Numerical Methods: Finite Difference Approach
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Lecture – 13
Solution of elliptic equations

Welcome to the lecture series on Numerical Methods: The Finite Difference Approach. In the last lecture, we have discussed 2 dimensional parabolic equations and in the present lecture, we will start about these elliptic equations. So, in the elliptic equation, we can just find like 2 generalized equation. So, first equation it is called Laplace equation and second equation it is called a Poisson equation.

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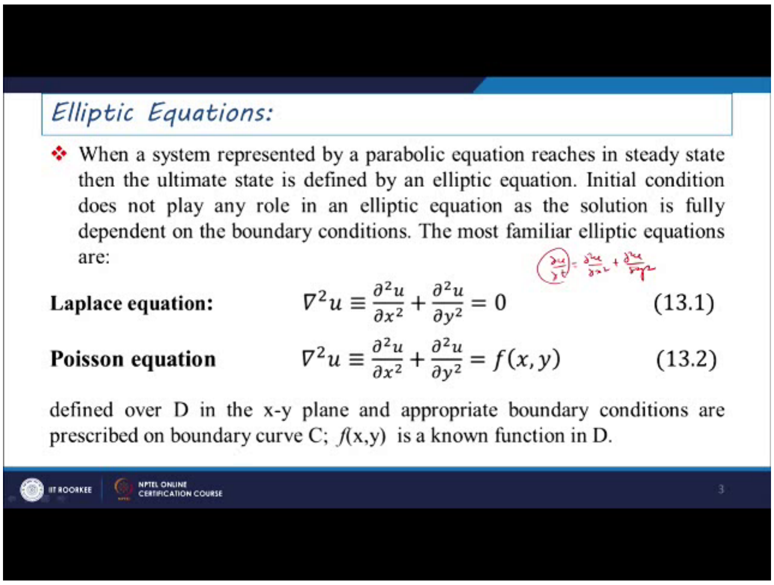

So, if you will just go for this Laplace equation here. So, if we are just representing any type of a parabolic equation. So, then when it reaches like steady state since in 2 dimensional nature, whenever we are just dealing with this parabolic equations.

So, that is written especially in the form of del u by del t; this equals to del square u by del x square plus del square u by del y square. And if you will just consider this system, it is in a steady state, then especially we can just consider this del u by del t term as a 0 there. And especially if you will just reduce this del u by del t term as 0 there, then this equation will be reduce to Laplace equation and which is nothing but the Elliptic equation of first kind we can just say.

And initial condition does not play any role in an Elliptic equation as the solution is a fully dependent on the boundary conditions. And the most familiar elliptic equations are as I have told that they are Laplace equation and Poisson equation. Especially Laplace equation is a signified as nebulous square u, this is nothing but del square u by del x square plus del square u by del y square, this equals to 0. And if you will just consider f of x y n plus of 0, then we can just consider this as a Poisson equation here.

So, whenever we are just to defining this class of equation, especially we should have to need like a domain where it should exist in the x y plane and the appropriate boundary conditions are needed to get the solution along the boundaries and especially the boundary is a denoted here as c and f of x y it should be known to us inside the domain. So, for solving this elliptic equation, we subdivide the domain into rectangular mesh and discretize the partial differential equations at the mesh points. This leads to a set of like linear system of equations which can be solved either by Gaussian elimination method or by Gauss Siedel method.

Since in Gaussian elimination method especially we can just say that, will have a set of unknowns and will have set of known values there. And in a iterative manner we can just get the solutions there. And in a Gauss Siedel method especially, you can just find that sum of this unknown values which can be known from the previous cycle of calculation or sum of this like n plus one time level of calculation, it can just take the immediate previous steps calculation values.

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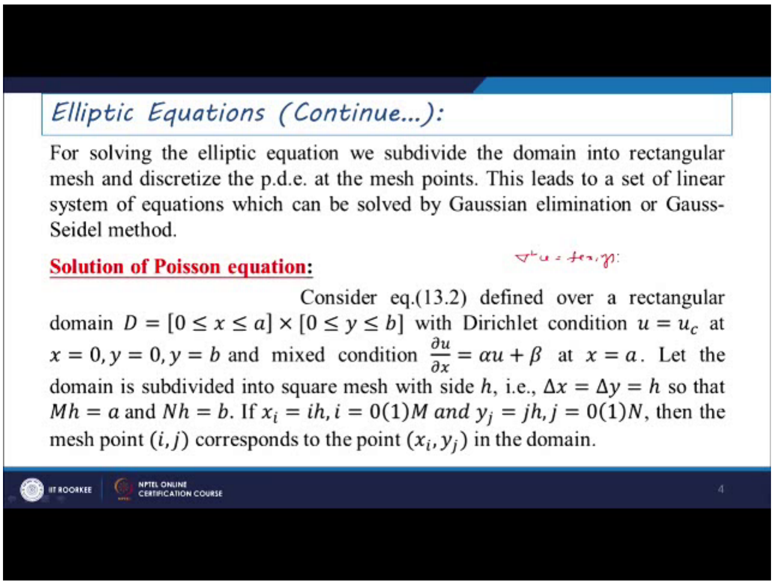

