Numerical Methods: Finite Difference Approach
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Lecture – 15
ADI method for solving elliptic equations

Welcome to the lecture series on numerical methods to finite difference approach. In this lecture series in the last lecture we have discussed elliptic equations, and their solution procedure based on like explicit approach and a implicit approach.

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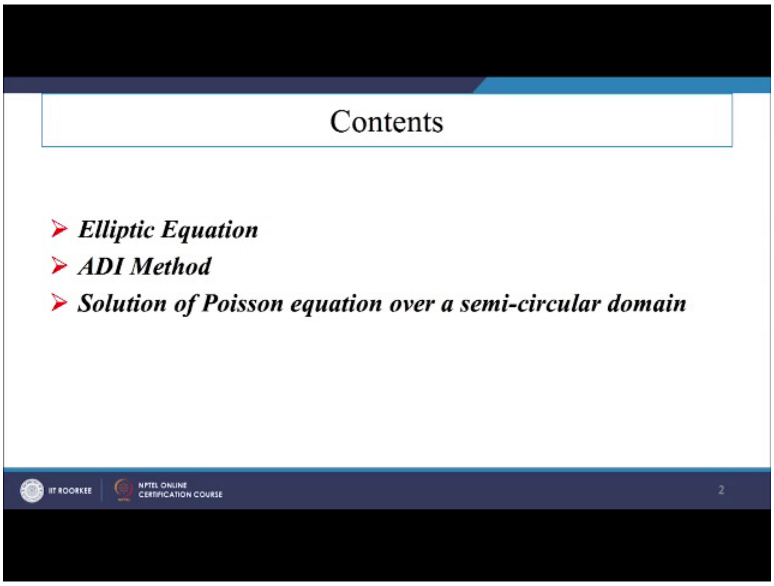

And in this lecture e will discuss about this alternating direction implicit method for a elliptic equations.

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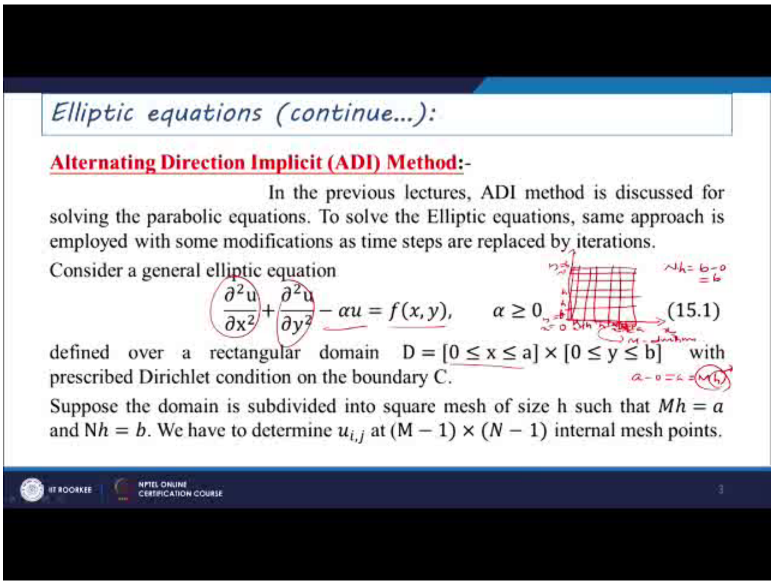

So, if you will just go for this ADI method here for a elliptic equation. So, alternating direction implicit method especially that is considers. So, this some of the steps calculation like in the x direction, we can just consider like a half step calculation and in the y direction we can just consider another half step calculation.

So, to solve this elliptic equation, if you will just use whatever we have discuss just discussed in the previous lectures for this parabolic equations especially, there we have just considered that if we are just considering this direction of a x moment in the n plus 1 steps and then we are just considering like y direction movement in n plus 2 steps then further we are just considering this x in the n plus 2 steps, and y in the n plus 3 steps there.

