Numerical Methods: Finite Difference Approach
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Lecture – 10
Approximation of derivative boundary conditions

Welcome to the lecture series on numerical methods finite difference approach and in this method, already we have discussed explicit scheme, implicit scheme and Crank Nicholson method, that is a semi implicit approach and their stability and convergence analysis and in the present lecture, we will discuss about this boundary condition treatment; that means, if suppose derivative boundary condition in the form of like either in the form of a Neumann boundary condition or in the Dirichlet boundary condition or in the form of like mixed derivative condition if it is just provided.

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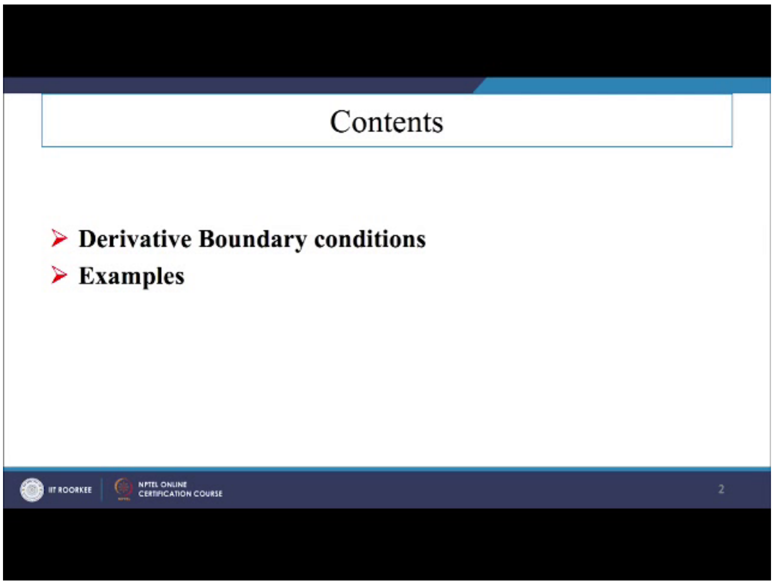

However we can just deal this boundary conditions that we will just discuss.

So, when we will just go for this like Dirichlet condition especially in that sense, this boundary conditions are prescribed along the boundary in the form of variable. So, it is easy to handle their itself. If the boundary conditions are given in the form of like Neumann condition or in the forms of mixed derivative conditions, then it is difficult to handle sometimes.

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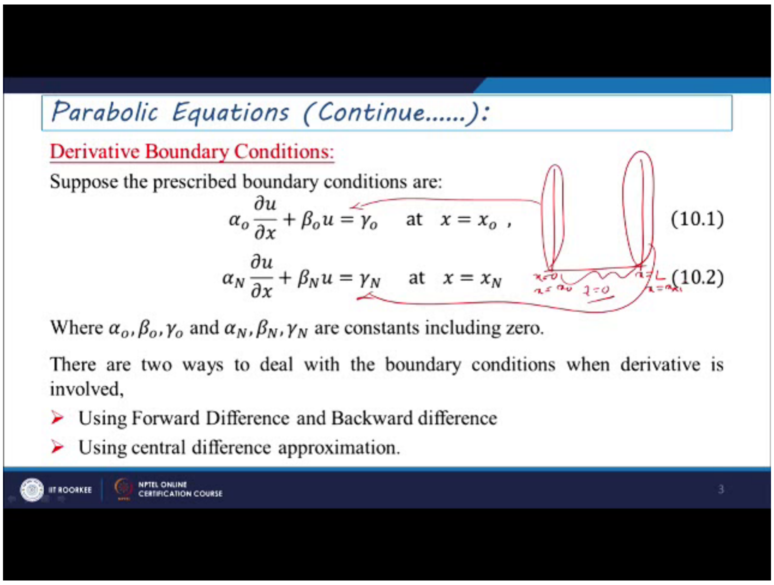

Suppose the prescribed boundary conditions are given in a mixed derivative condition and mixed derivative condition especially we have just discussed that one in the form of like alpha 0 del u by del x plus beta 0 u. this equals to gamma 0 at x equal to x 0, since we will have this boundaries that is in the sense.

So, at x equals to 0 you should have to prescribe a value and x equals to L we should have to prescribe a value since the along this boundary this will be supplied by this initial condition. and if at x equals to, this mixed derivative condition that is just represented in the form of like a mixed derivative term here that is alpha 0 del u by del x plus beta 0 u equals to gamma 0 and along the last boundary it is also a mentioned this boundary condition in the form of like alpha N del u by del x plus beta N u this equals to gamma N at x equals x N.

