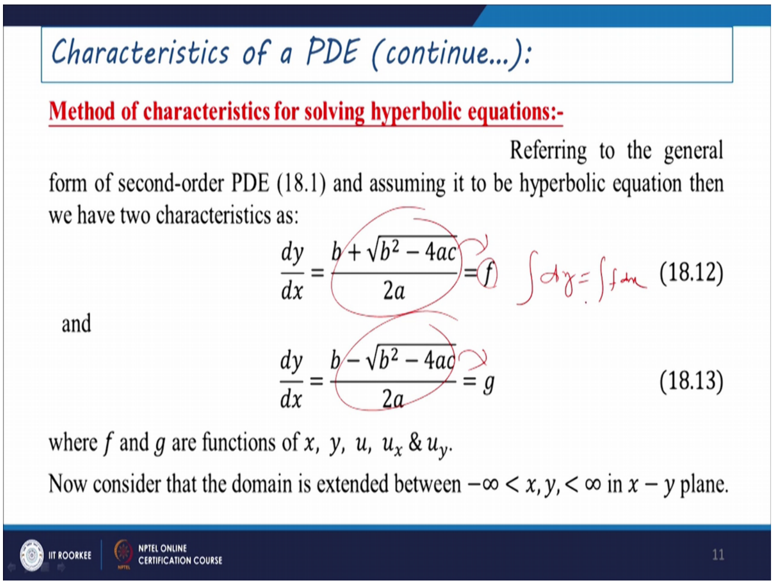


And if you will just go for this like characteristics for solving this hyperbolic equations, referring to the general form of the second order partial differential equation which is just defined in the beginning of this slide and assuming, that it should be hyperbolic a inform then the two characteristics are as like dy by dx this is just expressed as like a b plus square root of b square minus 4ac by 2a one and another one it is just expressed as b minus square root of b squared minus 4ac by 2a which is just a written in this form.

If you will just signify this first function as f here and second function as g here, where f and g are functions of you can just considered as x, y, u, u x or u y there. So, if you will just consider this domain is extended between like a open space like x - y plane. So, which is just a defined from minus infinity to infinity in the x – y plane.

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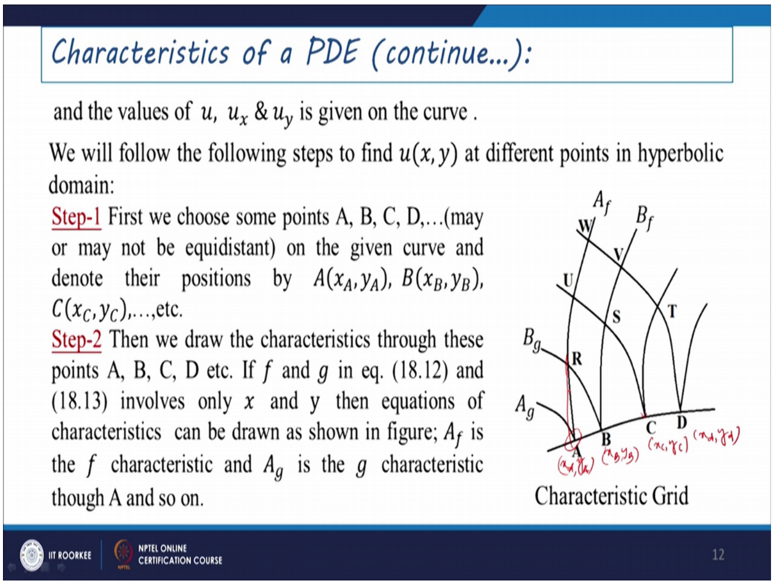


Then we can have the values like u, u x, u y are defined along this curve and to determine these points. So, if you will just go for like different steps the steps can be like described as follows here.

So, first step we will just choose these points like on the curve here that is characteristic grids we can just form and suppose if you will just to choose these points like A, B, C, D which may not be equidistance if you will just consider in a regular domain on the given curve and denote their position. Suppose A is like x A, y A here and B as like suppose x B, y B here and C as like a x C, y C here and D as suppose x D, y D here..

And then we can just draw these characteristics through these points A, B, C, D and if f and g; suppose which is just expressed in the equations like 18.12 here that is one is just taking like B plus another one is just taken as B minus there.

Which just involves so, only x and y then the equations of characteristics can be drawn as shown in the figure as here. That is A f the f characteristic and A g is the g characteristics. So, which is just defined at the point A, likewise we will have this a characteristic at the point B like B f and B g.

