**Numerical Methods: Finite Difference Approach**

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**Lecture – 16**
**Solution of Hyperbolic equations**

Welcome to the lecture series on numerical methods a finite difference approach. In the last lecture, we have discussed about this solution by using different schemes for elliptic equation; in the present lecture, we will discuss about to these hyperbolic equations and these hyperbolic equations can be solved by using a different technique is as we have discussed earlier.

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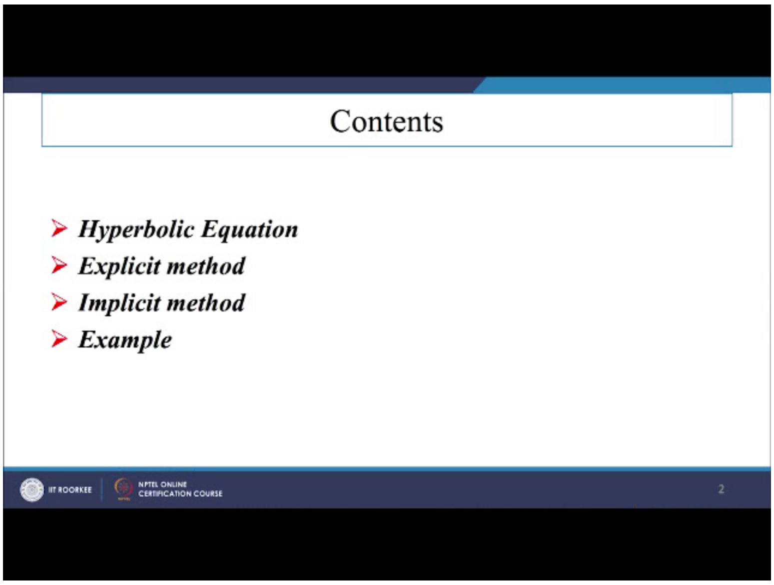


So, in this lecture, we will just discuss about this like explicit and implicit method, based on this hyperbolic equations.

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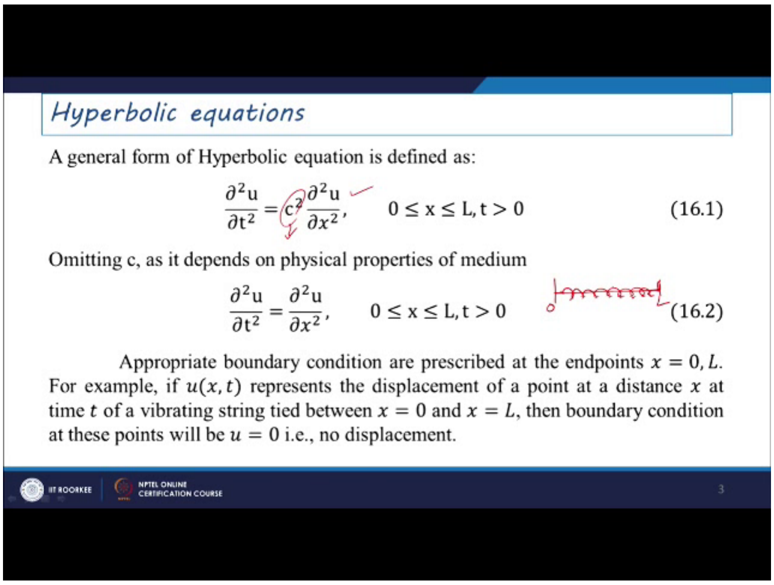


So, especially if you will just go for a generalized a hyperbolic equation. So, this hyperbolic equation can be written in the form like del square u by del x square this equals to c square del square u by del x square. So, especially this is just written like in a heat transfer problem or any like we can just considered as a mechanical problem, where c especially this depends on this physical constraint there; either it can be a c can be a defined as a specific heat coefficient or it can be like any type of special constraints depends on the physical scenario of the problem.

So, if you will just omit c since it depends on the physical properties of the medium. So, we can just write this equation in the form of del square u by del t square this equals to del square u by del x square and this domain it can be lies like a 0 to L here for t greater than 0; this boundary condition if you will just see here this requires at the end points x equals to 0 and x equals to L here.

For example, if you will just consider like u of x t here as a physical problem suppose, represents the displacement of a point at a distance x at time t of a vibrating string tied between x equals to 0 and x equals to L; then boundary condition at these points will be u equals to 0 there is no displacement if you directly if you will just see here; so if you will just consider this domain like 0 to L here and we are just considering a big string here.