And if you will just go for this a solution of Poisson equation here, if you will just consider this equation which is a written in equation 13.2 here, that is in the form of like Laplace square u that is nothing but f of x y which should be defined over a rectangular domain d where d is bounded between like x from 0 to a and y is from 0 to b with the dirichlet condition u equals to u , since in case of dirichlet condition, we have to prescribe u as the particular values at x equals to 0, y equals to 0, y equals to b suppose and the mixed derivative condition that is a del u by del x equals to alpha u plus beta at x equals to a there.

So, let the domain is a subdivided into square mesh with side h that is we can just consider del x equals to del y equals to h, so that your total size it will be represented as or the total length in the x axis it can be just represented as m h is equals to a and in the y direction, the total length from 0 to the last boundary of y can be represented as a m h and if each of this independent grid points we are just denoting it as like in the x axis as x i there which can be written as a ih since i is varying from 0, 1, 2 up to m here.

So, this 0 bracketed 1 m means, the point will be start from 0 incremented by 1 and it will move off to the point m there. Similarly y j can be dented as jh where j h varying from 0 to m there, then the mesh points i j corresponds to the point x i, y j in the domain.

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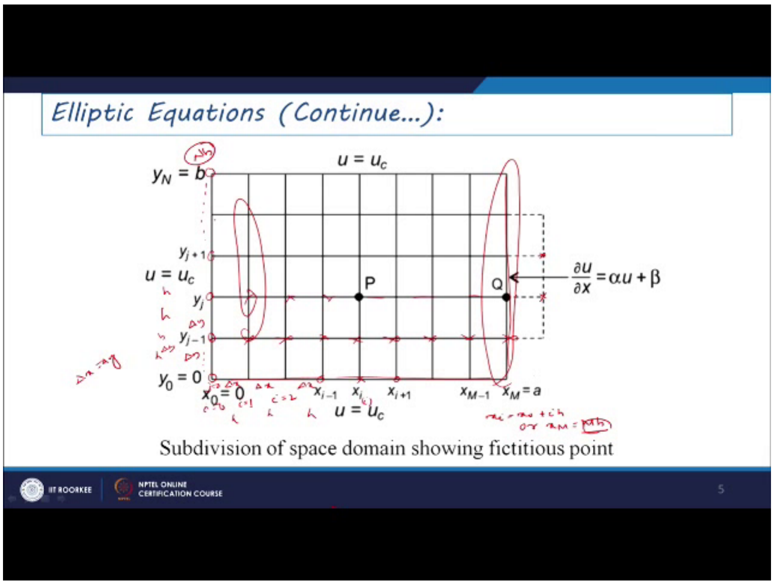

So, if you will just signify in the domain here, then especially we are just writing x naught equals to 0 since your starting point is i equals to 0 here, then i equals to 1, then i equals to 2.

So, likewise if we will just consider ith cell here, then especially this node will be denoted as x i here and it is previous node will be like x i minus and like for most node is like x i plus 1 here. Similarly if you just go for y direction here, y 0 or y naught equals to 0 here. So, this is the first boundary and at the end we can just consider y n equals to b here and if you will just consider a particular node point suppose jth node here, then the particular value of y with respect to j node it will be done that as y j here and it is previous node will be y j minus 1 and this for most node will be likewise j plus 1. And this will just continue here 0, 1, 2, 3 likewise it will just go to j minus 1, then j and j plus 1, then it will just continue up to y n equals to b there.

Since in the boundary condition, it is a prescribed that we will have like the dirichlet condition u equals to u c along x equals to 0, y equals to 0 and y equals to b there. And we will have like mixed derivative condition at x equals to a here. So, whenever we will just consider this mixed derivative condition, we require a fictitious cell in. So, if you will just consider the central difference scheme along this boundary here, it requires one more cell, after this boundary. So, that is why we have just considered this fictitious boundary along this last boundary of x equals to a here.