Since our domain starts from x equals to x naught 2 x equals to x N here and especially if you will just consider these coefficients that is as alpha 0, beta 0 and alpha and beta and all are constants. sometimes it may be prescribed with 0 will is also there and this gamma 0 and gamma N are also constants here. And sometimes also they are also given as 0 values. Since you usually if the normal derivative condition is 0 suppose then you can just write del u by del N equals to 0 along the boundaries.

So, whenever we will have this boundary conditions in a derivative form or in mixed derivative form, then 2 ways we can just find the solution for this internal and discretized equations or we can just use it directly this boundaries discretization scheme there itself to get the solutions.

So, these methods are especially you can just use either this forward difference or backward difference method or the central difference approximations. So, if you just go for this forward and backward difference approximation here, at suppose the beginning of the point like along the first boundary suppose, at x equals 2 x naught suppose.

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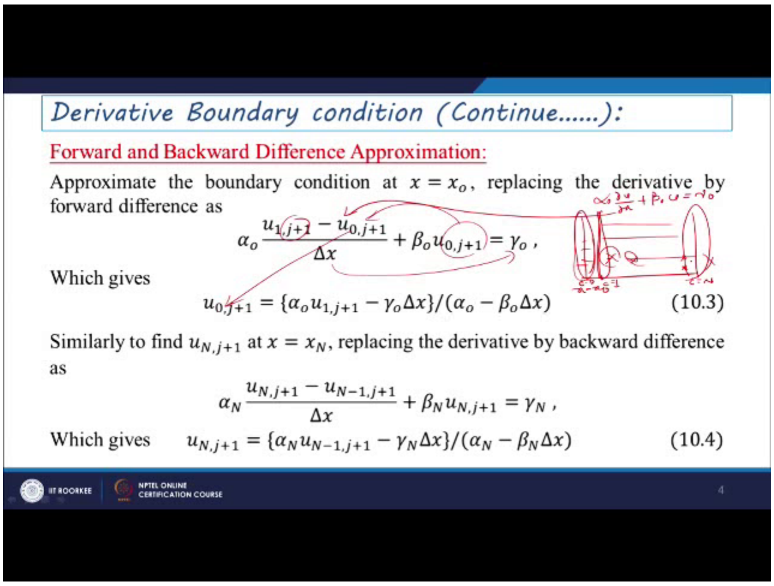

So, if you just replace this derivative that derivative condition especially it is just written as alpha 0 del u by del x plus beta 0 u this equals to gamma 0 beta. So, if we will use this like the forward differencing approximation here. So, we can just write this one as alpha 0 u1 j plus 1 minus u 0 j plus 1 by delta x plus beta 0 u 0 j plus 1 this equals to gamma 0.

Since along this boundary if you just see here. So, i is starting as a 0 here and the last boundary you will have i equals to N there. So, i equals to 0 means we can just starts this boundary like your condition as i equals to 0 and i equals to 1. So, especially at i equal to 1 you have to compute these values and this is also given in the derivative condition here and if you will just move forward to one step there, then we can just find at each of these time step levels. So, that is why we have just written this one as a j plus 1th level. So, at each of this time step level. So, we can just use this formula to get this discretized scheme.

So, that is why if you will just write this one in the form of like alpha 0 u 1 j plus 1 minus u 0 j plus 1 where delta x plus beta 0 u 0 j plus 1 this equals to gamma 0 here. then we can just separate this boundary will you for you as in the form of like alpha 0 u 1 j plus 1 minus gamma 0 delta x divided by alpha 0 minus beta 0 delta x here since u 0 j plus 1 it is just taken common from both this terms here.

And especially you will have a previous iterated value or some initial conditions it has been provided for inside the domain and from that initial values of this computed domain, this boundary can take this value and it can be updated in each of these alteration levels. Similarly, to find a suppose to the last boundary value that is at u n j plus 1 here at each of these time levels. Then we can just define this derivative in a backward sense.

Since we do not have any other domain afterwards there, so, we can just use this previous or backward step to get the solution. So, if you just use your backward step here then we can just write this point as a n minus 1 point there for j plus 1th level. So, this discretized scheme can be written as in the form of like alpha 1 N u N j plus 1 minus u N minus 1 j plus 1 by delta x plus beta N u N j plus 1 is equals to gamma N.

Especially people are using like, at the inlet or along the first boundary this forward marching and at the last boundary are at the outlet especially it is just used as the backward difference approach. and if you will just go for central difference approximation, then we have to consider some fictitious boundary.

Since especially for this we need like 3 cells at the boundary if you will just retreat.