So, likewise we are just combining the steps to for this calculation of a values or a variables at different points. So, in this method if we want to solve like elliptic equations, same approach can be employed with some modifications as time steps are replaced by iterations here. So, suppose you if you will just consider a general elliptic equation that is in the form of like del square u by del x square plus del square u by del y square minus alpha u, this equals to f of x y where alpha greater than 0 and which is defined over a domain suppose x lies between 0 to a and y lies between 0 to b.

If you will just see here the domain if you will just define this can be the domain, since this is the y direction here this is the x direction and starting point x equals to 0 here and x equals to a here and y equals to 0 then y equals to b here. So, since the boundary conditions it will be prescribed at these values there, and these values here. So, that is why we will start x equals to 0 here, then we will just start at x equals to one to x equals to n minus 1. If you will just divide this domain into suppose like in the x direction if you will just divide into m grids and in the n direction if you will just subdivide into n grids here.

So, this is suppose m direction here sorry in the x direction we are just sub dividing into m dividends or m divisions, then our calculation will vary from like 1 to m minus 1 and j will vary from one to n minus 1. And if this like step size is considered as h here. So, then we can just divide this domain in the form like this one here. So, h is each of the space size or the grid length space here, and if we will just consider the same space size in the y direction also then we can just consider this is as the h size for each of this grid lengths and then the total length can be determined as like a total length we can just write that is as a minus 0, this is equals to Mh since h represents here the small grid length there.

Since m parts it has been subdivided. So, the total length can be written as Mh there and if you will just move in the y direction same can be considered; that means, that in the y direction total division is N here, and the each of these small grid size is like a h. So, Nh this can be written as the total length that is b minus 0 equals to b there. So, that is why you have just written here Nh equals to b, and Mh equals to a here and we have to determine uij at each of this grid points here which is not known to us.

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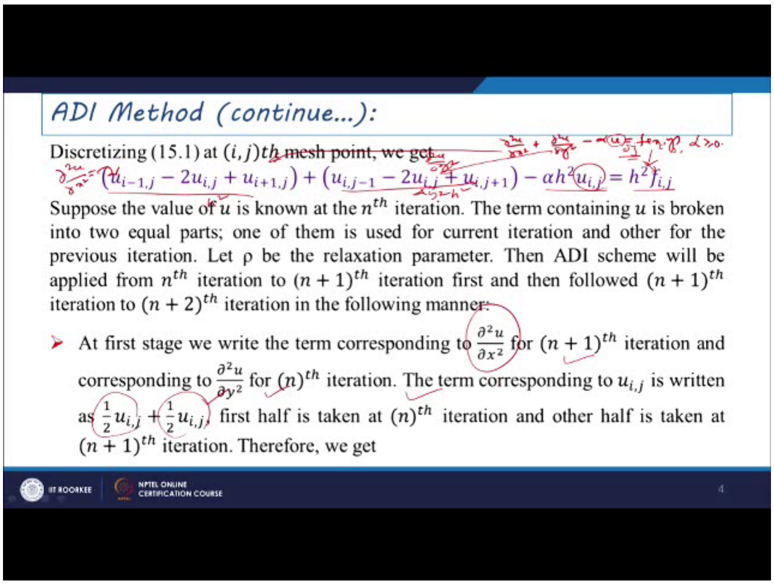

So, if you will just go for this discretization of this equation, that it is just represented here in the form like del square u by del x square, plus del square u by del y square minus alpha u, this equals to f of x y for alpha greater than equal to 0.

So, if you will just use the central difference approximation del square u by del x square this can be written in the form of like ui minus 1 j minus 2 uij plus ui plus 1 j by del x square or you can just write by h square here. So, then if you will just a discretize this equation here, this can be written in the form of u ij minus 1, minus 2 uij plus uij plus 1 by del y square this is the complete representation of del square u by del y square and since alpha is there.

So, u is defined at a ij grid points or at the ij grid points, we are discretizing this total partial differential equation. So, that is why this point is written as a uij there, and the last point it is just written as a f of xy. So, that is why this can be defined at xi and yj point there. Since your del x square is h square and del y square is h square. So, we can just write it in a modified form that h square can be taken and multiplied at this numerator side here that is in the form of alpha h square uij and this is h square of ij here. Suppose the value of a u is known at the n th iteration, previous iteration it should be known to us or initial guess we can just provide all these values either in the form if it is just provided, then we can just write that values and if it is not known to us at the beginning we can just consider that values as 0.

