Numerical Methods: Finite Difference Approach
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Lecture – 11
Solution of two dimensional parabolic equations

Welcome to the lecture series on numerical methods to finite difference approach. And in this approach in the last lectures, we have discussed one dimensional parabolic equations and their solution methods.

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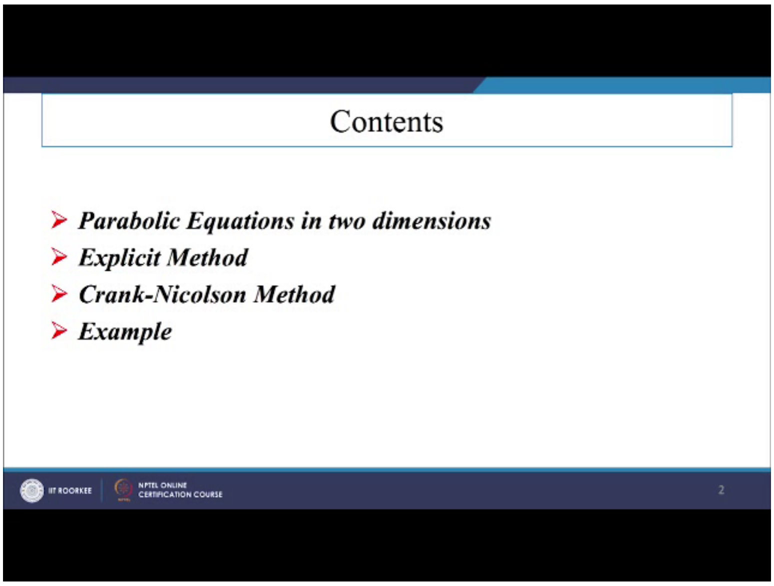

And in the present lecture we will just go for these parabolic equations in two dimensional, and we can just use like explicit method, implicit method or like semi implicit method that is a Crank Nicholson method to find the solutions.

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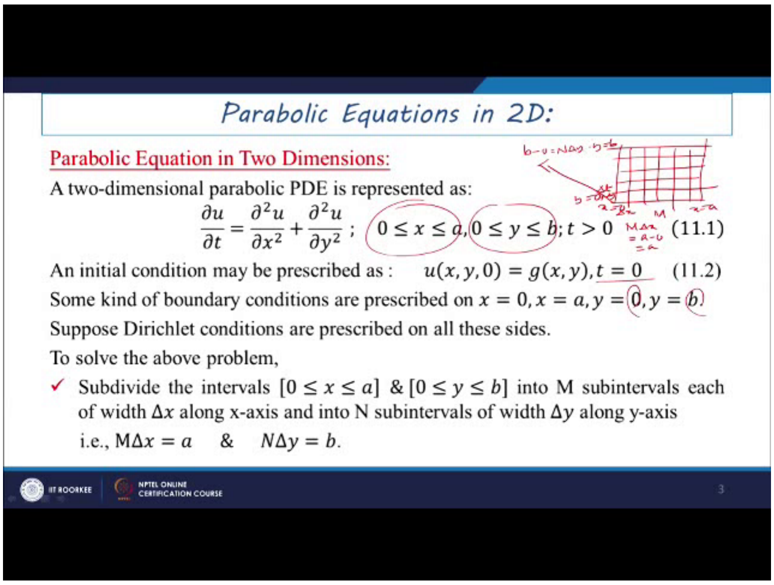

So, if will just go for this parabolic equations in two dimension, then this equation can be written in the form like del u by del t, these equals to del square u by del x square plus del square u by del y square, where we will have this a boundary that is just defined within this region like x lies between 0 to n, y lies between 0 to b.

So, if you will just see here this is a two dimensional problem means we will have like three coordinates here. So, one is x coordinate, another one is y coordinate, and another one is t coordinate here or a time coordinate here. But especially since space coordinates we are just varying in the x direction and y direction. So, that is why it is called the parabolic equation in two dimensions; and if you will just deal such type of equations. So, we need the boundary conditions at two boundaries that is especially if you we can just define the boundary in the form like x equals to 0 to x equals to a here, then y equals to 0 to y equals to b here, then we will have a third increment that is in the form of t there.

So, especially we can just move in the x direction, that is incremented with the space length or this grid size del x here, and if you we want to move in the y direction we should have to consider this is grid size as del y there, and if we want to move to the next approach from the initial approach or this initial steps to the next time step level, we have to move this time step in the direction of del t there.

So, for this problem especially since we need a like a three conditions here. So, first condition is it can be prescribed u of xy 0, this equals to g of x y at t equals to 0 that is initially this condition should be provided at each of the grid points, then some kind of boundary condition. So, that is a prescribed at x equals to 0 and x equals to a, then some kind of boundary condition. So, we should have to know at y equals to 0 and y equals to b. Suppose in this problem Dirichlet conditions are prescribed at all these sites; this means that u is specified or u is given at all of these boundaries. To solve this above problem if you will just go for the solution method, first we have to subdivide this intervals x lies between 0 to a, and y lies between 0 to b into m sub intervals in the x direction and n sub intervals in the y direction and each is off width del x in the x direction and del y in the y direction.