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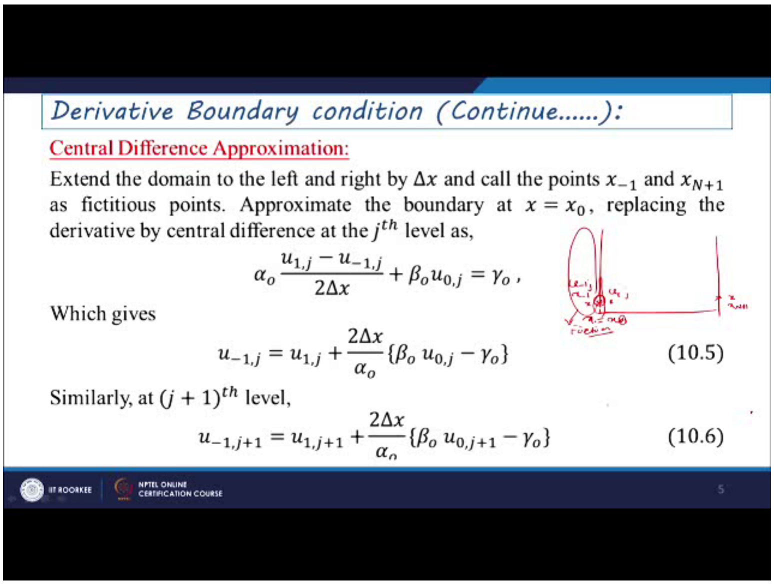

Since if you will just consider like first boundary x equals to x naught here for the central difference keep we need like this inside 0.1 and outside 0.1, but there is no existence of domain. So, that is why this is called your fictitious domain here. So, artificially you can just consider this domain their itself and you can just consider that point as the point.

So, for this you have to extend the domain like in the left or in the right if this condition is asked or if it is asked to use the central difference approximation at the first boundary or at the last boundary. So, you have to extend these boundaries. So, if you will just to extend this boundary since x 0 is the last point here. So, the previous point it can be signified as x of minus 1 and the immediate next point of this last boundary it can be written in the form of x N plus 1 here and at x equals to x 0 if you will just approximate this derivative by the central difference approximation.

We can just write this one as u of 1 j minus u of minus 1 j. this divided by 2 of delta x into alpha 0 plus beta 0 u 0 j this equals to gamma 0. Since 0 is the boundary condition we are just considering here for this initial step which gives since u of minus 1, this is not known to us and fictitiously we have just considered this boundary here. We can just write this one as u of minus 1 j, this equals to u 1 j plus 2 delta x by alpha 0 ,because 0 u, 0 j minus gamma of 0.

And similarly, if you will just consider at j plus 1th level here, then we can just write this one as a u of minus 1 j plus 1 this is equal to u 1 j plus 1 plus 2 delta x by alpha 0 into beta 0, u 0, j plus 1 minus gamma 0 here. So, if you will just go for the central difference approximation here. So, consider suppose u of minus 1 j, which is known by suppose this previous equation if you will just see that is in the form of u of minus 1 j this equals to u 1, j plus ,2 delta x by alpha 0,beta 0, u 0 j minus gamma 0 here.

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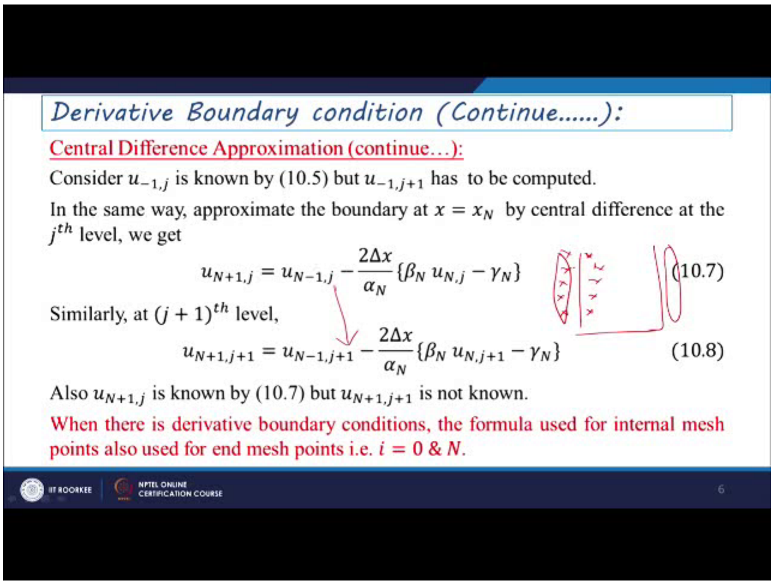

So, then we can just compute u of minus 1 j and in the same way we can just approximate this boundary condition at x equals 2 x N by central difference at jth level that is at u 1 plus 1 j this is nothing but u n minus 1 j minus 2 delta x by alpha and beta and u and j minus gamma N. Similarly, since each of these time steps we can just incremented 1 by 1. So, that is why it can be written in the form of like j plus 1th level as un minus 1 j plus 1 here, minus 2 delta x by alpha and beta n un j plus 1 minus gamma N.