And we can start the iteration and each of these iteration, we can just consider this previous iterated value it should be known to us. Hence the term containing u here it can be broken into 2 equal parts. So, one part can be used for current iteration and other can be used for previous iteration let we are just using here ADI scheme with successive over relaxation method or a successive under relaxation method. Since in the earlier formulation we are just independently using this ADI method without using SOR method there, but here we try to implement SOR method in a combined form with this ADI scheme to get a improved solution.

So, suppose rho be the relaxation parameter, then ADI scheme will be applied from n th iteration to n plus 1th iteration first then followed n plus 1th to n plus 2th iteration. In the following manner that we have just written here at the first test we write the term corresponding to del square u by del x square since this movement in the x direction here. For n plus 1th iteration and corresponding to del square y u by del y square for n th iteration. And the term containing are corresponding to uij is written as since two parts we are just subdividing. So, that is why we can just write as half uij plus half uij where first half of a uij can be consider at a n th iteration, and the other half it can be considered at n plus 1th iteration.

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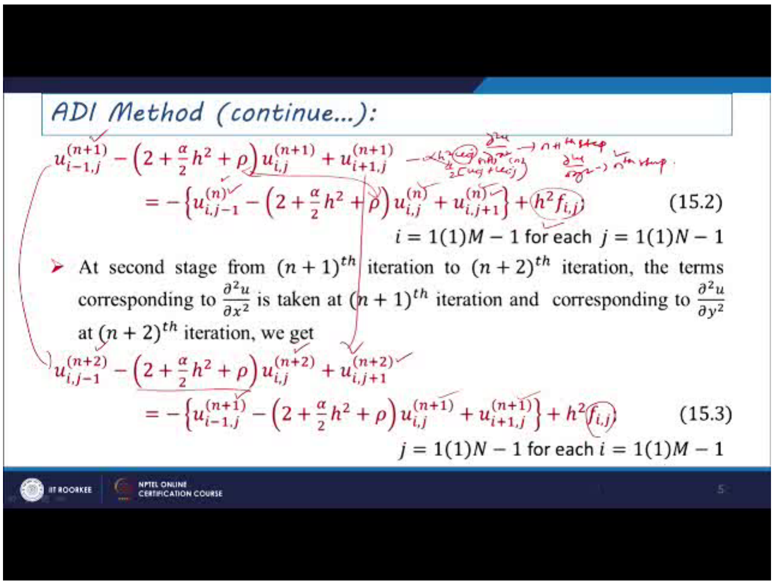

So, if you will just subdivided in this form, then we can just obtain this complete equation as in the form since if you will just see here rho is multiplied there. So, ui minus

1 j n plus 1, since a i sorry x we are just considering since a del square u by del x square term we will just consider at n plus 1th step, and del square u by del y square we will just consider at n th step there. So, that is why this n th step calculation if you will just see this contents are all of these terms, which is involving with the y discretization terms.

So, that is why if you will just see here. So, ui minus 1 j. So, minus 2 uij plus ui plus 1 j and plus 1 plus if you will just say here that we are just written that minus alpha h square uij. So, which is written in the form of half of uij plus uij, and half version it is just at a n plus 1 level half it will be at n th level. So, that is why it can be written as a like a alpha by 2, since alpha is multiplied there into h square uij to the power n plus 1, and here also alpha by 2 h square uij to the power n plus 1 n here.

And last factor if you will just see here that is as a h square fij it is just identity of there, but we have just added if you will just see here rho uij to the power n plus 1 and rho uij to the power n to get it updated from this SOR method there. So, at second stage from n plus 1th iteration to n plus 2th iteration the terms corresponding to del square u by del x square is taken at a n plus 1th step and corresponding to del square u by del y square at n plus 2th iteration since function is a dependent on these independent points or at that point itself that will take the functional value. So, that is why this will not get sense.