Hence so, since we are just considering here m subdivisions in the x direction. So, that is why you can just write this total length that is nothing, but m del x this can be defined as like a minus 0 as a there. And if you will just go for y direction here the total length it can be defined as a b equals to or b minus 0 this can be defined as n del y here. Since its grade space length is del y in the y direction here.

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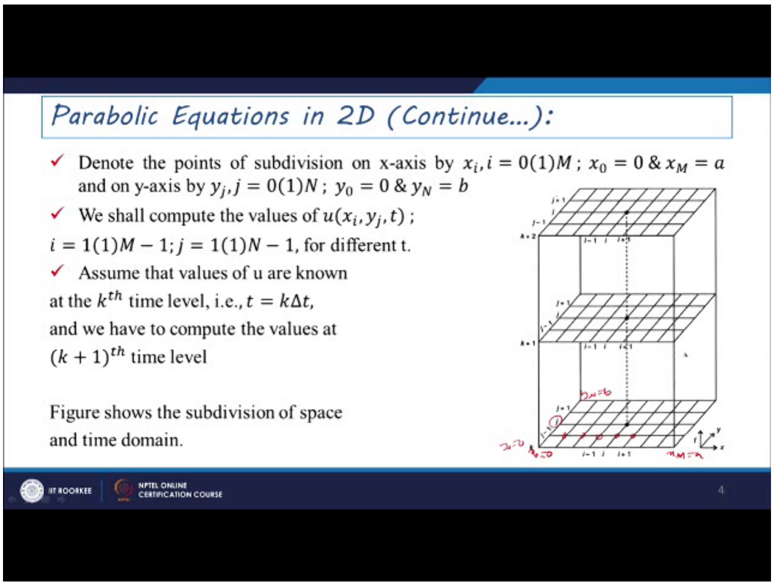

So, if you we want to see here then we can just fine that these three directions, it is just represented in the graphical sense here. So, x just represent in the like horizontal direction here and y is in the like plain section direction here and t is the vertical direction here. Since we are just denoting this subdivision of x axis as the I substitute. So, that is why we can just write this as x i here and your starting point here it will be x 0 equals to 0 and last point it will be xm equals to a here, and then the y axis since we are just a denoting here the j coordinates if you will just see.

So, that is why your starting coordinate here y 0 equals to 0 here, and last point yn equals to be there. And we shall compute all the unknowns that is placed on this grids here. For like variation of i equals to 1 to m minus 1 and j equals to 1 to m minus 1 for different t values.

So, assume that you are known to us at the kth time level. So, if kth level values are known this means that if the boundary values are known to us, then the next immediate level we can just proceed by considering this your boundary level values for the next iterated level calculations. So, if suppose k equals to 0 suppose here initial level, then all values are known to us along the first boundary layer, and further increments it can be calculated with the time as a increment and that is in the form of t equals to k del t here since in the first step we can just consider t equals to 0, then the second step we can just consider t 1 equals to del t, then third step we can just consider that as t 2 equals to 2 del t is. So, likewise we can just move.

So, this is the figure which is shows all this a time domain and also this a space grids that is formulated in the x and y coordinates.

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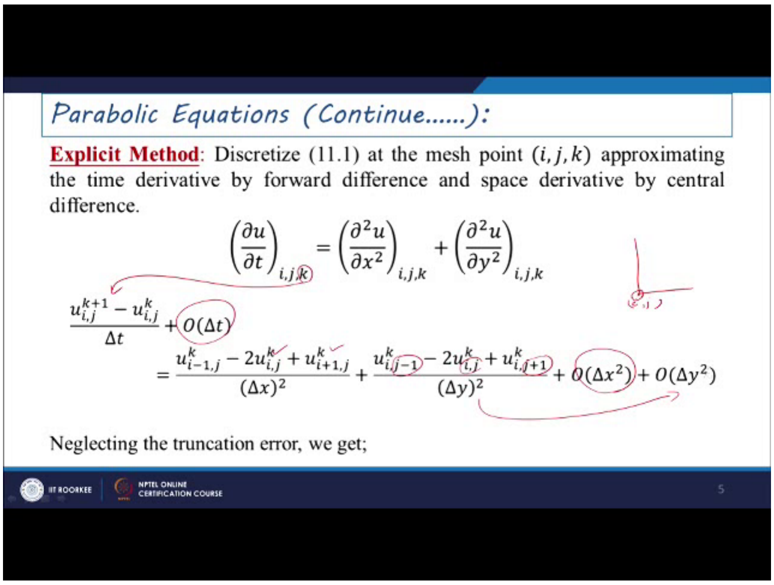

So, if you will just go for the solution in explicit method here and if we want to discretize this equation del u by del t equals to del square u by del x square plus del square u by del y square at the mash point I jk here. Since x represents the ith coordinate here and y axis represents the jth coordinate here and k represents the time level. And if you will just approximate this time derivative by a forward difference approach and a space derivative by central difference approximations here, then we can just write this del u by del terms since there is a time variation. So, we can just consider there is a variation in the k here.

So, that is why we are just writing uij at that particular grid point, since in the space coordinate if you will just see ij is the coordinate their. And time means it is just in the upper level it is just moving there. So, that is why we can just consider this space grid size is fixed there.