So, this subdivision especially this is just takes this grid sizes like del x in the x direction and del y in the y direction or especially we can just signify this one as h h h likewise and this one is since we are just considering here as del x equals to del y, we can just consider x equal sorry h equals to k there or del x equals to del y. So, each of this direction this is just this grid increments are coming in the form of like a size there.

So, that is why we can just write here x i equals to x 0 plus ih there or we can just write x m, this can be written as like mh there, the total distance we can just say. Since a starting point is 0 then h, then 2 h, 3 h, likewise if you will just move. So, final point we can just get it as m h there.

Similarly, if you will just go in the y direction, we can just find the last node point as n h there. So, if you will just go for this discretization of this Poisson equation there, that is nothing but, we are just writing as del square u by del x square plus del square u by del y square, this equals to f of x y.

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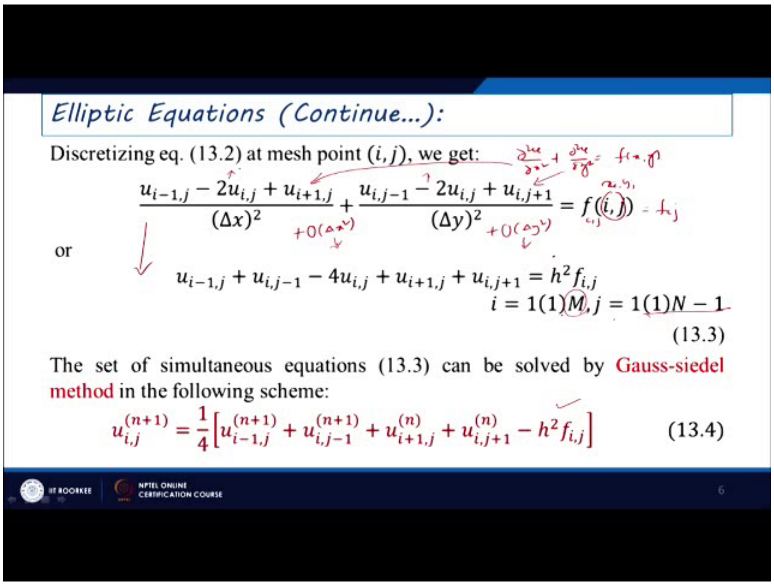

So, we can just signify that one, this point as x i and y j points 0, but especially we are just denoting here, that is in the form of like u of x i minus 1 y j. So, that is why we are just writing this one as f i,j or you can just signify this one as f i,j at this point here. So, if you will just go for the central difference approximation of this partial differential equation, the first term it will just give you like u i minus 1, j minus 2, u i j plus u i plus i j by del x square plus order of del x square term and plus, if you just go for the discretization of del square u by del y square, it can be represented in the form of u i,j minus 1 minus 2 u i,j plus u i,j plus 1 by del y square plus order of del y square here.

And if you will just neglect this higher powers of del x and del y here or we can just a neglect this error terms, there over. You can just get this total expansion as is in the form of like u i minus 1 j minus 2 u i,j plus u i plus 1 j by del x square plus u i j minus 1 minus 2 u i,j by del x square plus u i,j plus 1, this equals to nothing but f i,j here. since we are just denoting this one as x i y j there.

And if you will just write del x equals to h and del y equals to h here, so obviously, this del x square is nothing but h square and del y square equals h square and we have just write u i minus 1 j plus u i,j minus 1 minus 4 u i,j plus u i plus 1 j plus u i,j plus 1, this equals 2 h s square f i,j here. And if you will just take these values to be calculated at u i,j th level, then we can just write u i j at the unknowns level that is at n plus 1 step here, which can be separated and it can be written as like a 1 by 4 if you will just take to the right hand side here into u i minus 1 j which is a known from this previous step of calculation.

Then if you will just see here, u i,j minus 1 which is also can be calculated or it can be put from the previous step of calculation value. So, that is why it can be written as n plus oneth levels here, but if you will just see here, i plus 1 and j plus oneth level values which is not known to us that is why we can just put this values as u i plus 1 j m plus u i,j plus 1 m.

Since it can be predicted or it can be just put these values that as the previous step of calculated values here. So, then if you will just see here, minus h square f i,j which is already present there, that we can just keep it in that form. And in this step if you will just see that i is varying from 1 to m here and j is varying from 1 to n minus 1 and this is specifically if you just see here at y last boundary, we are just using the dirichlet boundary condition. So, that is why at nth step, this value is in known to us here and if you will just see here y 0 at zeroth level, these values are also known to us. So, that is why our computed value for j it will just vary from 1 to n minus 1 here.