Since, if we are just putting a fictitious boundary condition at the first boundary and along the last boundary here, then we can just find this boundary condition at each time step level to compute in a iterative manner or we can just update these values in an iterative manner if you will just to go from one time step to next time step they are over. also if u N plus 1 j is known by this previous equation, that is like, 10 point 7 here, but u n plus 1 j plus 1 is not known then, we can just move further when there is a derivative boundary condition that formula can be used for internal mesh points also used for end points that is i equals to 0 to n then itself.

So, sometimes if we will just use the central difference approximation for these explicit formulas.

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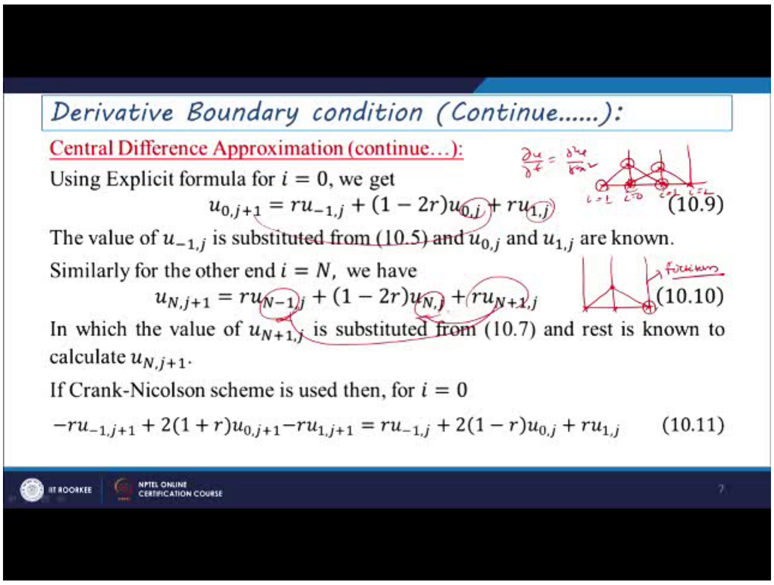

That is like, if i equals to 0 suppose we can just write this formula as, u 0 j plus 1 this is r u 1 minus 1 j plus 1 minus 2 r u 0 j plus r u 1 j there itself. So, this means that if we are just solving this equation that has del u by del t is equals to del square u by del x square here; obviously, we will have this explicit scheme or in the next step calculation if you will just see this molecule is a form in the form of like. So, this is the value which is to be computed and this value you have to consider here, for the computation of this points value and if you will just approximate this value at this point here. So, it should needed 2 more points there itself.

So, that is why if we are just computing this value at u 0 j plus 1 suppose this is the boundary here, then this requires this point, this point and this point. So, that is why it is a just written as r of ui minus 1 minus 1 j plus 1 minus 2 r u 0 j plus r u 1 j . and directly you can just write this as i equal to 0, i equals to minus 1, i equals to 1, i equals 2. And this value of u of minus 1 j is substituted from this previous calculation. That is from 10.5 if you will just see here that as defined as u 1 j plus u 0 j relationship.

So, we can just use this relationship as, in the form of here that is u 0 j and u 1 j is known to us and u of minus 1 j, it can be directly substituted and we can have the coefficients in the form of u 0 j and u 1 j there. So, directly we can just say that u 0 of j plus 1 this can be involved only the terms that is, 0 j and 1 j. and similarly for the other end like at the last boundary if you just go for this computation u n j plus 1, this can be written in the form of r of u n minus 1 j plus 1 minus 2 r, u n j plus r un plus 1 j.

last boundary if you just see here. So, this will just form these molecules like if this is the last boundary here. So, this will just consider these points and it is point for this calculation of explicit scheme there. So, this point it is a not known to us and a this point is considered from this fictitious boundary conditions. and this value can be computed from this our earlier computation central different scheme there itself which can be expressed in the form of like u n minus 1 j u n j and directly we can just establish these relations between these 2 points with this one and which can be replaced by these 2 points and combined form of n minus 1 and n point we can just obtain the value of u n j plus 1 there.

Since the more than like 2 points it is involved for the calculation. So, order of accuracy is more than the first order like forward difference scheme or backward difference scheme for these methods here. and if Crank Nicolson scheme is used to then for i equals to 0 we can just consider this one as a minus r, ui minus 1 j plus 1 plus 2 into 1 plus r u 0 j plus 1 minus r u 1 j plus 1 this equals to r u minus 1 j, plus 2 into 1 minus r u 0 j u 1 j here and this is minus 1and this minus 1.

So, these are these fictitious boundary conditions we have just consider here.