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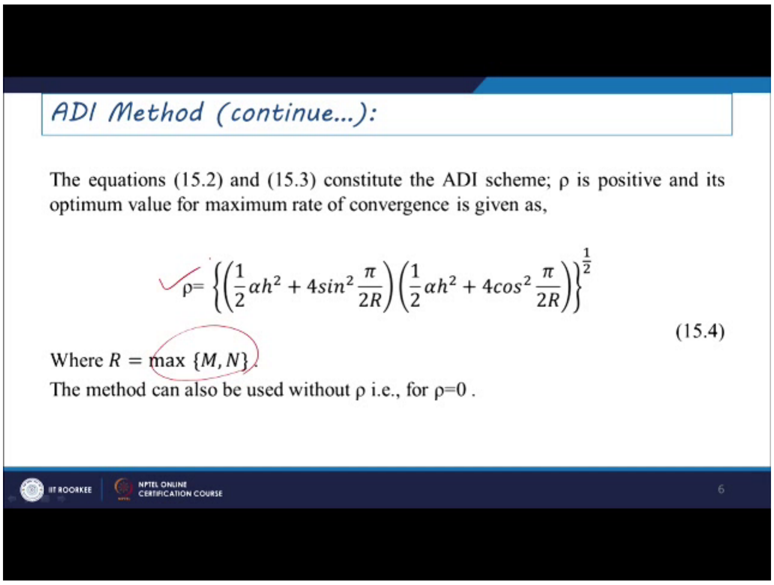

Here if you will just see our j is varying from one to n minus 1, and i is varying from one to m minus 1, since this way equations will be used for computation of the values at the unknown grid points. Since we are just using SOR method here we have to use this like relaxation parameter for these equations. So, especially in the last lecture we have discussed about successive over relaxation method, where this relaxation parameter especially it is used for a higher grid sizes as in the form of half alpha h square plus 4 sin square phi by 2R into half alpha h square plus 4 cos square pi by 2R, whole to the power half where R can be considered as a maximum of m and a n there, and always we will just consider rho is positive and its optimum value for maximum rate of convergence, it can be obtained from this equation here.

So, this method can be applicable when we will just consider rho equals to 0, if you will just consider rho equals to 0 in this equation. So, directly we can just write that one as the ADI scheme there itself. So, that is the like our general ADI scheme.

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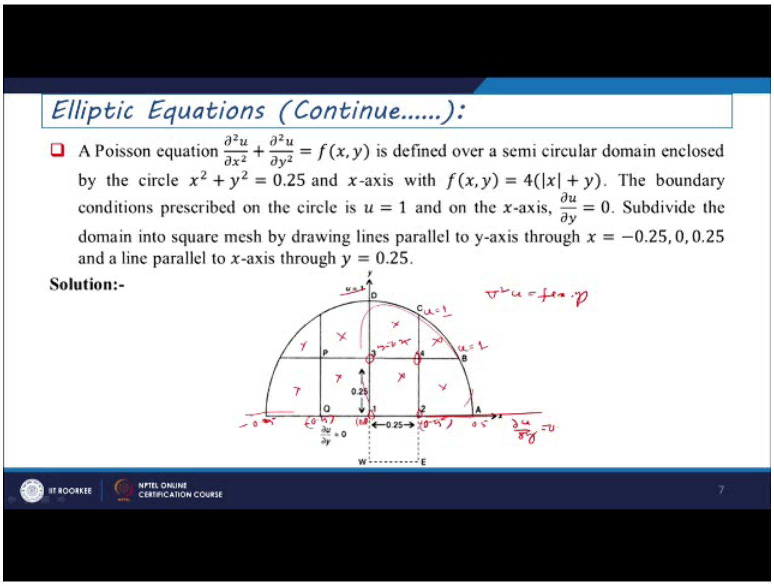

We are just considering here this problem which is like semicircular domain or irregular grid points can be chosen to find the solution. So, for this position equation the problem is given like a del square u by del x square, plus del square u by del y square this equals to f of xy is defined over a semicircular domain enclosed by the circle x square plus y square this equals to 0.25 and x axis with f of xy equals to 4 into mod of a x plus y.