So, that is why we are just writing uij to the power k plus 1 minus uij power of k, see since this k plus 1 represents these velocities or this a temperature at the time level 1 there if you will just consider k equals to 0 there. And if you will just consider like uij 0 means it is can be taken from these initial conditions, and that is why it is just written in the form of uij to the power k plus 1 minus uij to the power k by del t. And the order of approximation already we have discussed that it will just consider as a order of del t here first order approximation, it will just take since we are just considering here this forward difference approximation. And if you will just go for these a space coordinates here in the x direction, we can just write this central difference approximation at kth level as ui minus 1 j, k minus 2 uij k and then ui plus 1 j k here by del x square and its order of approximation is a order of 2.

So, that is why it is just written as order of x square here and if you will just go for like the del square u by del y square term. So, then we can just take this central difference scheme at ij coordinate. So, it will just consider one step backward and one step forward there. So, that is why it is just considering j minus 1 point and j plus 1 together with a j point there.

So, its order of approximation it is just it is has consider also second order of approximation. So, that is why this higher order terms it can just occupied with this a power of order two afterwards. So, if will just neglect this higher powers of x and higher powers of y, that is in the form of in del x and del y and also this first order term of del t.

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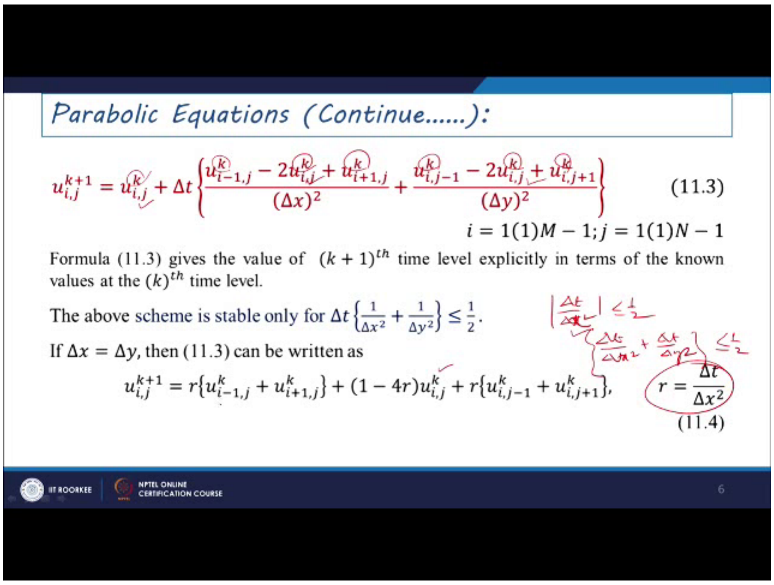

Then we can obtain this expansion is uij power k plus 1 this equals to uij power k plus delta since we are just writing uij to the power k plus 1 minus uij by del t. So, that is why del t can be multiplied in the right hand side and uij can be taken to the right hand side.

Since if you will just see right hand side, we are just keeping all of these known values there and i is varying from 1 to m minus 1 and j is varying from 1 to n minus 1 there. So, if we will just use this formula at k plus oneth level, then this scheme will be stable whenever we will have del t into 1 by del x square plus 1 by del y square, it should be less are equal to half. Since you already we have discussed that explicit scheme is stable whenever we will have like del t by del x square it should be less or equal to half.

Since a for a like one dimensional sense, especially we are just writing or one dimensional problem especially we are just writing del t by del x square it should be less or equal to half for stability of this explicit scheme. So, that is why when it is extended to two dimensional sense he have seen two coordinates are involved like a del t by del x square and del t by del y square. So, we can just consider. So, both these values some should be less or equal to half here.

So, if in a particular sense we will just consider like del x equals to del y equal grid space or equal grid length here, then we can just find that uij to the power k plus 1 it can be written as since a uij to the power k it is a present here, uij to the power k present here, uij to the power k present here. So, that is why it can just a consider here as 1 minus 4 r into uij to the power k here.

Since r is defined as a del t by del x square here, and if you will just write this a like a i minus 1 and i plus 1 terms here. So, that can be written in the form of r into ui minus 1 jk plus ui plus 1 jk here. And if you will just write like j space coordinates j minus 1 and j plus 1. So, it can be written as r into uij minus 1 plus uij plus 1 k here.

So, for the further computation if you will just implement Crank Nicolson scheme for this two dimensional parabolic equations, then we have to discretize this equation at the mesh point ij k plus half.