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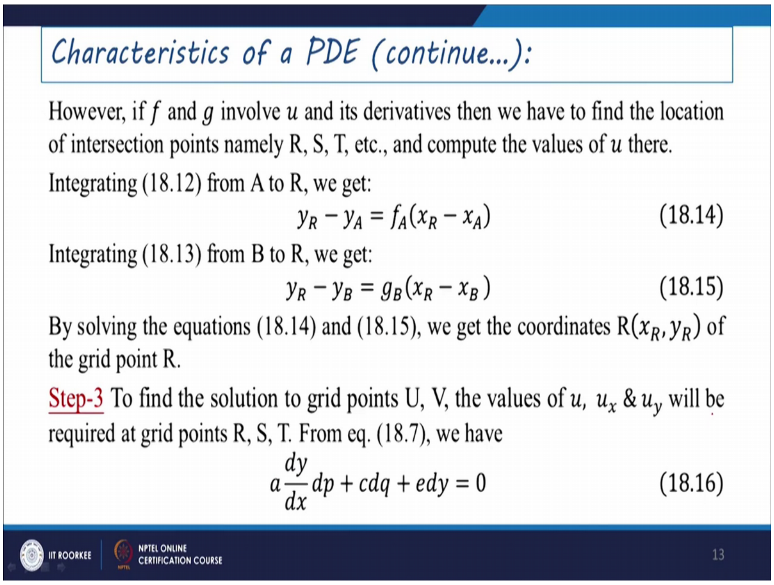


If you will just find here f and g involves this u and its derivatives then exactly we can find the location of intersection points namely like if you will just see here that is the points as R, S, T here.

So, to determine these points here we can just integrate this equation like 18.12 from like a A to R here if you will just see point A to R, yes this one. So, we can just integrate this a differential equation there. So, that will just give you like y R minus y A this equals to f A into x R minus x A there..

Since this a equation if you will just see 18.12 here, yes this one dy by dx that is nothing, but defined a as function f here if you will just integrate simply here. So, integration of dy we can just write this one as integration of f dx.

Since the point it is just defined at the point a itself there. So, that is why this functional value we can just assume this one as f here there and this integration dx that is just a varying from x R minus x A there and similarly if you will just integrate a equation like 18.13 from B to R. We can just get it as like y R minus y B this is equals to g B into x R minus x B there.

So, by solving these equations like 18.14 and 18.15 here we can get the coordinates of R that is as x R, y R of the grid point are there itself. To find the solution to the grid points U, , the values of u, u x and u y will be required at the grid points like R, S, T. .

So, for that we will just solve these equations that is in the form of like a dy by dx dp plus cdq plus edy equals to 0 and if you will just go for this integration of 18.16 equation here.

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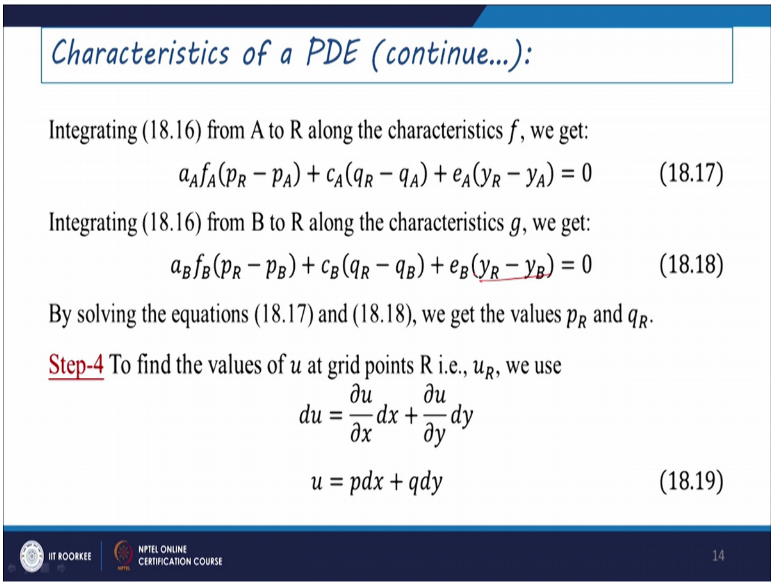


So, then we can just find here a dy by dx that is nothing, but it is just defined as the function f there. So, that is why it can be taken as a A f A and this integration range that is just varying dp integration, that is why we have just considered as a p R minus p A there. Then if you will just say so, cdq dq means q range is just varying from R to A there itself. So, c A q R minus q A plus e is the point defined at the point a there itself. So, e A plus dy is the range. So, y R minus y A these equals to 0 there and similarly if you will just integrate this equation 18.16 from B to R along the characteristics curve g. We can just get this one as like a B f B p R minus p B plus c B and q R minus q B plus e B y R minus y B there.