So, that is why if you will just see for i values; So, i value it will just vary from 1 to m here since we are just using a derivative boundary condition at the last boundary here. So, that is why this last boundary value it is not known to us. So, that is why we can just vary these values since x 0 value at zeroth level it is known to us. Initially since it has been a prescribed to us, either it can be given from the initial conditions or it can be calculated from the boundary conditions. So, that is why this step of values it is not known to us. So, that is why we can just use this derivative is boundary conditions that at the previous step and this fictitious step values and then we can just evaluate this step at like mth position.

So, that is why this complete write up of this equation can be written in the form of like u i,j n plus 1, this equals to 1 by 4 into u i minus 1 j at n plus oneth level plus u i,j minus 1 at n plus oneth level plus u i plus 1 j at nth level plus u i,j plus 1 at nth level. And this f of i j value that is nothing but f of x i y j, it can be considered from this original equation and it can directly put the values over here.

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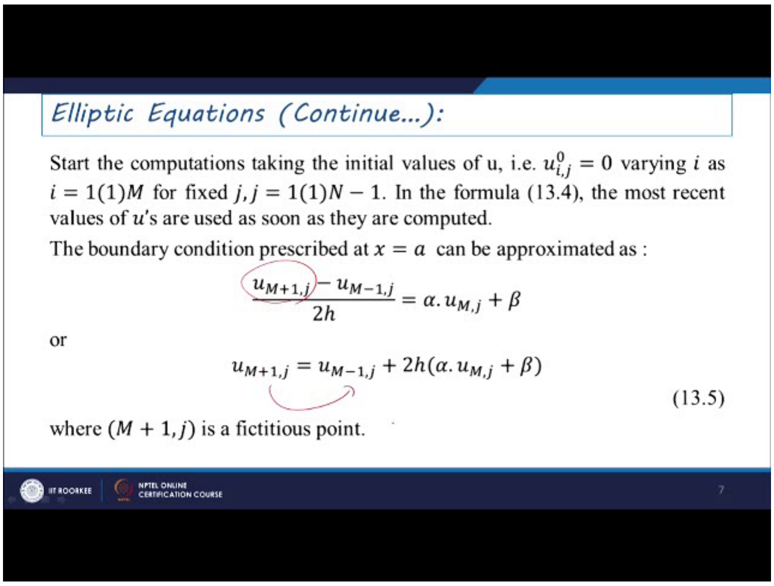

So, if you will just go for this further computations taking the initial values of u, that is in the form of like u i j 0 equals to 0 varying from like i is 1 to m for a fixed j and j equals to 1 to n minus 1. So, each of this fixed I, we will just keep it off and one by one j step, we will just move it up this means that, you can just see that whenever we will just go for the calculation here. So, first step i equal to for j equals to 0 here, all the values of i is known to us. But for the further calculation, if you will just go for like here like u del y increment here, then all this calculation for fixed j, it will just move for this cycle here. So, that is why this i will vary for a fixed j, again this j will be fixed, again this i will be vary at these points. So, likewise the computationally proceeding or the calculation will be proceeding.

So, if you will just consider this boundary conditions there in the formula 13.4 especially if you will just see here. So 13.4, it requires like i minus 1 and one more point if you will just see here i plus 1 and j minus 1 and j plus 1 here. So that is why, we can just consider this boundary and the prescribed a point that is x equals to a can be approximated since we are just using the central difference approximation here which gives this like order of approximation as a order of h square here.

So, especially we can just write this last boundary that is in the form like, since x m is the last boundary here, that is why we are just writing u m plus 1 j minus u m minus m minus 1 j by 2 h, this equals to alpha u m j since this is the particular point alpha u plus beta.

So, and where this u m plus 1 j value, this is the fictitious cell value which is not known to us. So, which can be updated by considering this u m minus 1 j value which is existing inside the domain there and which can be just added whenever this value will be computed from this previous cycle of calculation. If you will just go for a practical problem here, this problem we if we are just considering here as like del square u by del x square plus.