The boundary conditions prescribed on the circle, is a u equals to 1 on the and on the x axis del u by del y equals to 0. So, since the problem is asked that we are just writing this equation that is in the form of Nablus square u equals to f of xy, and above this is semicircular domain we are just assuming u equals to 1 here and along x axis we are just considering this as a del u by del y equals to 0 there. So, if you will just subdivide this domain into square mesh by drawing suppose the parallel lines to along the y axis through x equals to minus 0.25, 0 and 0.25.

Since this defines is a circle with a centre at a origin, this is like minus your equation is varying from like 0.25. So, this is just a taken in the form of like 0.25 then 0.25. So, that is why this final point will be 0.5 here this will be 0.5 here. And if you will just consider here that is like the domains is subdivided into like a two different lines drawn along the y axis through x equals to minus 0.25 here, and x equals to 0.25 here, and one more line it is just plotted at a y equals to 0.25 here.

So, then we will have like regions 1 2 3 4 5 6 7 8 here, but since the center line symmetry it is just used for this problem. So, if you will just calculate one half of this domain here. So, other half we can just considered as this symmetry condition there itself.

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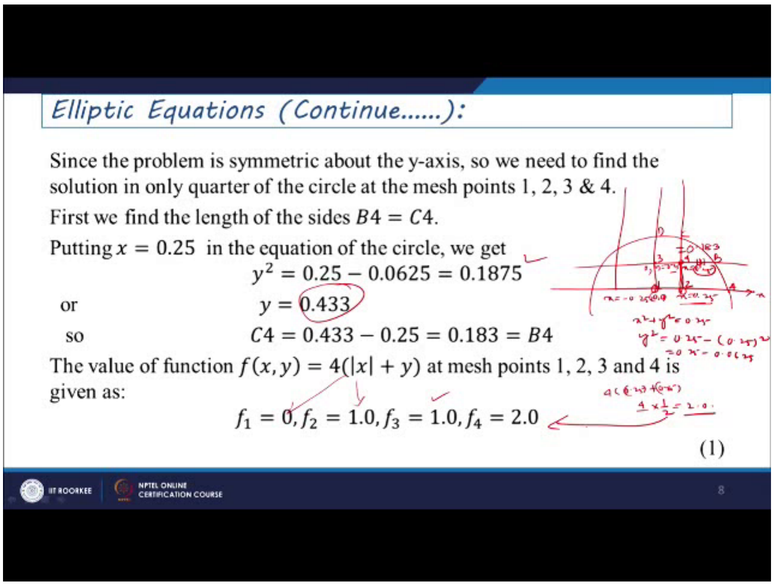

This problem is symmetric about y axis. So, we need to have to find the solution in the quarter of the circle, at the mesh points if you will just see here that is at a 1 2 3 4. So, this point, this point, this point and this point here.

Since this value is known u equals to 1 here, u equals to 1 here, u equals to sorry this is like your derivative boundary condition it is just a given here and along this line here and this point can be considered as the symmetricity. So, that is why one point can be taken from this point. So, if you will just go for this calculation of this problem here then we can just find that first we have to solve to get the sides b 4 equals 2, c 4 there. So, if you will just put like x equals to 0.25 here at that grid point, if you will just start computation of this domain here.

So, it is just a given that this domain is subdivided into two parallel lines x equals to minus 0.25, and x equals to 0.25 here and we will have a parallel line here. So, the points are signified as a this is the x direction here. So, A is the point, B is the point, C is the point, here D is the point here this is a originate is just chosen and your length points are chosen as 1 2 this is a 3 this is 4 here. So, we are just finding this length of the sides B 4 and C 4 since both are equal there. To find this one if you will just consider like x square plus y square this equals to it is just given 0.25 and at this line if you will just consider x equals to 0.25, it is just given here x coordinate we have to find y coordinate at that point.

So, that is why you can just consider y square this equals to 0.25 minus 0.25 square there. So, which is nothing, but 0.25 minus 0.0625, so which is just a given as a 0.1875. If you take this square root of this one then we can just find 0.433. So, that is why if you will just define here C 4, C 4 is nothing, but this distance is like 0.25 here. So, 0.433 minus 0.25 the remaining part it is just given as a 0.183 here.