So, same way just point has been just getting it is changed. So, by solving these two equations like 18.17 and 18.18, we get the values of p R and q R. To find the values of u at the grid points like R that is u R, we can just use this formula that is as du equals to del u by del x into dx plus del u by del y into dy there, where if you will just find the solution here u is nothing, but a pdx plus qdy.

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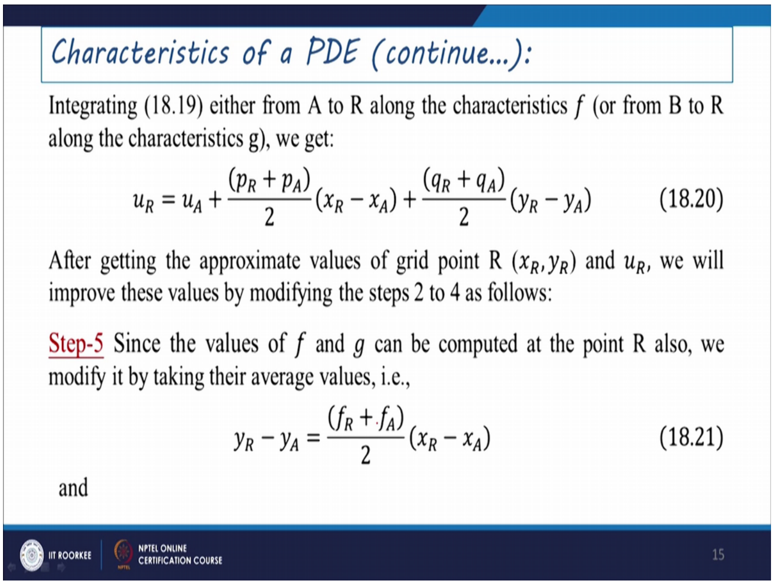


And; obviously, if you will just integrate this equation from A to R along the characteristics a curve f or from B to R along the characteristics curve G, we can just get u R equals to u A plus p R plus p A by 2 into x R minus x A; that is the distance from like point A to R. in the x direction and if you will just write for this a q position there. So, it can be written as q R plus q A by 2.

So, we are just considering this average of two values there if you will just see the curve there. So, A and B it is just considered and both these curves so, that are just intersecting at the point are there itself. So, that is why we are just considering p R plus p A by 2 there and similarly since both these curves they are just intersecting at the point like R and A for q value also..

So, that is why we have just considered q R plus q A by 2 into y R minus y A there. After getting the approximate values of grade point R here x R to y R and u R, we will improve these values by modifying the steps 2 to 4 as follows.

Since the values of f and g as computed like at the points R also, we modify it by taking the average value. So, that is like y R minus y A if you will just consider; since earlier we have just considered as f A there. So, which is just nothing, but the average we have just considered as f R plus f A by 2 into x R minus x A here.

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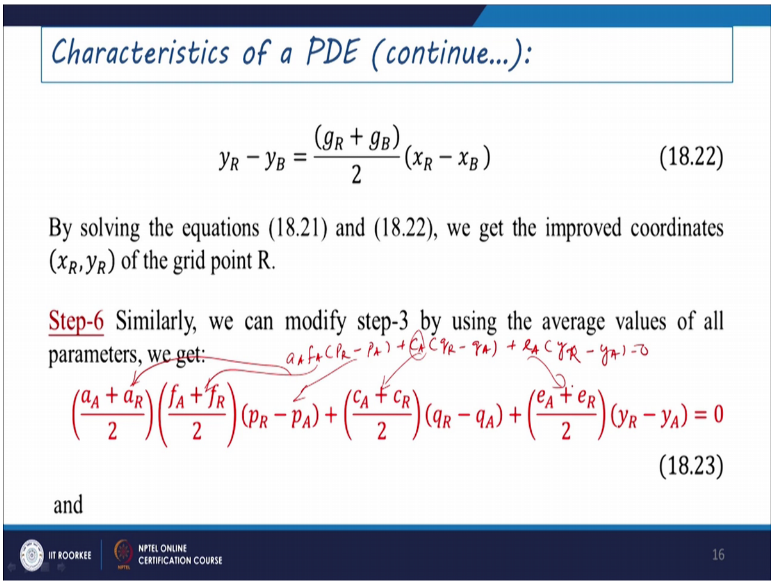