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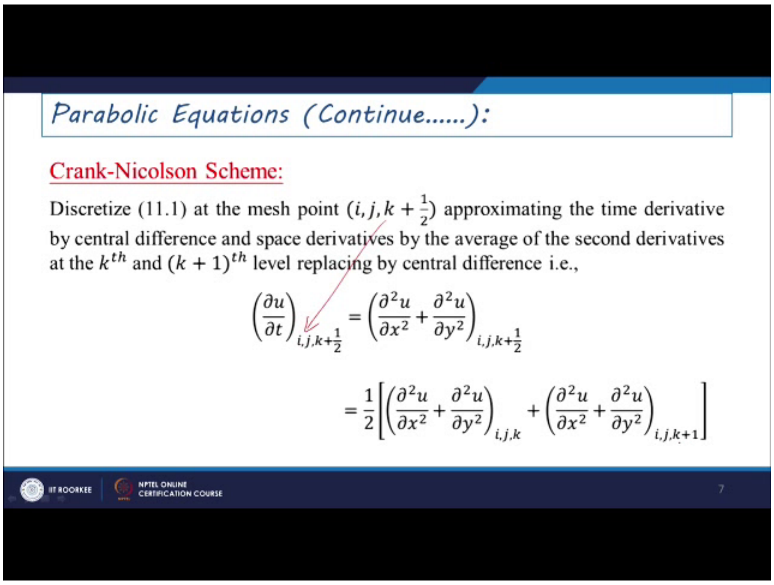

Since we are just considering half of the time level there, and then we are just taking the average. So, if you will just approximate this time derivative by central difference and space derivatives by average of second derivatives at kth level, and k plus oneth level by replacing the central difference. That is especially if you will just write here del u by del t at the grid point ijk plus half.

So, this will just give you this u discretization at k plus oneth level, minus you add the kth level divided by your 2 del t. So, and the space coordinate if you will just see here ijk plus half here. So, if you will just take this averages at the grid points ijk and ijk plus 1, especially we can just average the space coordinates in x and y direction there. So, if you will just consider this central difference for a time derivative here.

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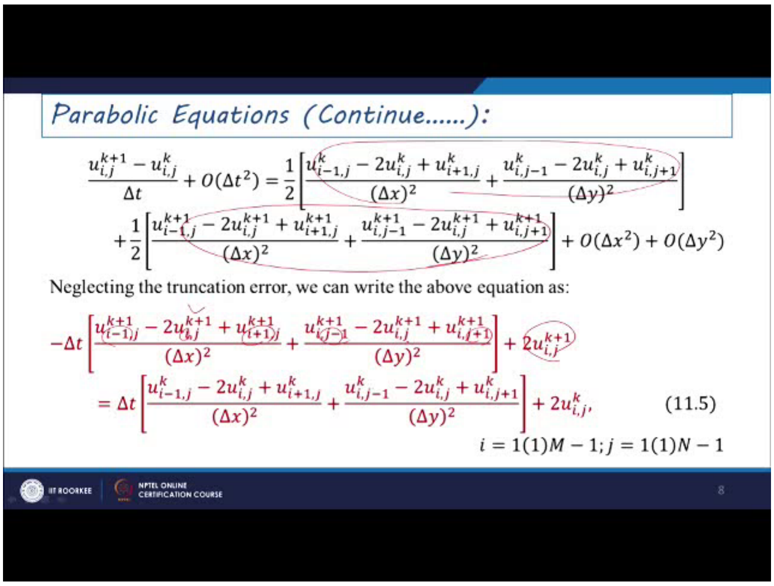

So, then it can just represented uij at k plus oneth level, minus uij at kth level divided by del t and since it is just takes the central difference approximations.

So, this order of approximation for like a space coordinate, it you can just consider order of del x square plus order of del y square here. So, that is why these average schemes it can be considered as a half into uij, ui minus 1 j k, minus 2 uyjk, plus ui plus 1 jk divided by del x square plus uij minus 1 k minus 2 uijk plus uij plus 1 k here by del a y square.

So, this is a especially if you will just see we are just considering this one in the kth level here, and if you will just see all these points we are just approximating at k plus oneth level here and this order of approximation especially, it is just considered in the form of like order of del x square plus order of del y square here. So, if you will just neglect this truncation error, we can just write the above equation that is minus del t since if you will just see here we want to separate this k plus oneth storms to the left hand side, and kth term to the right hand side since a kth level values are known to us. So, that is why we want to kept that one in the right hand side here. And if you will just see here so, some of these values that is u i minus 1 coordinate values and ith coordinate values and I plus oneth coordinate values all are unknown in the k plus oneth level. And some of these coordinates if you will just see j minus 1 and j plus 1 this is also unknown values here and especially if you will just see here that is a 2 uij k plus 1 and almost here it is also occurring like minus 2 uij k plus 1 and minus 2 uijk plus 1.

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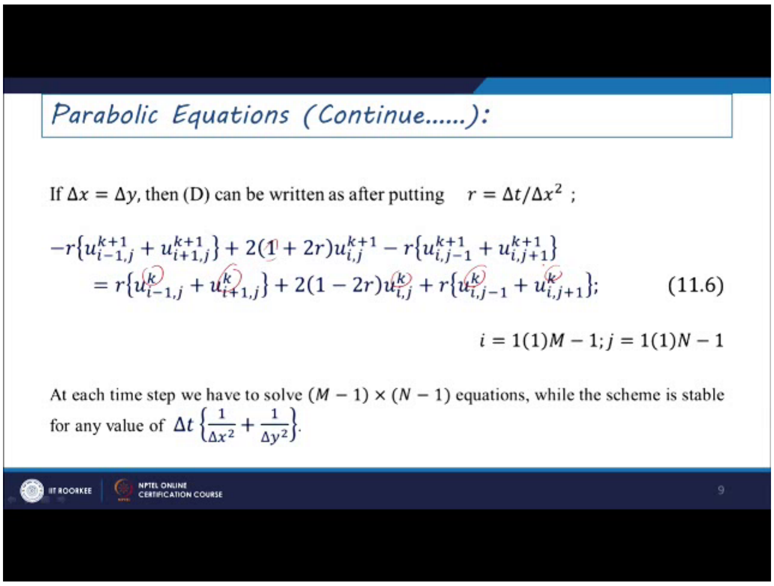

We can just separate it out and we can just write in a combined form as, if the del x equals del y since we are just considering here uniform criticizes.