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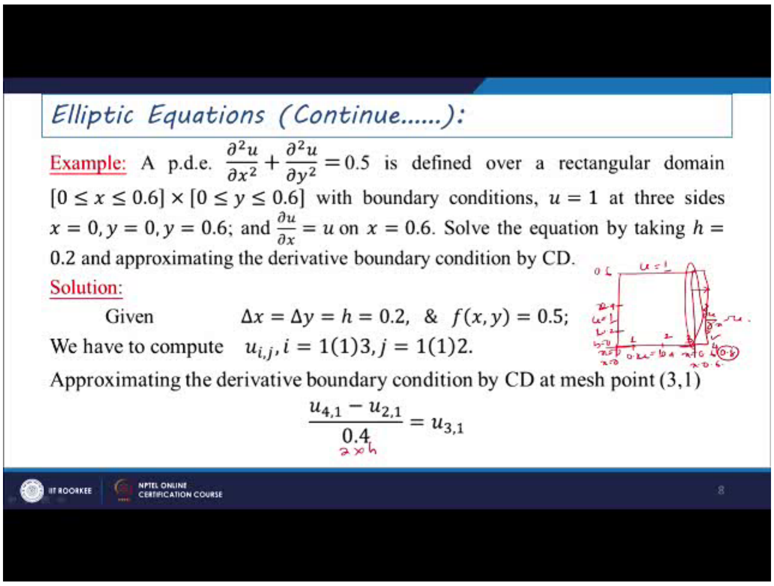

Del square u by del y square, this equals to 0.5 is defined over a rectangular domain where this domain is defined within the range like x is like 0 to 0.6 and y is lying between 0 to 0.6 with boundary conditions u equals to 1 at 3 sides, x equals to 0, y equals to 0 and y equals to 0.6 and del u by del x equals to u on x equals to 0.6. And the question is asked you to solve this equation by taking h equals to 0.2 and approximating the derivative boundary conditions by a central difference approximation.

Since the question is you, given this a boundary is existing like in the form of like x equals to 0 to x equals to 0.6 and y is starting at 0 to 0.6 here. So, this is a closed domain and the boundary conditions it is just written in the form of like 3 sides x equals to 0 and x equals 2 like 0.6, we have like here del u by del x equals to u and at y equals to 0. So, this boundary, we can just consider as this one as u equals to 1 here and this one u equals to 1 and this one u equals to 1.

So, if you will just see, since this derivative boundary condition it is just given along x equals to 0.6 here or at the last boundary. We can just consider the central difference approximation along this boundary as since if you just see here that h equals to 0.2 if you will just consider, we can just move like 0, 0.2 then 0.4 and the 0.6 and we have to consider one fictitious cell here, which will vary like one more extra point here. And if you will just see here, j will vary from like 1, 2 only, since last boundary last boundary 0 here. So, 0.2 than 0.4 and last is 0.6 here.

So, that is why j can be vary from only one and 2 points and i will vary from like 1, 2, 3 points here. Since the 4 is the fictitious boundary which can be just updated from this a previous step values. And if you will just approximate this derivative a boundary condition at this last boundary here, the last boundary it is just represented at x equals to 4 there or we can just consider this one as a 0.8 here. So, that is why, this cell can be considered as like u of 4, 1 minus u of 2 1 by 0.4 this equals to u 3,1.

Since we are just considering this one as like last boundary since we are just considering for y equals to 1 here or you can just denote this one as a u of 4 j minus u of 2 j by 2 into h here; h is given as a 0.2, that is why it is just taken as 0.4 here which can be represented as u 3 j here. So, if you will just move in that form, then we will have this domain structure like this one.

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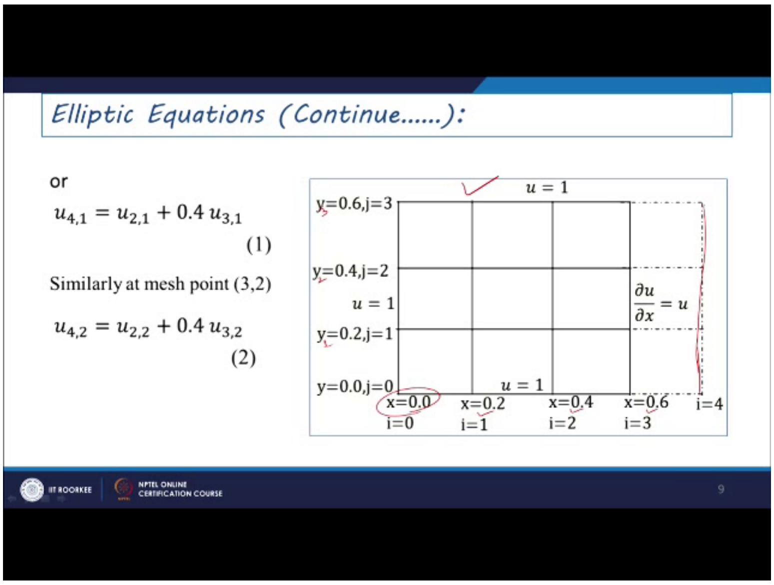

So, here I have just denoted as a in a clear form, x equals to 0.0 here and a starting point is a x equals to 0.2 and incremented by 0.2 that is why second is like x equals to 0.4, third is x equals 0.6 and this is the fictitious boundary here. And j is also varying like j equals to 1 that just takes a value like y 1 equals to 0.2 and y 2 equals to 0.4. So, last boundary y 3 equals to 0.6 here.