So, this is nothing, but b 4 equals to also 0.183 and the value of the function f of xy this is just given as like a 4 into mod x plus y at the mesh point 1 2 3 4 here. If you will just see here. So, 1 just take the point as a 0 0 there. So, if you will just point 0 here. So, this will just take the 0 value there. So, f 1 is here and if you will go for like f 2 point here, f 2 is nothing, but x equals to 0.25 here. So, if you point put here 4 into 0.25 y as 0. So, this will just take like a 4 by two point sorry 4 into 0.25 that is nothing, but 1 by 4. So, that side is one.

And f 3 point if you will just to go here. So, this is nothing, but a x coordinate is a 0 here, but y coordinate equals to 0.25. So, that is nothing, but f 3 equals to 1.0. And a for f 4 point if you will just see. So, this is nothing, but this point here. So, that is why this coordinate if you will just take here. So, 4 into if you will just see here x point is like

0.25 plus 0.25. So, that is nothing, but 0.5 there. So, 4 y into 1 by 2. So, that services are taking 2.0.

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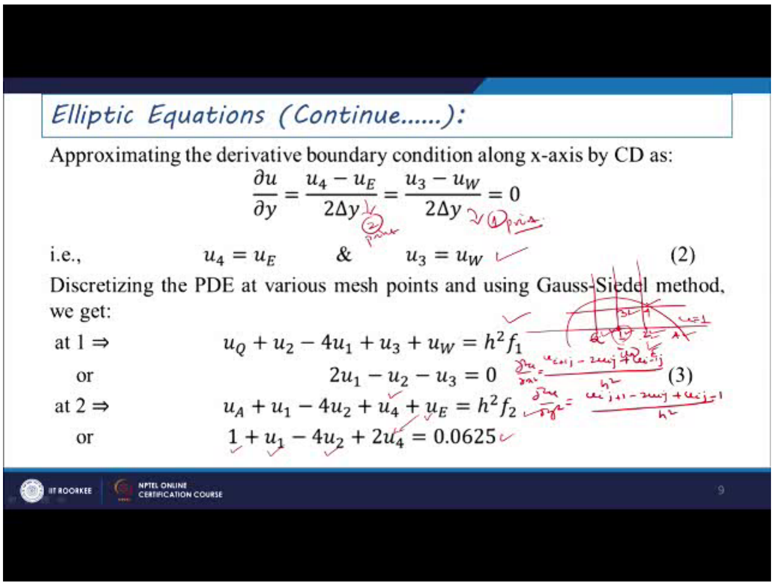

So, if you will just go for this derivative condition which is a defined at along the x axis here. So, we can just find that a del u by del y that is nothing, but this last point which is defined as a u 4. So, for this derivative condition if you will just see in the picture here we have just defined here like a fictitious domain there. So, this fictitious domain we are just giving this w as the west coordinate here and east part it is just written as a e here.

So, for this if you will just take this like difference of u 4 minus u E by 2 h or the length here. So, then we can just obtain this a derivative at this point. So, if you will just go like a del u by del y, this can be written in the form of a u 4 minus u E by two del y this can be written also as a u 3 minus u W by two del y,at the point if you will just see this is the value at the point 2 and this is the value at the point 1 there, this equals to 0 there.

So, that is why u 4 equals to u E and a u 3 equals to uW. So, this is at a point 2 this is at point 1 and if you will just discretize this partial differential equation at various mesh points using gauss-Siedel method, then we can just get that point 1 if you will just see 0 point one means we can just consider like a q is the previous point there and two is the secondary point there. So, if you use these points at a different levels. So, then we can just find since our domain if you will just see. So, the domain is defined in this form here.

So, if you will just see here this is nothing, but your point like 1 here, this is the point 2 here this is the point a here and this is the point like q here. So, if you will just take the point of discretization as a one here. So, we can find Q minus like 2 u one and this is like minus 2 point here. So, sorry this is plus 2 point here. So, that is why we can just write this a discretized equation as in the form like u i plus 1, j minus 2 uij plus ui minus 1 j by h square for a del square u by del x square, term and similarly we can just write del square u by del y square equals to ui j plus 1 minus 2 ui say plus ui j minus 1 by h square here.