So, this means that whatever they say grid length we are just considering in the x direction the same grid space we are just considering in the y direction here. So, after putting all these values we can just find that minus r ui minus 1 j k plus 1, plus ui plus 1 j k plus 1 plus 2 into 1 plus 2 r since it is like 4 terms, that is present in the form of ui j to the power k plus 1 here, and if you will just see here one term that is just represented here 2 uij k plus 1 without the multiplication of r term, which is defined as a del t by del x square here.

So, that is why they say 2 is a present here and minus r uij minus 1 k plus 1 plus uij plus 1, k plus 1 here and the right hand side all the non-terms that is in the power of k all are present here only. So, if you will just vary these values from i equals to one to m minus 1 and j equals to 1 to n minus 1. So, we can have like m minus 1 into n minus 1 system of equations, and each time step we have to solve this m minus 1 cross n minus 1 system of equations, where the scheme should be stable and for the stability of this scheme we should have to choose that del t and del t into 1 by del x square plus 1 by del y square it should not be restricted with any value for the Crank Nicholson scheme since already we have defined that for the explicit approach this should be less or equal to half there.

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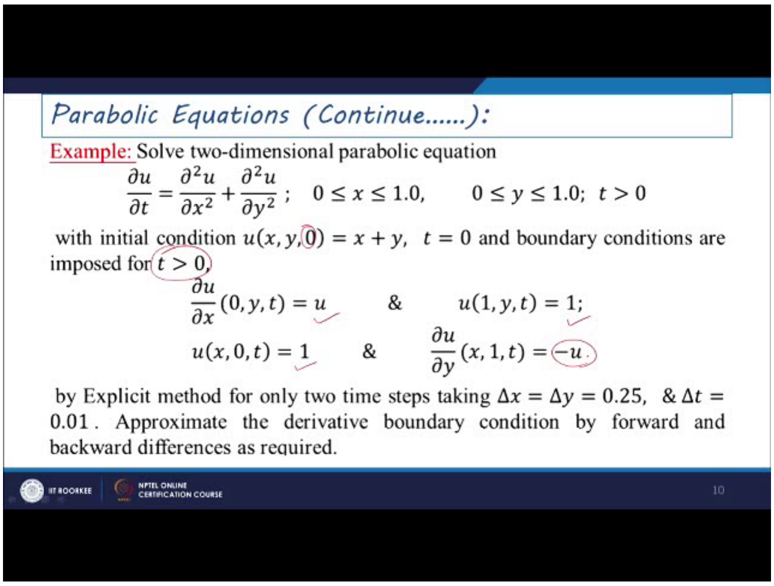

And if you will just go for the solution of parabolic equation, so, with a specified boundary condition suppose if we are just a considering here as del u by del t equals to del square u by del x square plus del square u by del y square, where x is a lying between 0 to 1 and y is lying between 0 to 1, for t greater than 0. With the initial condition initial condition means we are just stop prescribing all of this space coordinate values at p equals to 0 point, that is especially defined as x plus y here.

And the boundary conditions are imposed for t greater than 0 since the boundary is fixed and it will not changed with respect to time. So, that is why it is just written as a t greater than 0 here and which is defined as in the space coordinates in the x direction at x equals to 0, this normal derivative or del u by del x equals to u here, and at the last boundary that is x equals to one we are just choosing, u of 1 y p this equals to 1 here, and for y equals to 0 it is just a given that u of x 0 t that is one here and the derivative condition that is del u by del y at the last boundary that is just consider as minus u del.

And it is a asked that find this solution using explicit method for only two time steps, taking del x equals two del y this equals to 0.25 with time increment del t as 0.01. And approximate the derivative boundary condition by forward and backward differences as required. Since already we have shown that whenever we are using this derivative boundary conditions at this initial boundaries or at the first boundary we are using forward difference approximations and at the last boundary if you are just using then we are just using this backward difference approximations.

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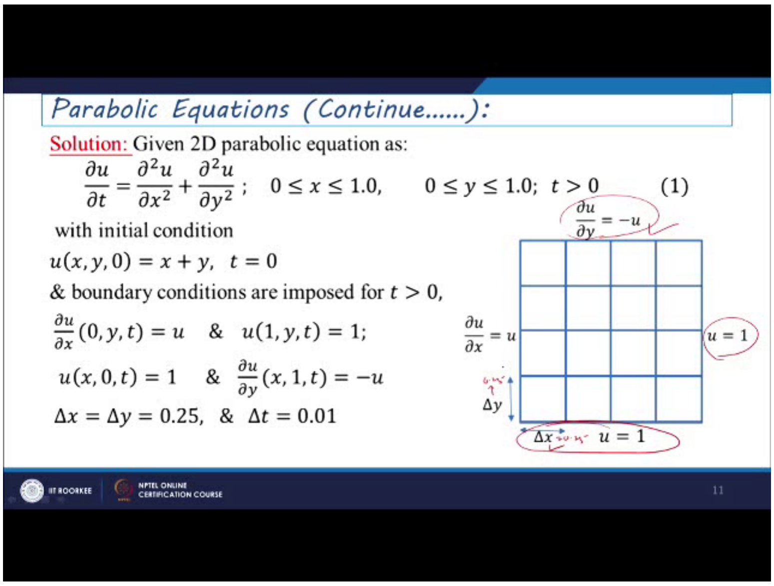