And if you will just varying this boundary is j is variation here, we can just write like u 4, 1 as u 2,1 plus 0.4 u 3,1 here, then u 4,2 this can be written as u 2 plus 0.4 u 3,2. Since a, if you will just see here, we have just kept it as a 0.4 as in the division form and we are just taking this one in the right hand side is a multiplied form there and a discretizing this partial differential equation at various mess points like i j, i is varying from 1 to 3 and j is varying from 1 to 2 here.

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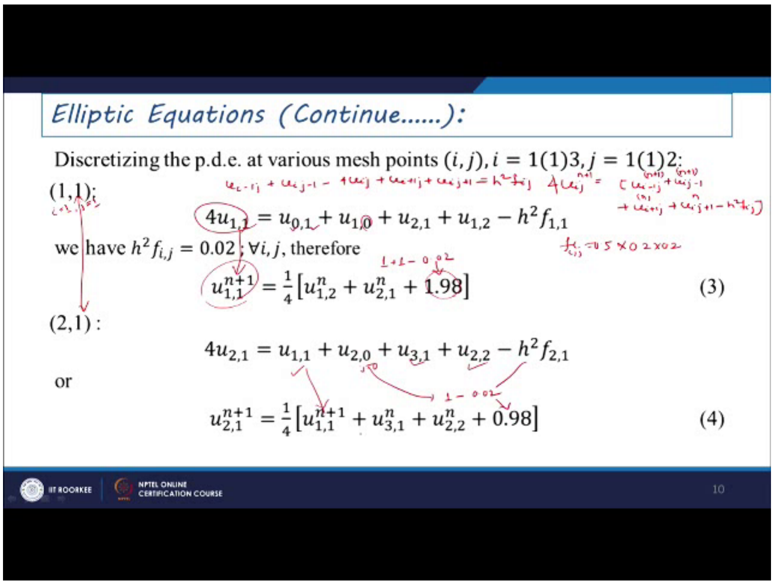

Then we can just find this discretized value as the first one, if you just see here, we will have like u i minus 1 j plus u i j minus 1 minus 4 u i,j plus u i plus 1 j plus u i,j plus 1, this equals to h square f i,j. And if you will just consider like starting point is suppose i equals to 1 here and j equals to 1.

So, that is why if you will just a separate this term here that is in the form of u i,j n plus 1 which is written especially in the form of 1 by 4, u i minus 1 j power n plus 1 here plus u i,j minus 1 power as n plus 1 here. So, plus u i plus 1 j power n here plus u i,j plus 1 power n here minus h square f i,j.

So, if you will just multiply 4 in this side, we can just write this one as in the form like 4 into u i j, u i j to the power n plus 1; this can be written in this form here. And since i equals to 1 here j equals to 1, especially 4 u 1 and i minus 1 means you can just write 1 minus 1 this is 0 here and j is 1 here, then u i 1 is there, j minus 1 is just taking 0 value here, then u 2,1 since a 1 plus 1, this is 2 and j 1 here, then u 1 j plus 1 means 1 plus 1 this is 2 here minus h a square f 1.

And whenever we will just consider like h equals to 0.2 here, we will have like f i,j, f i,j is just defined as 0.5 here. So 0.5, if you will just consider into 0.2 into 0.2 especially we are just considered. So, that is why this value is coming as 0.02 here and for all i j therefore, we can just write this one as a u 1 to the power n plus 1, since a this left hand side is unknown here.

And first two u 0,1 and u 1 0 especially if you will just see, then you can just find that u 0,1, we can just consider since a 0 is the like i equals to 0 is the this boundary here, 0 1 it is just prescribed as 1 value u equals to 1 is a given here. And if you will just see 1 0 also this is just given the value as 1 there. So, that is why 1 plus 1 this is 2 here. So, 2 minus this is a like 0.02. So, that is why this is just given as 1.8.

So, if you will just put these values here, especially I can just write this 1 plus 1 minus this is 0.02. So, that is why this just gives the value 1.98 here. Similarly if you will just go for like is 2,1 cell here, since we are just varying like i fixings j there itself, that is why I am just keeping fixed j here. So, j equals to 1 only and i is varying like 1, 2, 3, likewise it will just vary. So, if you will just go further there is a 2,1 cell here, I can just find here as if you will see here this formulation here.