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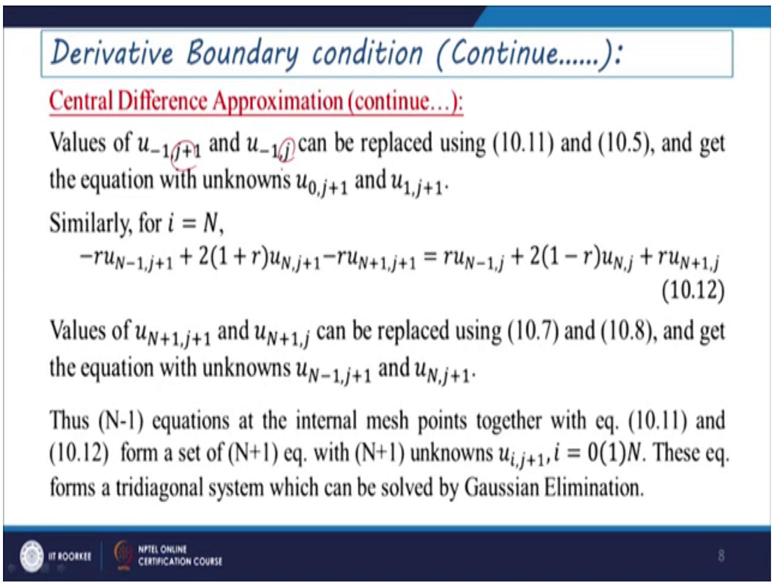

And these values can be replaced by using our earlier formations which we have just put that as u of minus 1 j plus 1 and u of minus 1 j there itself, since j plus 1 j are the time step variation there and these coefficients minus 1 remain fixed there. So, along these boundaries.

So, if you will just use this a equation for these unknowns of ah computations like u 0 j plus 1 a 1 j plus 1. So, we can just obtain the values there. similarly for i equals to n if we can just go for this Crank Nicolson scheme then, we can just obtain this fictitious boundary as in the form of n plus 1 j plus 1 here and this last one this will also give you n plus 1 j. So, these boundary conditions can be replaced in terms of like Nj and n minus 1 j by considering this earlier definition of a boundaries and we can get the equations with unknowns like u n minus 1 j plus 1 and u n j plus 1.

So, hence we will have like n minus 1 equations with the internal mess points and we can just form a set of n plus 1 equation with n plus 1 unknowns ,that is in the form of like uij plus 1 and these equations from a tridiagonal system which can be solved by Gaussian elimination method.

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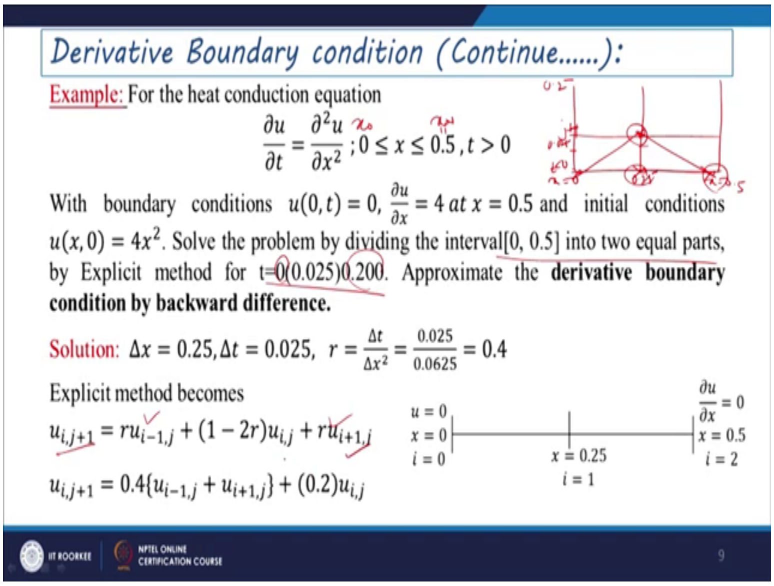

So, for the heat conduction equation, if you will just consider this equation that is nothing but del u by del t equals to del square u by del x square where x lies between x 0 equals to 0 and x N equals to 0.5 for t greater than 0 and the initial condition is a given as u of x 0 is equals to 4 x square. So, we can just formulate this domain as in the form of like x equals to 0 here x equals to 0.5 and we can just discretize this domain that is in the order like, if in the see here x equals to 0.25, 2 equal parts since the question is asked here. There is 0 to 0.5 into 2 equal parts. So, we can just write this one as 0.25. So, final point is 0.5.