So, at the different points we have to provide this u data there itself at fixed time that is t equals to 0 there and we will have these positions that should be defined correspondingly also there. So, that is why we are just defining this boundary condition at these points will be u equals to 0, since there is no displacement is occurring at the constant tied of the string at 2 the points there.

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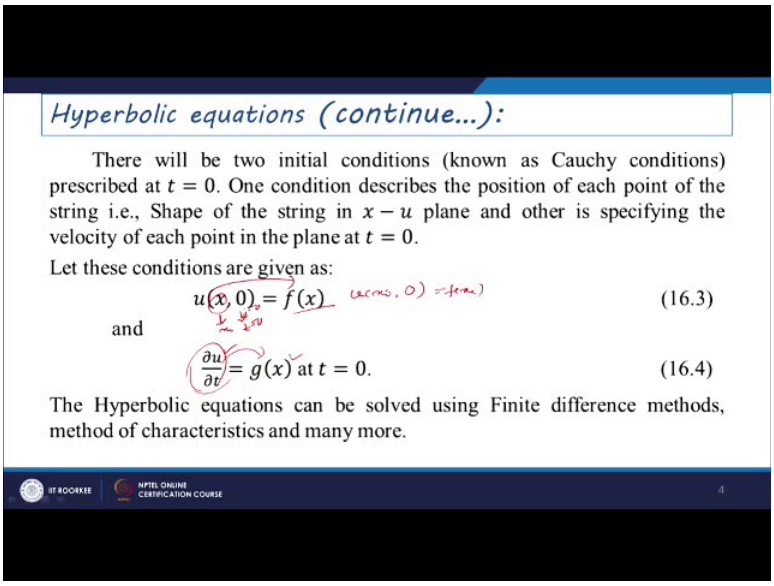


But if you will just see there will be 2 initial conditions. So, that can be provided in the form of Cauchy boundary condition prescribed at p equals to 0, 1 condition describes this position of each point of the string, that is a safe of the strings in xu plane and other is specifying the velocity of each point in the plane at t equals to 0, let these conditions are given by u of x 0 is f of x and del u by del t this equals to g of x at t equals to 0.

So, if you will just see here time t equals to 0 u is redefined as a function of x there and if you will just see here del u by del t that is the time derivative especially, it is just considered as a functional value since whenever time will get change with respect to like positions. So, you will have a change that depends on the functional values of z there. So, this hyperbolic equation can be solved using finite difference methods or method of our characteristics or many more methods it is available.

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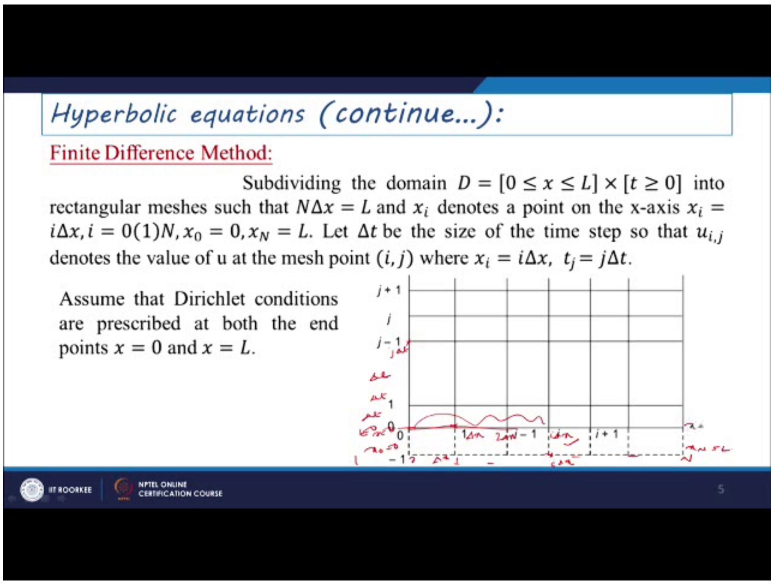


But if you will just go for a finite difference method here then we have to subdivide this domain that is defined as like x equals to 0 to L there. So, for t greater than equals to 0 into like if you will just take here finite number of grids that is suppose 0 1 2 up to n here points here then we can just define x 0 equals to 0 here and xn equals to L there and the del t is the size of these time steps. So, you can just consider t equals to 0 at this level in this line t equals to 0 then, it will be incremented by del t likewise and after like I steps we can just get this grid position as in the form of I del x there and the time position as the z del t there.