So, this means say i minus 1 this is nothing but 2 minus 1 is 1 here, then 2 here, then j minus 1 means 0 here then u of 3,2 plus 1 this is 3, then 1 here plus u 2 and this is j plus 1 is 2 here minus h square f 2 1. And 2,0 especially if you will just see, this means that j equals to 0 here, j equals to 0; if you will see this boundary condition is fixed as u equals to 1 there. So, that is why this value is a 1 minus this value is like 0.02. So, that is why this 0.98 it is coming and all other values are unknown to us.

And this u 1 value, we can just predict it out since u 1 it is computed from this previous cycle here. So, this can be put it there for the next of operation.

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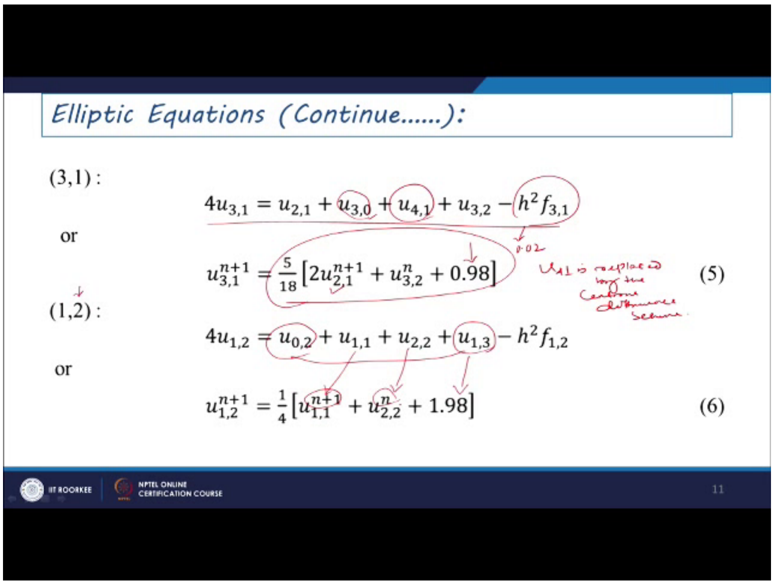

And if you will just go for i equals to 3 there are keeping fixed j there, we can just have this series expansion as in this form here 4 u 3,1 is equal to u 2,1 plus u 3,0 plus u 4,1 plus u 3,2 minus h square f 3 1 here. And if you will just see here, u 3,0 if you especially if you just see here, this is a nothing but j equals to 0 means this is u equals to 1 and this is nothing but you can have the value like 0.02. So, that is why this difference is coming as the 0. 98 here.

And if you just see here u 4,1 cell, that is nothing but if you will just go for our previous calculation u 4,,1 is defined as u 2,1 plus 0.4 u 3,1. So, which is a just determined in the form of u 2,1 and u 3,1. So that is why, we are have like 2 u 2,1 here and if you will just go for like 1 term added it over here for u 3,1, that can be taken to the left hand side and which can be eliminated directly and which has this formulation.

So, especially u 4,1 is replaced by the central difference scheme. And finally, if you will just go for like u f 1 to here that is a keeping j as 2 since a j computation for like 1, we have completed. So, that is why we are just moving one more further step for j. So, if j equals to 2 for i equals to 1 here. So, this can be written as u 4 of u 1,2 this equals to u 0,2 plus u 1 u 2 plus u 1,3 minus h square f 1 2 here.

So, if you will just see here, this means that j equals to 3 we will have this boundary condition u equals to 1 there and we will have like u 0,2 we will have u equals to 1. So, that is why 1 plus 1 this is nothing but 2 here. So, 2 minus this is 0.02, that is why this is just giving 1 0.98 here. And rest of this are can be updated from this previous cycle of calculation or from the known values. So, u 1,1 n plus 1 it is known to us, so that is why we have just written it as n plus 1 here and this is unknown to us. So, that is why we have just written it as n from here.