And if this equation is a just given with these conditions, then we can just have formatted in a manner that address it can required one more extra condition or a fictitious boundary at the last boundaries in set y, it is just a given as a del u by del y equals to minus u at y equals to 1, and at the first boundary it is just given a derivative boundary condition at x equals to 0 that is as a del u by del x equals to u.

So, that is why at these points if you will just use like central difference approximations, it requires a extra fifties a grid space for the calculation. So, if you will just write all these boundary conditions in a compacted form. So, it can be written as like del u by del x at 0 y t as u and u of 1 yt equals to1 and u of x 0 t this has 1 here.

Since the x 0 t means say at y equals to 0 we are just considering this boundary here and if you will just consider like x equals to one here that is nothing, but we are just considering a sorry this is x equals to one if you are just considering this is one here. And if it is considered as y equals to 1 here we have just considering this is minus u here, and this space length that has considered as a del x here, which is defined as a 0.25 here, and del y is the space length or the grid space which is a consider in the y direction as 0.25 here also.

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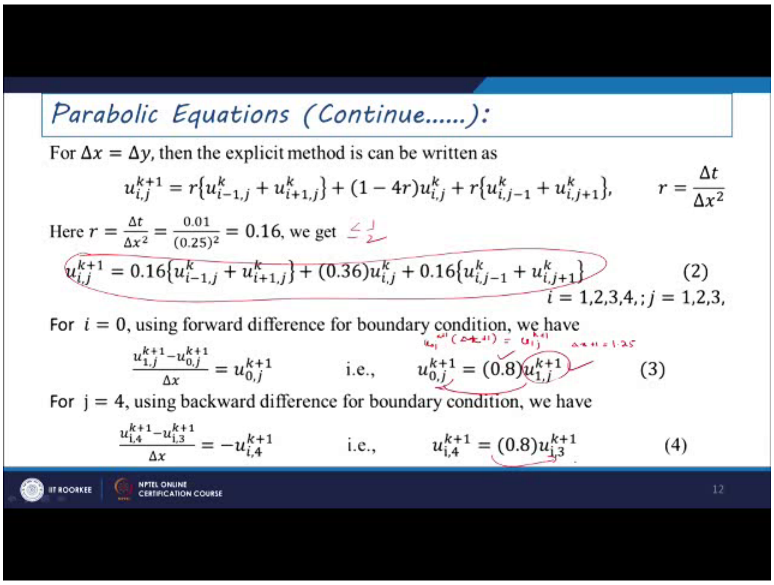

So, if you will just proceed with these values then this formulation can be written as like for del x equals to del y since uniform grid size you already it is given in the problem, then this explicit method can be written as ui j to the power k plus 1, this has minus a sorry r ui minus 1 j k plus ui plus 1 j k, plus 1 minus 4 r uij k plus r into ui uij minus 1 k plus ui j plus 1 k here where r is define as a del t by del x square. And we have to keep it in mind that whenever this r values should be less or equal to half then the system will provide as a solution here for this explicit scheme. And for this if you will just find this r value here r can be written as a del t has a by del x square, which is defined as a 0.01 by 0.25 whole square that is nothing, but 0.16.

And if you will just use these values, so, we can just get this value as a like uy z to the power k plus 1 that is defined as here 0.16, since it is less or equal to half you can just say . So, that is why you can just use and this is explicit approach or to get the solutions. So, after putting all these values like r values; so we are just reducing this equation in this form here. And since the question is given like two steps we have to move here. So, i is varying from 1 2 3 4 since a one more boundary it is required in the j direction also. So, j is varying from 1 to 3 here. So, for i equals to 0, if you will just use like forward difference along the boundary, we will have like u nj minus u 0 j at k plus oneth level by del x this is just defined at u 0 j at k plus oneth level.

Since the boundary initial boundary condition it is not provided exactly at the point u 0 j. So, u 0 j can be taken this value as in the form of like 0.8, u 1 j to the power k plus 1 here. If you will just see here that is this equation is separated in the form like in since del x is the first multiplied with here. So, we can just write as a u 0 j k plus 1, that is del x plus 1. So, this equals two especially u one j k plus 1 here. So, that is why they say if you will just divide this one as del x plus 1 here. So, del x is a defined as a 0.25 here. So, 0.25. So, 1.25. So, 1 by 1.25 that is not especially 0.8 here.