Since especially if you will just take here del x that is nothing, but the distance here we are just considering. So, for first 1 we can just write this distance is del x here from origin, second we can just write 2 delta x. So, likewise if you will just move after like I point, so we can just get this on as I delta x there and similarly we can just proceed it for j direction that is in the time scaling we can just calculate as j delta there. So, assume that there is let conditions are prescribed at both the ends points, that is x equals to 0 and x equals to L there.

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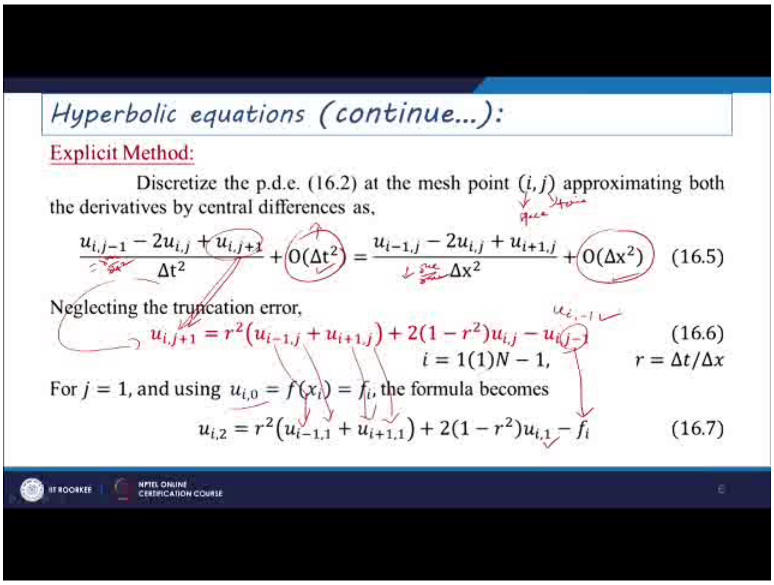


So, for this explicit method if you just discretize this partial differential equation at the mesh point i j, suppose i is the space coordinate and j is the time coordinate, then we can just write this 1 as a ui j minus 1 minus 2 uij plus uij plus 1 by del t square plus order of del t square this is equals to ui minus 1 j minus 2 uij plus ui plus 1 j by del x 2 square plus order of for del x square.

So, this is nothing, but your a del square u by del t square term, this is nothing but del square u by del x square term. So, both these if you will just see this just gives us a second degree approximated value, this is just giving a second degree error estimated value there or if you will just neglect this higher powers of for del t, that is in the form of del t square and higher power for del x square, then we can just obtain this series expansion as in this form here.

So, if you will just neglect this a truncation error term that is in the form of order del of t square and order of a del x square here, then we can just often this partial differential equation in a linear form as uij plus 1, since we want to find the next iterated value calculation or like a next step value here at j equals to 0 these values can be known from this initial conditions, then we have to proceed step 1 here to get all this unknown values.

So, for this see if you will just use for j equals to 1, this is the first time step and using ui 0 equals to f of xi this as fi since at the 0 th level, if you will just see here t equals to 0 this only I has the variation. So, j remains in 0 there itself. So, that is why we can just write this 1 as ui 0 and this last point; if you will just see here that is just defined as in the form of f of x here u of x0. So, that is why j equals to 0 if you will just consider and x at a different I point. So, you can just consider as x I there. So, that is why it can be written in the form of ui xi and t at 0 th level this can be defined as f of xi here and if you will just use this 1 in the formula then for j equals to 1 if you will just write. So, we can just write this 1 as ui since j plus one. So, that is why this will just give you j plus 1 means this is just giving you ui 2 here. So, r square so I as I there itself. So, ui minus 1 j is 1 here . So, then ui plus 1 j is 1 there, so then 2 into 1 minus r square. So, ui j is 1 here minus it can be written in the form of since ui 0 it is just written as fi here. So, ui 0 it is just a replaced by fi there.