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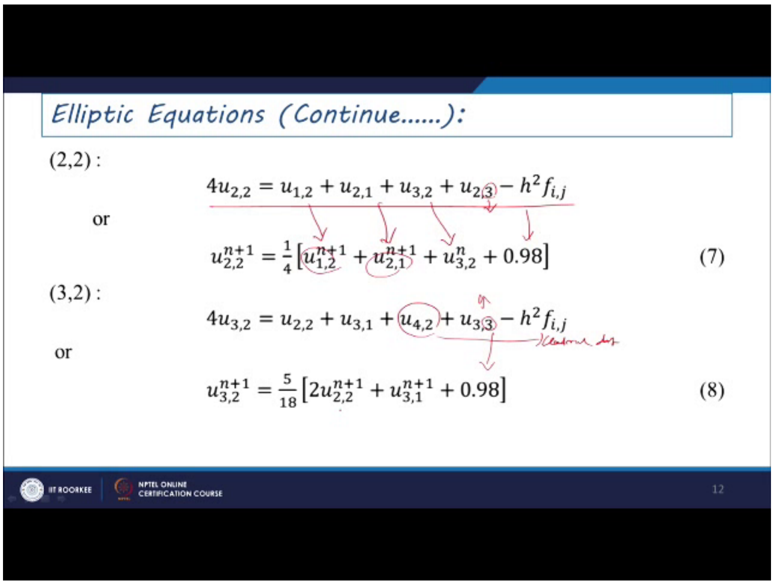

Similarly, if you just go for like I equals to 2 for a fixed j equals to 2 there. So, then we can just represent this equation in this form here. For j equals to 3 these values is known to us u equals to 1. So, that is a 1 minus 0.02, that is nothing, but 0.8 here. So, all other if you will just see here 1,2; 2,1 and u 3 is 2 is there. So, these values are known from this previous steps whatever we have just made it now. So, this can be updated from that step values there and the next if you will just go for i equals to 3 here, this can be written in the form of like u 2,2 u 3,1 plus u 4,2 plus u 3,3 minus h square f i,j here.

So, if you will just see here for j equals to 3, we will have this known value there. So, that is why 1 minus 0.02. So, that is why this is just giving you 0.98 here and u 4,2, if you will just see this can be replaced by the central difference approximation and where they see u 2,2 it will just come and u 3,2 will it will just come over there. So, u 3,2 it can be separated it out in from the 1 term from the left hand side and u 2 can be joined here and is 2 u 2,2 here.

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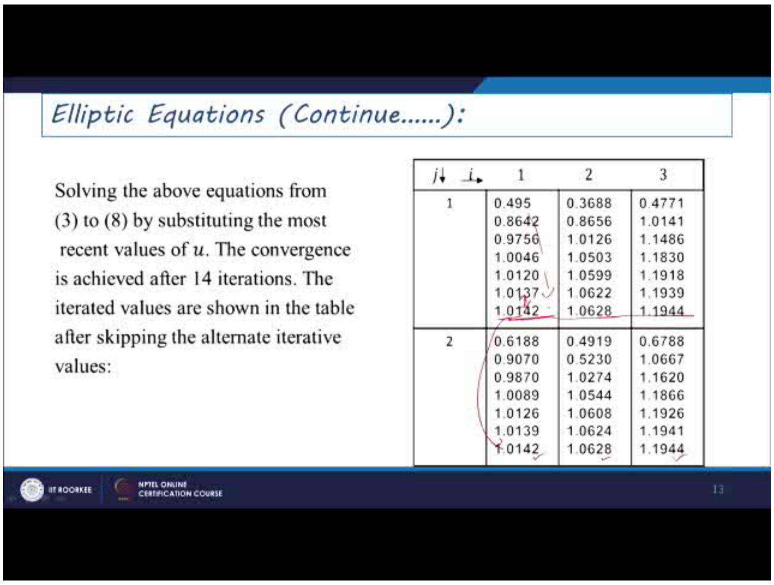

So, if you will just solve this above set of equations by substituting this most recent values of u, the convergence is achieved after 14 iterations, the alternate values we are just denoting here since i is varying like 1, 2, 3 here; for a fixed j, we can just iteratively get all these values here. So, if you will just see here like repeated values are obtaining like 1.0137 here and 1.0142.

So, after like third decimal places this convergence is occurring. So, that is why, we have just consider this one as the last iterated value here. And by considering these values again, we are just moving for the second iterative and we are just getting from these 2 steps that this difference is just minute here; this means first step we are just getting 1.0142 and from this second cycle of calculation we are just getting this one as the last values here.

So, that is why this just shows this convergence rate for this like a Gauss iterated methods. So, if you will just iterated values are shown in the table after skipping the alternative iterated values, if you will just consider here, then you can just find that this updation of the values you can just predict out from this table here. So, whenever we are just going this tabular values calculation here in each of this i step values if you will just see here, 0.495 minus 0.8642, likewise if you will just go.

So, finally, we can just find the difference of 1.0137 minus like 1.0142 here, the differences here. So, then we can just find this convergence rates. So, if you just visualized, this convergence is coming up to third order here. And similarly, if you just go for these steps you can just find this difference is coming up to third order also there. So, after this we can just conclude that this is the convergence iterated values for like these variables here.

Thank you for listen this lecture.