So, the problem statement is that ,for the heat conduction equation del u by del t equals 2 del square u by del x square x lies between 0 to 0.5 t greater than 0 with boundary conditions u of 0 t equals to 0 del u by del x equals to 4 at x equals to 0.5. and initial condition u of x 0 equals to 4 x square. And we have to solve this problem by dividing these interval 0 to 0.5 by 2 equal parts, by explicit method, for t equals to 00.025 and 0. 200.

So, approximate this derivative condition by backward difference, it is given us. So, we can just write del x equals to 0.25, del t equals to 0.025, if you will just see here time increment, it is just given like first time, it will just given as a 0 then 0.025,then it will added once more that is a 0 point like 05. So, up to 0.2 we have to move. So, this is nothing but t starting value is 0 here, ending value is 0.200 incremented by 0.025 and for this scheme for del t equals to 0.025. We can just obtain the r value as a delta by del x square, that is nothing but 0.025 by 0.0625 this is 0.4.

So, if you will just use explicit method, then we can just write this formulation is in the form of uij plus 1, since the molecule can be found in the form of like, this is the unknown here. So, these 2, 3 values it can just use to get this next approximate value. So, uij plus 1 it can be written as like ru minus 1 j plus 1 minus 2 r uij plus r u by i plus 1 j here.

So, next time step if we will just see here j plus 1th level i have just written here. So, this is just taking i minus 1 say, then u i j then u i plus 1 j here and if you will just use this r equals to del t by del x square here. So, these coefficients can be written in the form of 0 point 4 since r is written r involved with these 2 terms here. So, ui minus 1 j plus ui plus 1 j plus point 2 into uij here.

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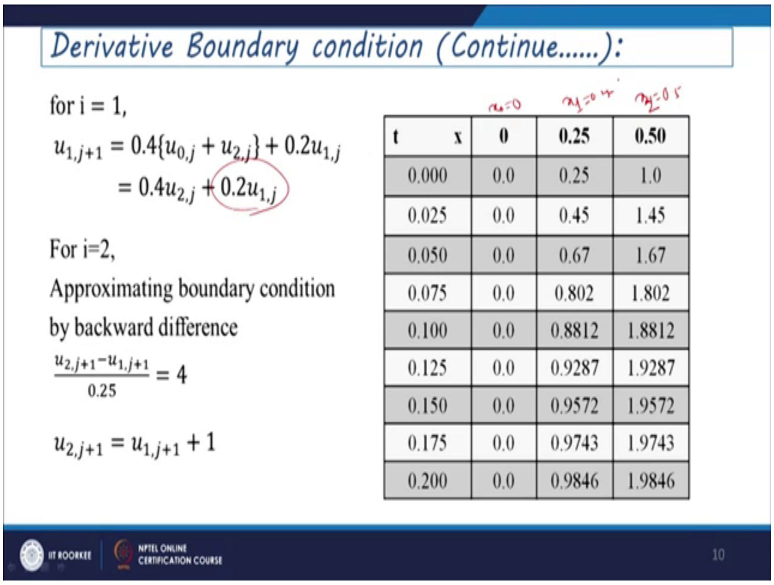

So, for i equals to one u 1 j plus 1, this can be written as 0.4 u 2 j plus 0.2 u 1 j here and for i equals to 2 approximately boundary condition by backward difference if you will just approximate then, we can just write u 2 j plus 1 minus u 1 j plus 1 divided by 0.25 this equals to 4 here. So, directly if you just put this value, we can just obtain this 1 u 2 j plus 1 as u 1j plus 1 plus 1 here and if you will just to go for this computation at all other steps. So, we can just obtain the values are in this form here that is t equals to 0 then 0.25, 0.5, 0.075, 0.1, then 0.125, 0.150, 0.175.

So, you will have like 3 steps, we are just completing here, that is at x equals to x 0, that is nothing but 0 and x N equals 0.5, we are just considering. So, that is why we can just consider this one as x 2 and x1 value. it is just taking as 0.25 here. So, that is why, this is very easy to compute an explicit approach, since 3 different points it is already asking you to find it out.

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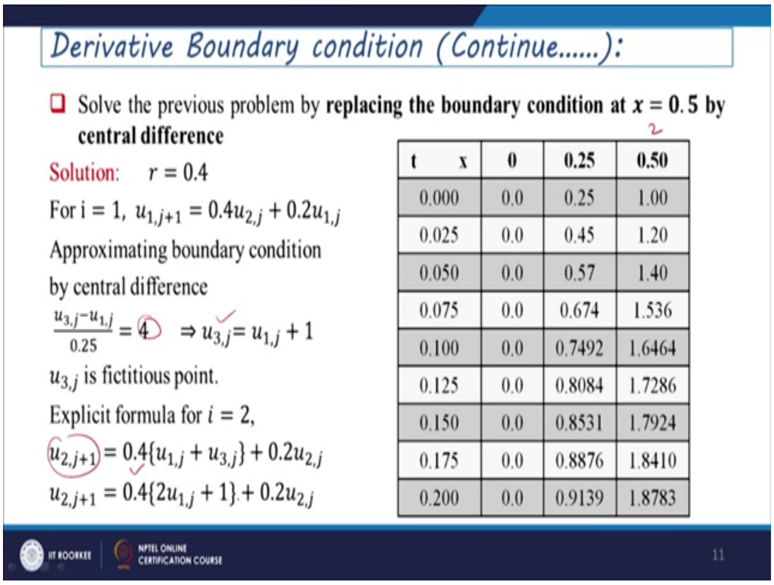