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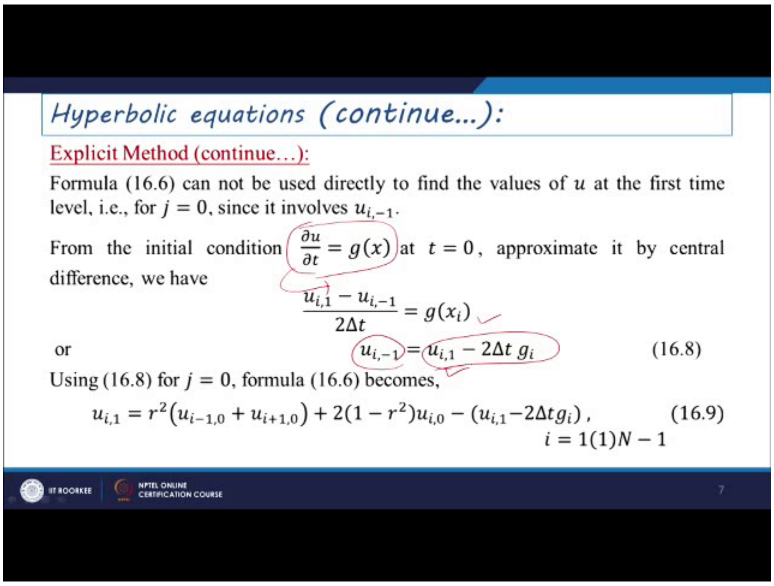


So, if you will just go for like this can be used to directly to find the values of u at the first time level that is for j equals to 0, since it involves a term if you will just see here ui minus 1 here from the initial condition there itself, that is in the form of del u by del t this is equals to g of x here at t equals to 0 and if you just write the central difference scheme for this formulation here, then we can just write this 1 as a ui 1 minus ui minus 1 by 2 del t this can be written as a g of x I here and ui minus 1 it can be written as a ui 1 minus 2 del t into z i.

So, for j equals to 0 if you will just see here if we will just put directly here j equals to 0 suppose, then this we will just give you like this point that is as u I minus 1 there itself .

So, this creates in the problem for the calculation of this equation there, so that is why we have to consider this initial condition to get this value that is as u of I minus 1, which can be written in the form of the domain values there itself ui 1 minus 2 del t into gi.

So, for j equals to 0 the formula like 6 point sixteen point six if you will just see here this can be written in the form of like ui 1 this equals to r square ui minus 1 this is 0 plus ui plus 1 0 plus 2 into 1 minus r square ui 0 minus ui minus 1 there. So, ui minus 1 that can be replaced by ui 1 minus 2 delta t gi there and I is varying from 1 to n minus o1.

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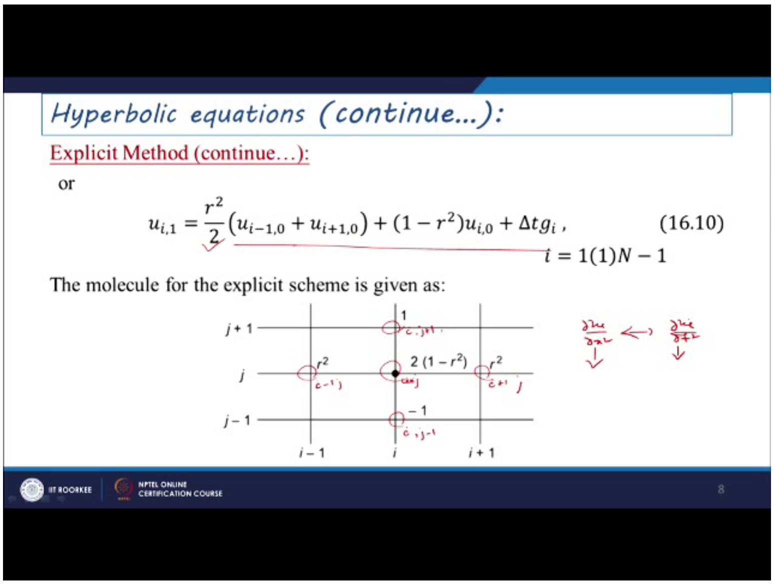


Or ui 1 it can be written in the form of r square by 2 ui minus 1 0 plus ui plus 1 0 plus 1 minus r square ui 0 plus delta gi I equals to 1 to n minus 1.

So, you can just check this values here, so all values if you will just see this is 0 and this is 0 this is 0; So, gi is unknown value here. So, only unknown value is e ui 1 here. So, that is why it can be taken to the left hand side and it can be separated. So, that is why this a 2 value is just and getting it out there and we are just obtaining this all these a unknown values on the right hand side here.