So, that is why if you will just combinely write this equation. So, at the first point if you will just see at point 1. So, your values it can be taken as like a u Q plus u 2 minus 4 u 1 plus if you will just see here this point is a nothing, but it is considered as a third point here, this is nothing, but fourth point. So, that is why this difference minus this difference we will just take this is nothing, but uW. So, that is why it is just considered as a u 3 plus u W this equals to h square f 1 there and if you will just consider the symmetricity condition here that is nothing, but you can just consider u Q if you will just see here, u Q is nothing, but u 2 point here, and a u W is nothing, but u 3 point here.

So, it can be doubled there itself. So, that is why we have just written like u Q plus we are just considering u 2. So, that is why two u 2 we can just consider and this like you 3 and u W it can be considered since symmetricity it is there. So, that is why you can just consider this is as two u 3 here. So, h square f 1. So, at point one we are just obtaining this f function as a 0 there. So, that is why if you will just take common here like a minus also, then we can just write two 4 u one minus 2 u 2 minus 2 u 3 equals to0. So, again if you will just take a two common. So, we can just write two un, minus u 2, minus u 3 equals to 0 there.

Similarly, at a point two if you will just calculate here. So, it can just consider a point 1 point a, and it can consider point 4 and it can consider point e here. So, that is why it can be written as like ua plus u one minus 2 u 2, plus u 4 minus 2 u 2 plus u E this equals to h square f 2 since you have two u it is just considered as a 1 here. So, that is why we can just consider 0.25 square. So, that is nothing, but 0.0625 and if you will just put all these values here since a along this boundary it is just a defined this value u equals to 1 there itself. So, that is why it is considered as a like ua equals to one and u one is there minus 4 u 2.

So, u E equals to u 4. So, that is why this is 2 u 4 it is just coming. So, this is the equation here.

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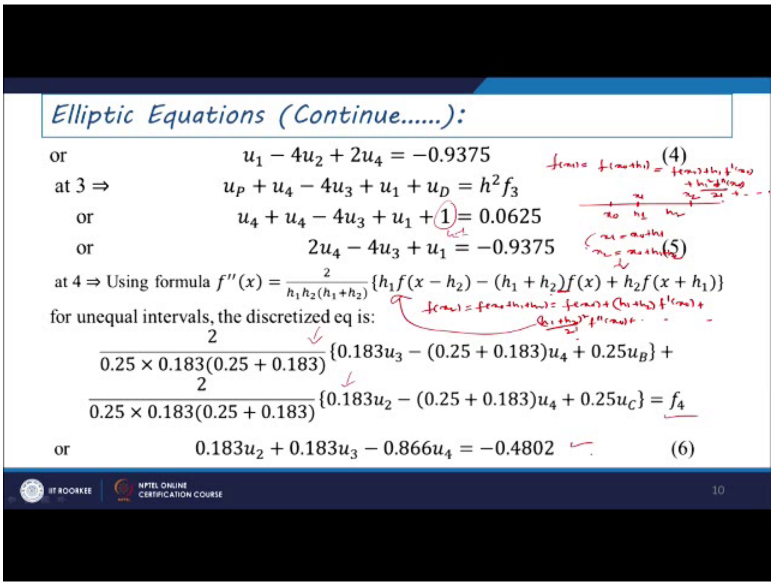

Similarly, if you will just find this point at a 3. So, you can just write that as up since a p is the point along this previous plane. So, that is why up plus u 4 minus 4, u 3 plus u 1 plus ud this equals to h square f 3 here. So, then if you will just put all these values here like up equals to u 4 symmetricity. So, u 4 plus u 4 minus 4 u 3 plus u 1 ud along the boundary it is just defined as a u equals to 1 there.