So, that is why that is why it is just written as a u 0 j to the power k plus 1, this is 0.8 u 1 j k plus 1 here and this value will be updated from this like the value that has been considered as u 1 j to the power k plus 1 here. And for j equals to 4 using backward difference for boundary condition, we can have like you i 4 to the power k plus 1 minus ui 3, k plus 1 here by del x this can be written as minus ui 4 to the power k plus 1 here and uy 4 to the power k plus 1 this can be also written as 0.8 and ui3 to the power k plus 1.

Same thing also here this boundary value this can taken the like inside computed value, and it can be updated in each of the time steps and it can be used in the computation process.

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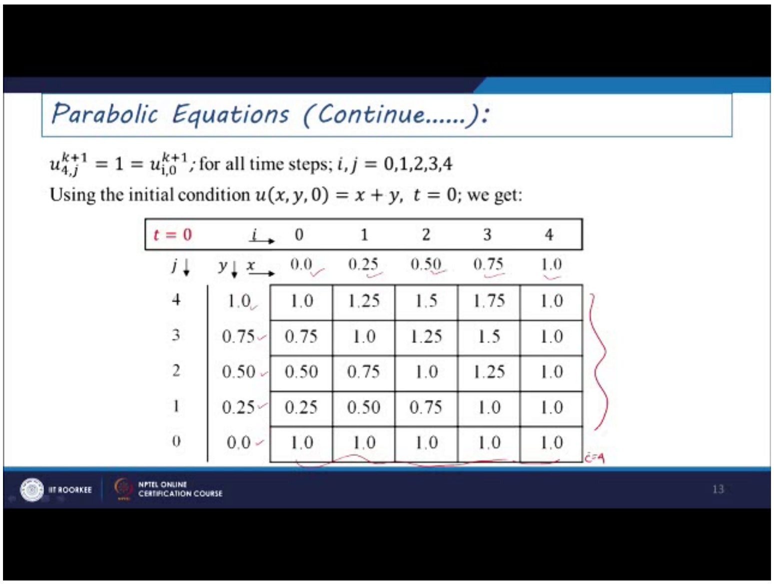

So, if you will just consider this value like a for all these a boundary values that is 4 u 4 j k plus 1 especially if you will just see here, that is nothing, but we are just fixing this last boundary is i equals to 4 here, and j is varying like j equals to 1 2 3 4 especially we are just swearing. You know all of these values we will have this boundary condition as one there, and especially if you will just see here the same condition for like y equals to 0 here.

So, we are also fixing this same boundary condition along this boundary here. So, that is why they say boundary condition values are written in the form of u 4 j k plus 1 this equals to 1 equals to ui 0 k plus 1 for all time steps, and i j are varying from 0 1 2 3 4 there and if you will just use this initial condition like u of x y 0 as x plus y for t equals to 0, we can just get since if you will just see here I is marching as in the form of here as 0 1 2 3 4 and jj also marching here 0 1 2 3 4 and especially the values for y is prescribed as like 1.0, 0.75, 0 0.50, 0.25 and 0 here and the values are given as like if you will just see here like x it is just prescribed has a like 0, 0.25, 0.5, 0.75 and 1 here. And if you will just consider like all of these coordinates here like u of one one coordinate, suppose if I will just consider here.

So, then it can be written as like x 1 plus y 1 here. So, all of these values that is just prescribed that is just written in the summing from over here; this means that if you will just consider like i equals to 0 and j equals to 0, the values are here like 0.0 and this is also 0.0 here. And if you will just compute all these values at time step t equals to 0 here especially and put all these values, then we can just obtain this tabular values in this form here. Since this is nothing, but a sum of x and y we are just considering so, whatever this values x and y it is just given here that has been computed in the terms of u and it is just putting in the table here.

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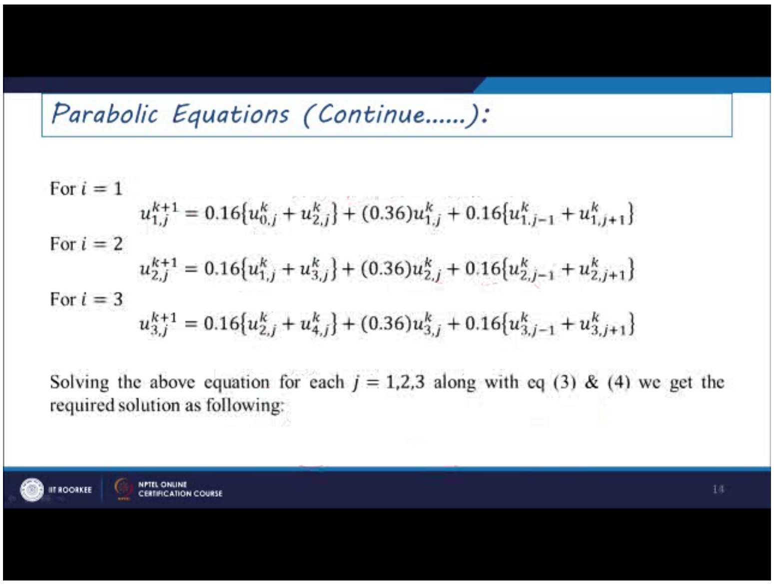

So, once this table has been formulated. So, that just given also this u coordinates or the u values at each of these grid points. So, then we can just use this a grid points values for the further calculation of the values. So, for i equals to 1, if you will just write the scheme can be written in the form of unj k plus 1 this can be written as 0.16 into u 0 j k plus u 2 j k plus 0.36 u 1 j k plus 0.16, u 1 j minus 1 k plus u 1 j plus 1 k here.