So, if you will just go for this molecule here. So, we can just find this 1 as since in the x direction we will have 1 coefficients that is in the form of like del square u by del x square and if you just go for time direction here we will have 1 terms del square by del t square here. So, this will just give you 1 terms this will just give you 1 terms here. So, that is why this molecule c if it is just formulated. So, it can be just a taken in the form if you will just see here that has a j plus oneth level jth level and j minus 1 level. So, this coefficient is taking r square here 2 into 1 minus r square this is r square here this is as a minus 1 here this is 2 into 1 minus r square this is as a 1 coefficient here. So, if you just see this is nothing, but your uij coefficient, i minus 1 j this is i plus 1 j, this is nothing, but ui j minus 1 ui j plus 1.

. So, that is why this is molecule this can be found from this equation here.

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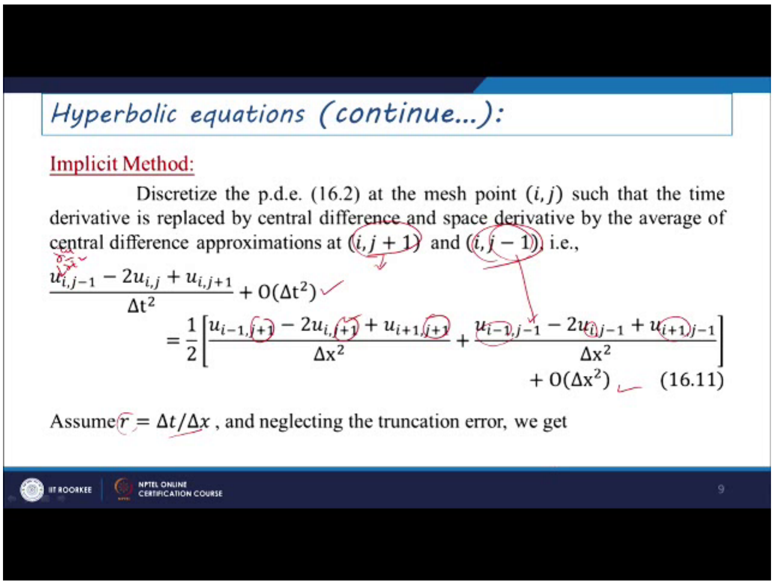


And if you will just go for a discretization in terms of for like implicit method here at the mace point i j. So, that the time derivative is replaced by the central difference and space derivative by average of central difference approximations. Since you already we have considered this one in the crank Nicholson method that we are just taking the average of this time scales for this like space derivatives there.

So, that is why the same treatment we will just make for these equations here also. So, that is why if you will just take this a time difference as the central difference approximation that is nothing, but 2 del square u by del t square as in the form of ui j minus 1 minus 2 u ij plus ui j plus 1 by del t square, plus order of a delta t square here. This equals to if you will just take the average at 2 different points at ij plus 1 point and ij minus 1 point there, we can just write this u terms as in the form of i j plus 1 point if you will just take the central difference, then we can just write i minus 1 i and i plus 1 and j plus 1 remains fixed.

Similarly, if you just go for i j minus 1, j minus 1 remain fixed. So, I will be varied from i minus 1 i and i plus 1 plus order for del x square term here plus order of a delta t squared from here we can just define this value r as in the form of a delta t by del x and if you will just neglect the higher powers of del x and del t there itself.

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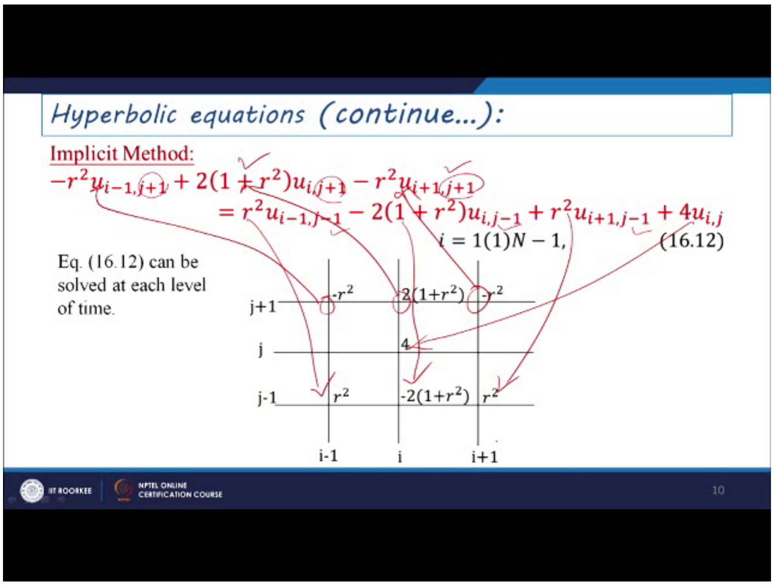


We can just obtain minus r square ui minus 1 j plus 1 plus 2 into 1 plus r square, ui j plus 1 minus r square ui plus 1 j plus 1, this equals to r square u minus 1 j minus 1 minus 2 into 1 plus r square ui j minus 1 plus r square ui plus 1 j minus 1 plus 4 uij.