Similarly, if you will just go for like y R minus y B so, same point if you will just consider here along the curve g there. So, you can just write that one is a g R plus g B by 2. So, that will just take this average of g function at that point so, into x R minus x B and if you will just solve this equation say 18.21 and 18.22; we get the improved coordinates x R, y R of this grid point R.

Similarly, we can just modify like a step 3 if you will just see here by taking this average. So, we can just write this one as like a A plus a R by 2 into f A plus f R by 2. Since you both this function so, we have just written in a step 3 here that has like a A f A p R minus p A plus c A q R minus q A plus e A y R minus y A this equals to 0.

So, each of these points that are just a changed by this taking these averages here, p R minus p A itself it is just there then c A is replaced by c A plus c R by 2 and similarly e A is replaced by e A plus e R by 2 here and similarly if you will just replace like a second equation of step 3.

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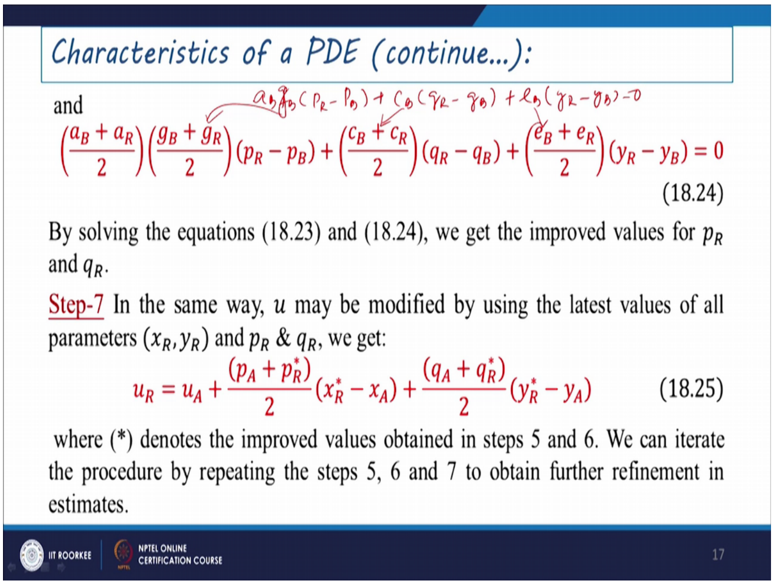


So, that is nothing, but like a B f B p R minus p B plus c B q R minus q B plus e B y R minus y P there this equals to 0. So, that is why this e B is replaced by e B plus e R by 2 c B is replaced by c B plus c R by 2 and here we have like this is nothing, but a g B. So, g B is replaced by g B plus g R by 2 there.

So, if you will just solve these equations so, you can get the improved values of p R and q R there. In the same way u can also be modified by using the least values of all these parameters like x R, y R and p R and q R and if you will just write this one, we can just write in improved words on here..

That is u R equals to u A plus p A plus p R star, since star just we are just denoting this one as the improved values obtained in step 5 and 6 for this like p R star and q R star here and x R star and y R star here. We can just iterate this procedure by repeating like steps 5, 6 and 7 to get the further refinement of these values here.

So, based on this see if you will just go for a practical example here.

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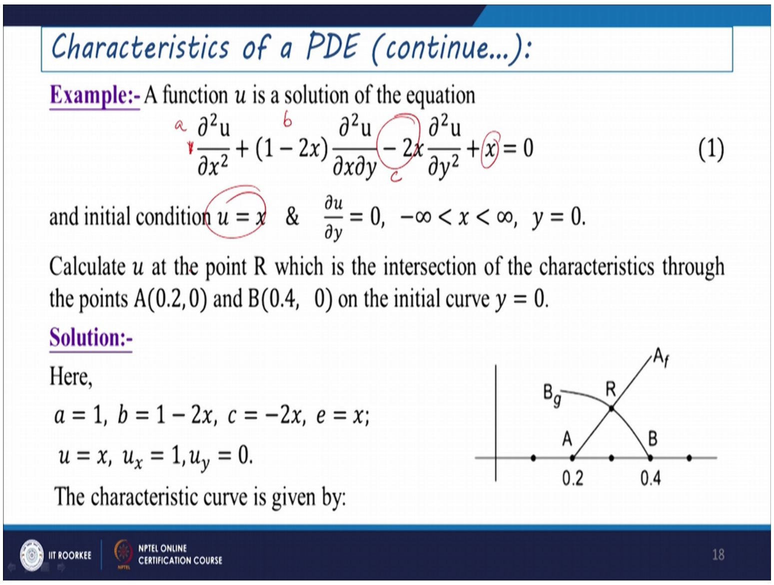