So, if you will just to go for this a problem, by replacing the boundary condition at x equals to 0.5 by a central difference scheme here, then we can just write for i equals to 1, the scheme as what u of 1j plus 1 this equals to 0.4 u 2 j plus 0.2 u 1 j here and by central difference approximation, we have to consider a fictitious point there. So, especially the fictitious point, since your last boundary point is a 2 here. So, we have just considered u 3 j minus u 1 j this by 0.25 is 4 here and u 3 j this can be written as u 1 j plus 1 there.

So, if you will just use explicitly this formula for i equals to 2, here we can just obtain this one as u 2 j plus 1.this is nothing but 0.4 u 1j plus u 3 j plus 0.2 into u 2 j and u 2 j plus 1 this can be written as 0.42 u 1 j plus 1 plus 0.2 into u 2,0. So, if all of these points, if you will just use this values, then simultaneously, we can just obtain this values, that is just figured in this tabular form.

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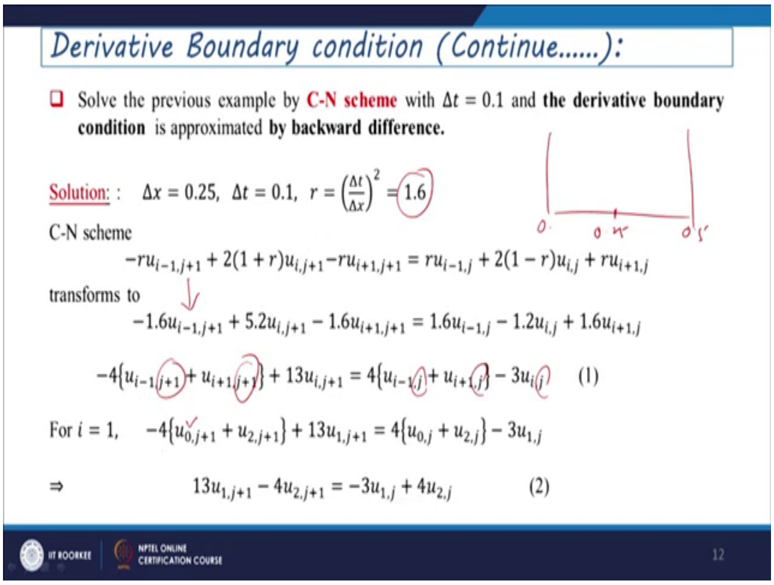

And if you will just use these schemes for Crank Nicolson scheme, with del t equals to 0.1 the derivative boundary condition is approximated by a backward difference scheme here. So, del x can be written as 0.25 since and the domain is defined in the form of like 0 then 0.5 then 0.25 here.

So, we can just write this formulation in terms of Crank Nicolson scheme as a minus r ui minus 1 j plus 1 plus 2 into 1 plus r ui j plus 1 minus r ui plus 1 j plus 1 this equals to r u i minus 1 j plus 2 into 1 minus r uij plus r u i plus 1 j. and if you will just put the coefficient r here that is in the form of 1.6 here, then we can just write this formulation in this form here and if you will just separate this unknown terms in the left hand side and the known terms in the right hand side, we can just write this equation as a minus 4 ui minus 1 j plus 1 plus ui plus 1 j plus 1 plus 13 uij plus 1 this equals to 4 into ui minus 1 j plus ui plus 1 j minus 3 u i j.

If you will just see here j plus 1 is the unknownth level and jth is the know level here. for i equals to 1 if you will just put here. So, u 0 j plus 1 this is nothing but the boundary values and u 2 j plus 1 and then 13 u 1j plus 1 there is equals to 4 u 0 j plus u 2 j minus 3 1j here and this implies that, we can just replace this one as in the form of like 13 u 1 j plus 1 minus 4 u 2 j plus 1 this equals to minus 3 1 j plus 4 u 2 j.