So, that is why this value can be written in the form of like 2 u 4 minus 4 u 3 plus u 1 equals to minus 0.9375 here. So, since this space is defined is in irregular domains have here grid sizes are not uniform. So, we can just use this formula for like derivatives of unequal spaced points. So, if you will just follow like numerical methods lecture of a ours. So, then you can just find that one. So, especially in this case we are just using like two different points, suppose if x 0 is the point there and h 1 is the first grid length here, then h t2 is the second grid space here.

So, like two different points I am just writing here. So, x 1can be written as like x 0 plus h 1 and x 2can be written as like x 0 plus h 1 plus h 2, then if you will just take Taylor series expansion at these two points like f of x 1 can be written in the form of a f of x 0, plus h 1 which can be written as f of x 0 plus h 1 f dash of x 0, plus h 1 square f double dash of x 0 by two factorial. So, plus rest of the points. Similarly if you will just expand the Taylor series expansion at this point here, then we can just write f of a x two this is nothing, but f of x 0 plus a h 1 plus h 2 which can be written as f of x 0 plus h 1 plus h 2 f dash of x 0 plus h 1 plus h 2 whole square by two factorial, f double dash of x 0 plus rest of the points.

So, if you will just eliminate here f dash of x 0 from both these equations. So, then we can just obtain this formula in this form here neglecting these higher powers of h 1 and h 2, and especially this second order differential formula it is just written as a 2 by h 1 h 2 into h 1 plus h 2, h 1 f of x minus h 2 minus h 1 plus h 2 f of x plus h 2 f of x plus h 1 here. So, for unequal intervals the discretized equation is like 2 by 0.25 into 0.183 into 0.25 plus 0.183 into 0.183 u 3 minus 0.25 plus 0.183u 4 here plus 0.25 uB.

So, this is u h 2 it is just there plus 2 y 0.25 into 0.183 into 0.25 plus 0.183, since we are just considering the second order derivative for a like a first del square u by del x a square, then we are just writing for del square u by del y square. So, this is nothing, but u 4 a f 4 there. So, if we will put these values then final form we are just obtaining this linear equation here, and if you will just considered these two equations 4 and 5.

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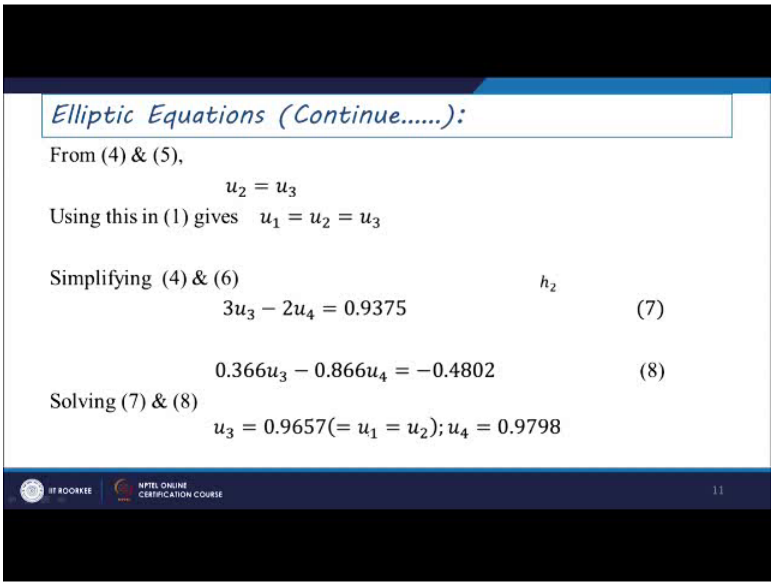

So, both these equations especially you can just find these are like 2 u 4 it remains same. So, u 1 remains same. So, that is why these coefficients if you will just compare here u 2 equals to u 3 we are just getting. And if you will just use this one in a first equation we can just get u 1 equals to u 2 equals to u 3 there, and if you just simplify equation 4 and 6 we can just obtain 3 u 3 minus 2 u 4, this 00.9375, and then especially if you will just solve this equation. So, simultaneously we can just get u 1 equals to u 2 that is nothing, but u 3 this you will just give you 00.9657 and u 4 equals to 0.9798 here.

Thank you for listen this lecture.