So, we can just visualize that how clearly zoster this a characteristic and the point of intersection we can just get it out. So, suppose this example we are just considering as like a function u is a solution of this equation del square u by del x square plus 1 minus 2x del square u by del x del y plus minus 2x del square u by del y square plus x equals to 0 suppose, with the initial condition u equals to x and del u by del y equals to 0 where your x plane is lying between like x line is lying between minus infinity to infinity and y equals to zero.

So, the question is ask the to calculate u at the point R which is the intersection of the characteristics through the points A at the point 0.2, 0 and B which is just defined as 0.4, 0 on the initial curve y equals to 0. If you will just consider this a coefficients of this characteristic equation here, this coefficients can be written as like a equals to 1 from this coefficient here and b as 1 minus 2x. Since this is nothing, but a r so, b s. So, c is nothing, but minus 2x here and e is nothing, but x here.

. So, we can just define u as the function of x here since the initial condition it is just given as x here and if you will just take the first partial derivative. So, u x can be written as 1 and u y equals to 0 so, the characteristic or we feel just right here.

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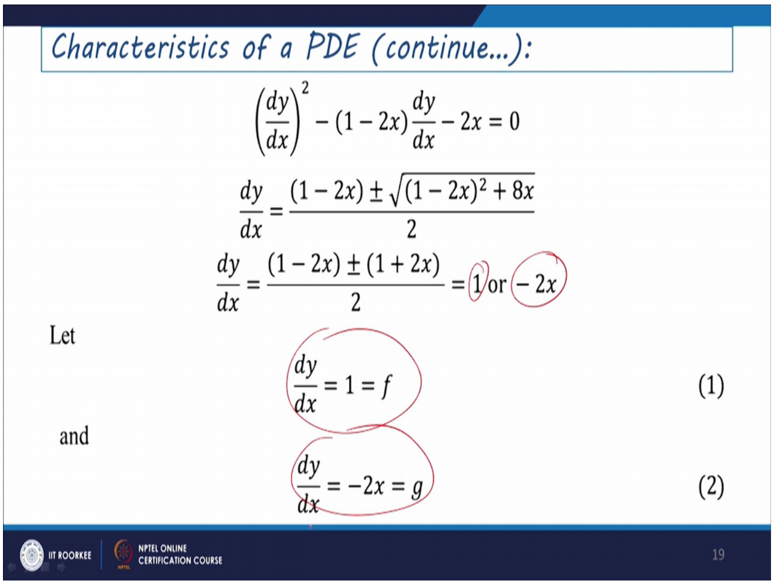


So, that can be written as a dy by dx whole square minus 1, minus 2x into dy by dx minus 2x equals to 0. So, if you will just find the roots so, that can be written in the form of dy by dx equals to 1 minus 2x plus or minus square root of 1 minus 2x whole square plus 8x by 2 and which can just provide the value 1 or minus 2x and if you will just consider like dy by dx equals to 1 which has f here and dy by dx equals to minus 2x as z here.