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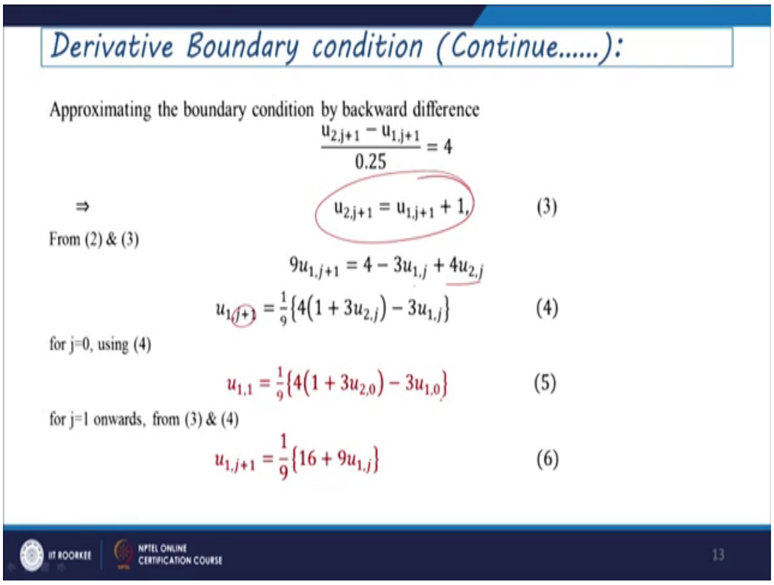

So, finally, if you will just approximate this boundary condition by a backward difference approach, we can just write u 2 j plus 1 minus u 1 j plus 1 by 0.25 this equals to 4 here. So, u 2 j plus 1 it can be written as a u 1 j plus 1 plus 1 here. So, from equation 2 and 3 we can just write 9 u 1 j plus 1 this equals to 4 minus 3 1 j plus 4 u 2 j and u 1 j plus 1 this can be written as 1 by 9 into 4 into 1 plus 3 u 2 j minus 3 1 j and for j equals to 0 here suppose, if you are just putting then u 1 it can be written as like 1 by 9, 4 into 1 plus 3 u 2,0 and minus 3 u 1,0 here. So, we can just write in the final form as 1 by 9, 16 plus 9 u 1 j there itself.

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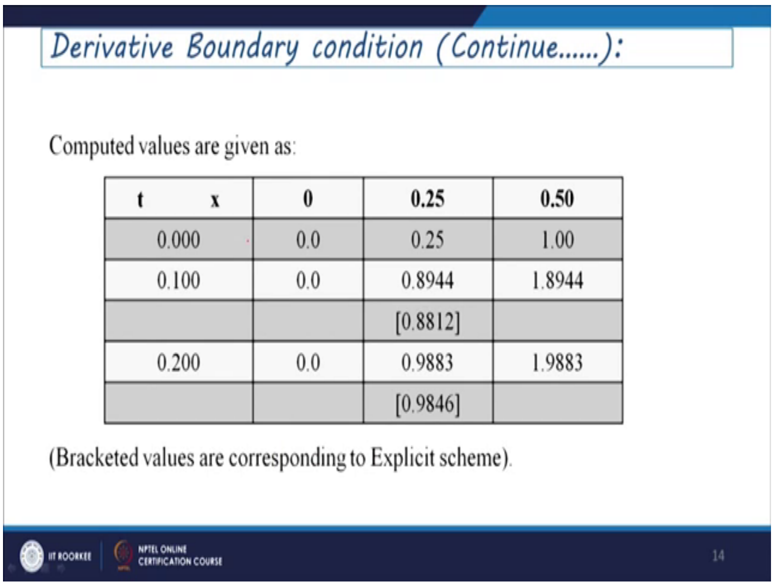

So, if you will just go for this a further competition. So, especially we are just are getting this derivative boundary condition just produces us like a 0.000 here. So, 0.0 so these are the initial conditions it is just provided to us, since at t equals to 0 we are just constrain these values. So, at 10.100 we are just finding these values at 0.0, 0.8944 and 1.894. But in the explicit scheme especially we are just getting this one in the form of 0.8812 here.

So, it is a too less compared to or it is a less compared to the Crank Nicolson scheme. And if you are just putting here as x equal t equals to 0.200 then we are just obtaining this boundary values as 0 since it is just a given in the problem that is a given initially. But along the last boundary if you will just see. So, this has just taken the value 1.9883 here. And if it is updated with this initial domain value it is just producing 0.9883 so, but the explicit scheme we are to obtain in this value at 0.9846.

So, if you just take these differences. So, then we can just find that this difference is a very minute there compared to the earlier one. So, this means that we when we are just moving towards these higher time steps or towards this like a more iterations then we are just obtaining this closer values of the accurate solution; this means that either you are just using explicit scheme or implicit scheme, but if you are just getting these more iterations or if you are just moving to higher time steps, then you are just approaching close towards the true solution.

Thank you for listen this lecture.