Especially if you will just see here this is a plus 1 coefficients, all terms we have just kept it in the left hand side since this turns it is unknown to us and this i minus 1 and j minus 1, j minus 1, j minus 1, all these values are at the previous time step calculated values. So, that is why we just try to keep these things in the right hand side and if you will just see this molecule here. So, j plus 1 coefficient if you will just see here, this is nothing, but minus r square at i minus 1th level here and a ith level this is just taken 1 2 into 1 plus r square, and at i plus 1 level it is just taking minus r square.

So, these coefficients if you will just see one coefficients and this is just taken from this values here; and if you will just go for uij coefficient here uij is nothing, but it is just associated with the coefficient as 4 here, and if you just go for like a j minus 1 step values. So, i minus 1 takes this coefficient is r square, this will just take as a minus 2 into 1 and 1 plus r square and this is just taking r squared term here. So, this forms the complete molecule for this equation.

So, if you will just go for a example based on this previous discussed methods.

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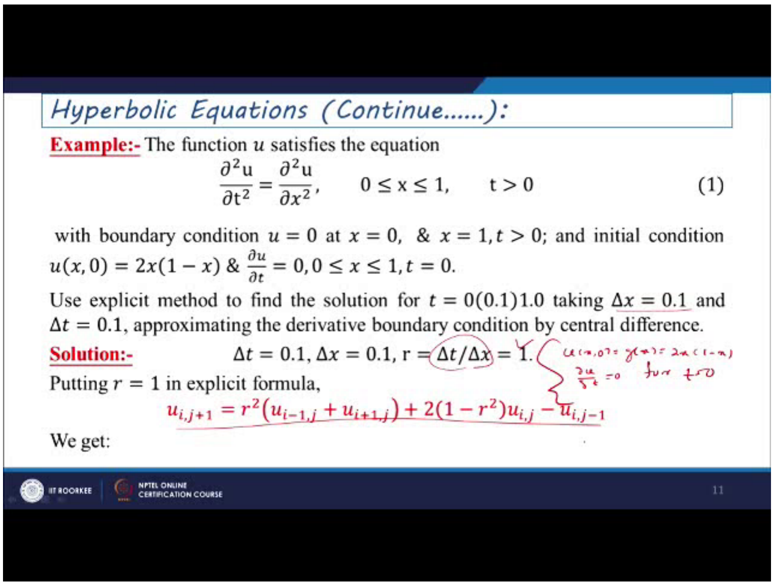


So, this function suppose you satisfies this equation that is in the form of a del square u by del t square this equals to del square u by del x square, where x range is lying between 0 to 1 and for all t greater than 0 with boundary conditions like u equals to 0 at x equals to 0, and x equals to 1 for t greater than 0 suppose. And the initial condition is prescribed as since already we have discussed this one u of x 0 this is just defined in the form of gx there. So, that is why gx can be considered as a 2 x into 1 minus x, and a del u by del t this equals to 0 suppose, for x lying between 0 to 1 here and t equals to 0.

So, this condition your initial conditions if you will just see here that is nothing, but it is just defined as u of x 0 that is g of x, which as in the form of 2 x into 1 minus x and another initial condition that is just given as like at a different positions del u by del t this equals to 0 for t equals to 0 both these conditions will be satisfied. And this conditions that is just given like we have to use explicit method to obtain the solution for t equals to 0 to 1.0 taking del x equals to 0.1 and del t equals to 0.1.

So, this means that t has like an 10 steps here, and del x also this will just take 10 different node points to get the solution here and we have to approximate this derivative by a central difference approximation. If you will just consider like a delta equals to 0 point 1 del x equals to 0.1 here. So, especially r is defined as a delta by del x. So, 0.1 by 0.1 this is just taking 1 theta. And if you will just put r equals to 1 in the explicit formula this explicit formula, it is just written in the form uij plus 1 this is equals to r is 1 there itself. So, that is why ui minus 1 j plus ui plus 1 j plus 2 into 1 minus r square uij, minus uij minus 1. So, this is a explicit formula that has a just to the find.