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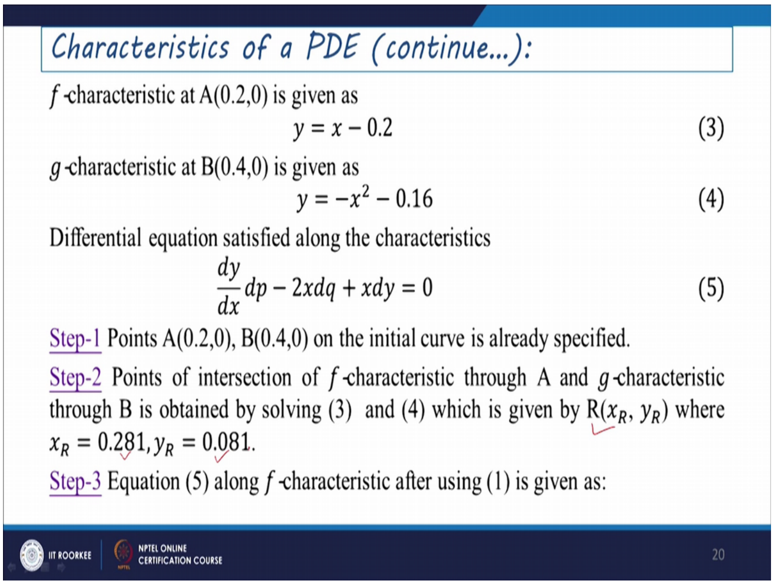

Then the solution so, we can just get it as like on the f characteristics at 0.20 as y equals 2x minus 0.2.

Since a we are just considering here as if you will just see this integration dy along this x here. So, this range is varying if you will just see the y R minus y A this is nothing, but f A into x R minus x A there. So, that is why it is just a taken the values as y equals to; since 0 is the point here y minus y 0 especially it can consider as x minus x 0.

So, x minus 0.2 here similarly if you will just find the solution for second equation here that is a minus 2x into dx so, its integration will be minus x square here. So, if the point is varying like from like 0.4 to 0 here so, y equals to minus x square minus this 0.4 whole square that is nothing, but 0.16 here.

Since this differential equation is satisfied along this characteristic so, you can just write this one as a dy by dx into dp minus 2xdq plus xdy equals to 0. So, if you will just follow these steps like points A as 0.2, 0 and B as 0.4, 0 on the initial curve is already specified and points of intersection of f characteristic through point A and g characteristic through B is obtained by solving like equation 3 and 4 here which is given by like the R coordinate as x R, y R where x R equals to 0.281 and y R as a 0.081 here and if you just follow here equation 5 here along f characteristic after using equation 1.

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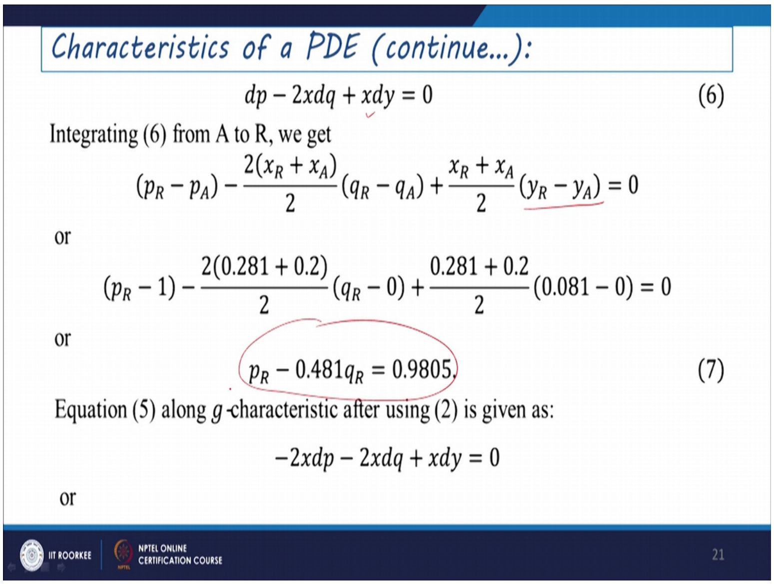

We can just get it as like dp minus 2xdq plus xdy equals to 0 and if you will just integrate this equation from A to R, we can just get this one as like dp that is nothing, but p R minus p A minus 2 into x R plus x A by 2 we are just taking the average here and q R minus q A plus x R plus x A by 2; x is the average we are just considering and a dy integration it is just varying from A to R. So, that is why it is just considered as y R minus y A equals to 0 here and if you will just put all these points here.

So, we can just write this one as like this point as a 1 and 0 especially we are just considering if you will just see. So, that is why it is just written as like p R minus 1 minus 2 into your average points so, we have already computed 0.281 and 0.2 it is just a given there. So, by 2 into q R minus 0 plus 0.281 plus 0.2 by 2 into 0.081 minus 0 this equals to 0; where if you will just solve this equation. So, this is just a taking this prom p R minus 0.481 q R this equals to 0.9850 here.