So, especially if you will just see here this scheme. So, that is a nothing, but it is just written as like uij to the power k plus 1, it can be written as a 0.16, since you only this coordinator values we are just putting here like i equals to 1 or i equals to 2 or i equals to 3 or i equals to 4 and j is varying like 1 2 3 4 there.

So, if you will just put all these values then for i equals to 1 2 3 here, we can just obtain this a expansion of this a equation as in the form of u 1 j, u 2 j and u 3 j here and if you will just go for the solution of these three equations for j equals to 1 2 3 then we can get the required solutions yes.

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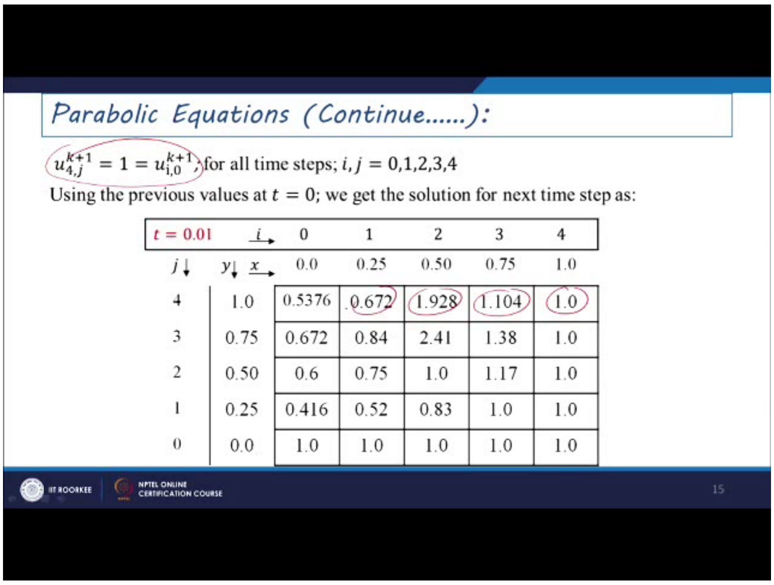

$t = 0.01$	$i \rightarrow$	0	1	2	3	4
$j \downarrow$	$y \downarrow$ $x \rightarrow$	0.0	0.25	0.50	0.75	1.0
4	1.0	0.5376	0.672	1.928	1.104	1.0
3	0.75	0.672	0.84	2.41	1.38	1.0
2	0.50	0.6	0.75	1.0	1.17	1.0
1	0.25	0.416	0.52	0.83	1.0	1.0
0	0.0	1.0	1.0	1.0	1.0	1.0

Like for u 4 j since these are all boundary condition. So, it is already fixed. So, for all time steps if you will just vary here like i j equals to 0 1 2 3 4 here.

So, for t equals to like 0.01 here since we are just moving to the next time step and the k plus oneth level we are just calculating all these values. So, in this step if you will just put all these values then we can just get these values or in this form here. So, 4 different levels if you will just see this represents these values of u at t equals to del t level that is 0.01th level and this values can be used for the further or the next step of calculation of u values for j and i coordinates.

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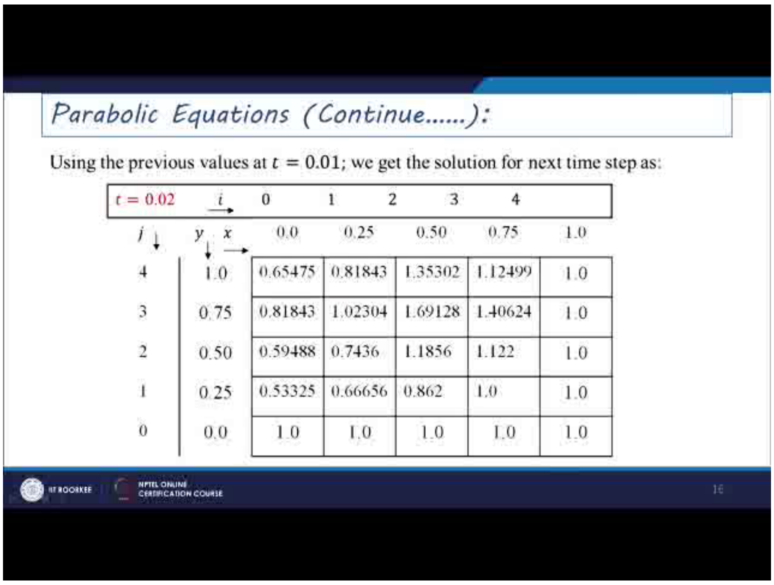

So, if you will just use these previous values at t equals to 0.01, we get the solution for next time steps that is in the form like t equals to 0.02. So, i is varying from 0 1 2 3 4. So, here and j is varying from like 0 1 2 3 4 here so, the values are coming since it is boundary values that are already fixed here. So, only these inner values it has been calculated. So, based on this a previous table values and using this formulation. So, we are not a changing anything there.

Thank you for listen this lecture.