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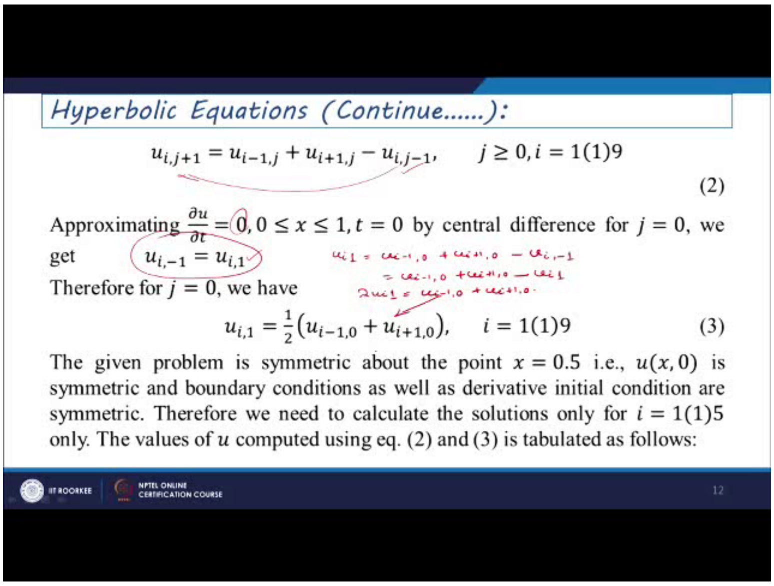


So, if we will just go for the solution for j greater equal to 0, and i is varying from 0 1 2 up to 9 here. So, at i equals to 0 the values is unknown to us, then our computation will we go from 1 to 9 points there, since at 0.1 also it is a known from this boundary conditions, and if you will just approximate this derivative that is a del u by del t equals to 0 where x lies between 0 to 1 here t equals to 0 by central difference approximation for j equals to 0, we can just obtain that is in the form of if you will just see here del u by del t is a written as ui plus 1 sorry ui 1 minus ui minus 1 by 2 del t equals 2 gxi there. And ui minus 1 this is defined as ui 1 minus 2 del t into gi there. So, gi it is just considered as a 0 there. So, that is why we can just write ui minus 1, this equals to ui 1 there.

So, it is just written ui minus 1 this equals to u ui 1, since your g of x is 0 there itself. So, for j equals to 0 we can just write also this equation as ui 1, this is equals to ui minus 1 0 plus ui plus 1 0, minus ui this is minus 1. So, that is why u i 1 it can be taken it out there. So, that is why it can be written as a half of ui minus 1 0, plus ui plus 1 0 since this will just consider at this level here.

So, directly if you will just write this equation this can be written as ui 1, this equals to ui minus 1 0 ui plus 1 0 minus ui minus 1 here this can be written as a ui minus 1 0 plus ui plus 1 0 minus this is ui 1. So, it is can be taken to the left hand side. So, 2 ui 1 this can be written as ui minus 1 0 plus ui 1 plus 0 here. So, that is why ui 1 can be written as a half into ui minus 1 0 plus ui plus 1 0.

And if you will just see the given problem is symmetric about the point x equals to 0.5 that is u of x 0 is a symmetric, and boundary conditions as well as derivative initial conditions are symmetry, since a this is a 2 x into 1 minus x, if you will just put here like a x equals to half here. So, automatically it can be like you will have the same condition as we can just get like a 0 and 1 and if you will just consider like 1 to 5 then we can just get this 1 at 0.75, since at 0.5 this remains same there itself.

So, that is why all symmetry city is existing. So, 1 half of this calculation we if you will just consider that is from i equals to 1 to 5 here, then other half can be considered as a same value here itself.