So, if you will just go for this equation 5 along g characteristics after using equation 2 it can be represented as like minus 2xdp minus 2xdq plus xdy equals to 0.

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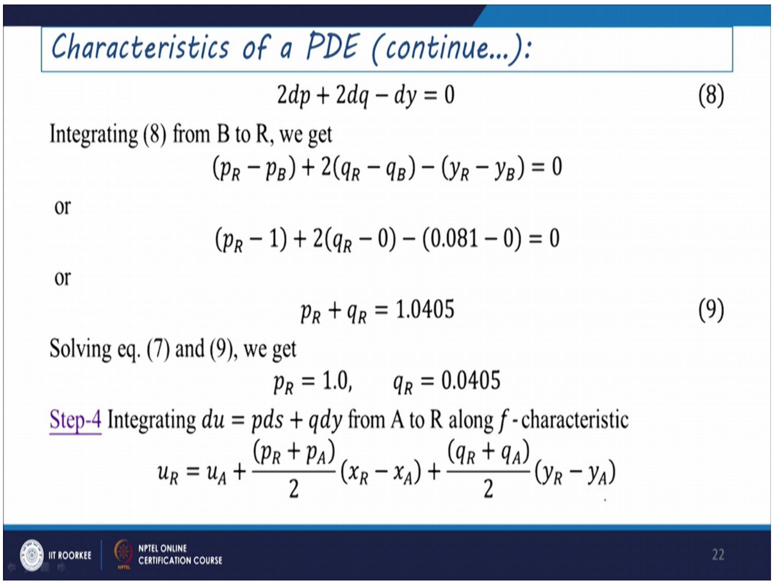


Which is nothing, but 2dp plus 2dq minus dy equals to 0, if you will just integrate this equation from B to R we can just get this equation as in the form of p R minus p B plus 2 into q R minus q B here. So, minus y R minus y B this equals to 0 or p R minus 1; since a p B is considered as 1 there plus 2 into q R minus 0 minus 0.081 minus 0 this equals to 0. So, p R plus q R this equals to 1.0405 here.

So, if you will just solve like equation 7 and 9 so, this is a two equations involving p R and q R here. We can just get the solution as p R equals to 1.0 and q R as 0.0405 here. Again if you will just integrate this equation du equals to pds plus qdy from A to R along f characteristics, we can just get this equation as u R equals to u A plus p R plus p A by 2 into x R minus x A plus q R plus q A by 2 into y R minus y A here..

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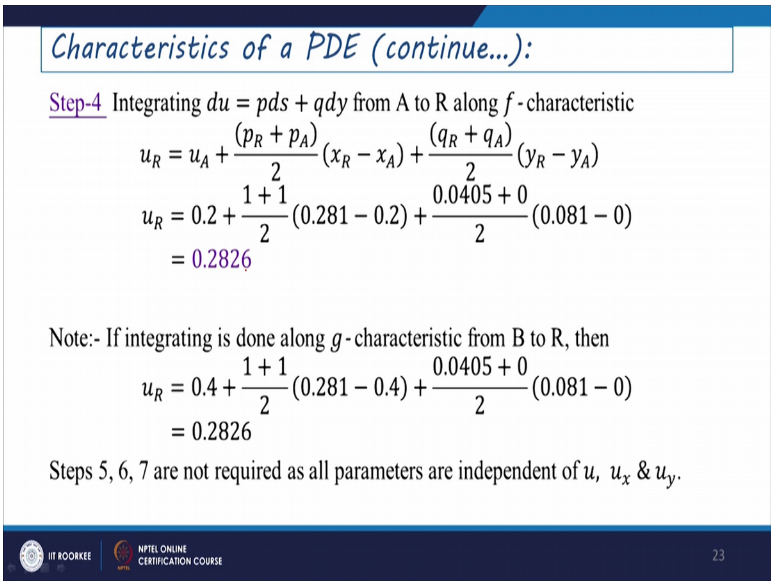


And again if you will just integrate du equals to this one from like pds plus qdy from A to R along f characteristics, we can just get it as u R equals to u A plus p R plus p A by 2 into x R minus x A plus q R plus q A by 2 into y R minus y A and if you will just put all these points.

So, you can just get u R as 0.2826 and one note we are just providing here if a integration is done along g characteristics from B to R then we can just get u R as 0.2826 here. So, steps like 5, 6, 7 are not required as all parameters are independence of u, u x, and u y here

Thank you for listen this lecture.