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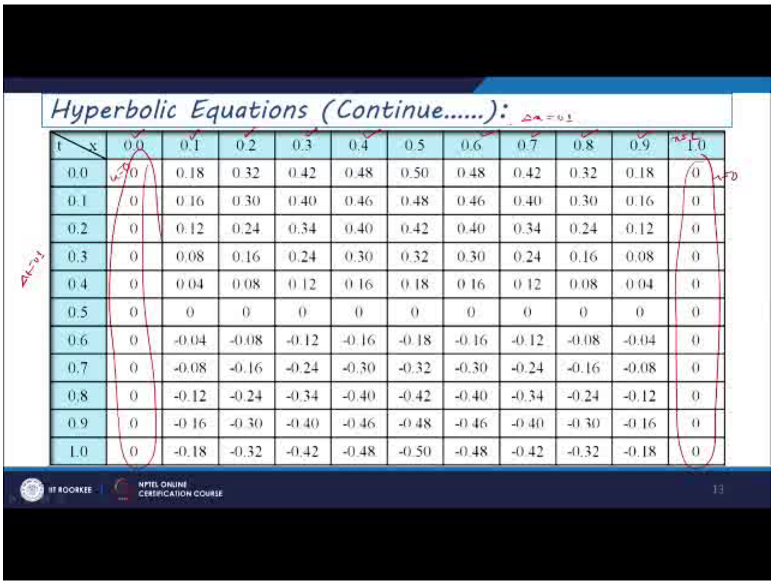


| t \ x | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0 | 0.18 | 0.32 | 0.42 | 0.48 | 0.50 | 0.48 | 0.42 | 0.32 | 0.18 | 0 |
| 0.1 | 0 | 0.16 | 0.30 | 0.40 | 0.46 | 0.48 | 0.46 | 0.40 | 0.30 | 0.16 | 0 |
| 0.2 | 0 | 0.12 | 0.24 | 0.34 | 0.40 | 0.42 | 0.40 | 0.34 | 0.24 | 0.12 | 0 |
| 0.3 | 0 | 0.08 | 0.16 | 0.24 | 0.30 | 0.32 | 0.30 | 0.24 | 0.16 | 0.08 | 0 |
| 0.4 | 0 | 0.04 | 0.08 | 0.12 | 0.16 | 0.18 | 0.16 | 0.12 | 0.08 | 0.04 | 0 |
| 0.5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.6 | 0 | -0.04 | -0.08 | -0.12 | -0.16 | -0.18 | -0.16 | -0.12 | -0.08 | -0.04 | 0 |
| 0.7 | 0 | -0.08 | -0.16 | -0.24 | -0.30 | -0.32 | -0.30 | -0.24 | -0.16 | -0.08 | 0 |
| 0.8 | 0 | -0.12 | -0.24 | -0.34 | -0.40 | -0.42 | -0.40 | -0.34 | -0.24 | -0.12 | 0 |
| 0.9 | 0 | -0.16 | -0.30 | -0.40 | -0.46 | -0.48 | -0.46 | -0.40 | -0.30 | -0.16 | 0 |
| 1.0 | 0 | -0.18 | -0.32 | -0.42 | -0.48 | -0.50 | -0.48 | -0.42 | -0.32 | -0.18 | 0 |

And if you just go for this computation the tabulated values, it can be taken these values as like so fast step if you will just see here. So, this you has just considered this values at t equals to 0, then 0.1, 0.2, 0.1, 0.4, 0.5, 0.6 up to 1 here since the increment is just taken as like a del t equals to 0.1.

Similarly in the x direction if you will just move here. So, del x can be considered as 0.1. So, that is why this is a has also taken level is 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1.0 and symmetricity means this boundary condition whatever it is just described at initial level, the same it is just a described at point x equals to 0 and x equals to 1 there itself.

So, it is given that u equals to 0 at x equals to 0 and x equals to 1. So, that is why we have just written u equals to 0 at all these boundaries that is prescribed at x equals to 0 and at x equals to 1, u equals to 0 here. And for 0.1 whatever the values you will just get it will just a justified at 0.9 and 0.2 whatever the value will just get and that is the same value you can just find 0.8, 0.2, 8.1 whatever the value you will just get, 0.7 you will just get this one, 0.4 whatever you will just get you will be satisfied at 0.6.

So, for the fast point calculation if you will just see here. So, this just takes the values as ui 1 here. So, j equals to 1 if you will just put these values from like i equals to 1 here. So, this can be considered as a 0 0 here plus ui as 1 here. So, u 2 0 here. So, if we will just consider this a first value as a u 0 0 plus u 2 0 by 2. So, that is nothing, but 0.18 here.

And similarly we can just get this 0.9 value that is 0.18 there, and we can just proceed in the same form that is a i will be vary from like a 1 2 1 4 5 6 sorry up to 5 if you will just to go then all points we can just and get these values there itself. So, based on this calculation if you will just calculate all these values you can just get in this form that is just to described here.

Thank you for listen this lecture.