Numerical Methods: Finite Difference Approach
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Lecture - 01
Introduction to numerical solutions

Welcome to the lecture series on Numerical Methods a Finite Difference Approach. In this lecture we will discuss various finite difference approach methods based on like its convergence and stability analysis. Afterwards several practical application oriented examples we will consider and solve that problems. At the beginning first we will just consider like a different class of ordinary differential equations and the basic methods used in finite difference methods that we will just consider. And whenever we will just start these course like numerical methods in finite difference approach, so why we are using this finite difference method that first we will discuss here. Why and how, it has been implemented in a system of equations or in the ordinary differential equation or in the set of partial differential equations that we will just discuss here over.

So, if we will just go for this in numerical methods here then first introduction section we will just consider that why we are just using these numerical methods or where it can be applicable or where we are just finding the difficulties that we can just use these numerical methods instead of analytical methods.

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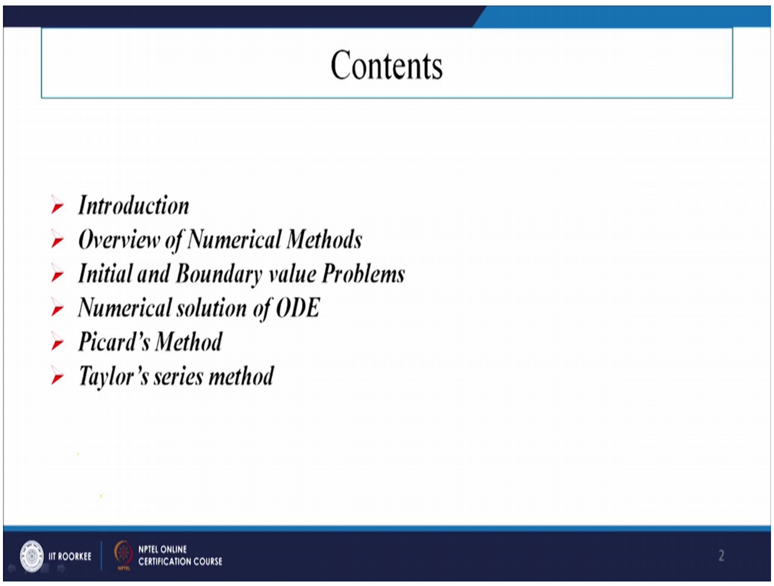

So, then the overview of numerical methods various class of numerical methods we will just consider and then based on that method. So, we will just go for like a different class of examples and by choosing different space size or like increment of the space length we can just solve these problems. Then this is a numerical solution of ordinary differential equations based on like Picard's method and Taylor series method we will just consider.

So, first thing I will just go for it like numerical method.

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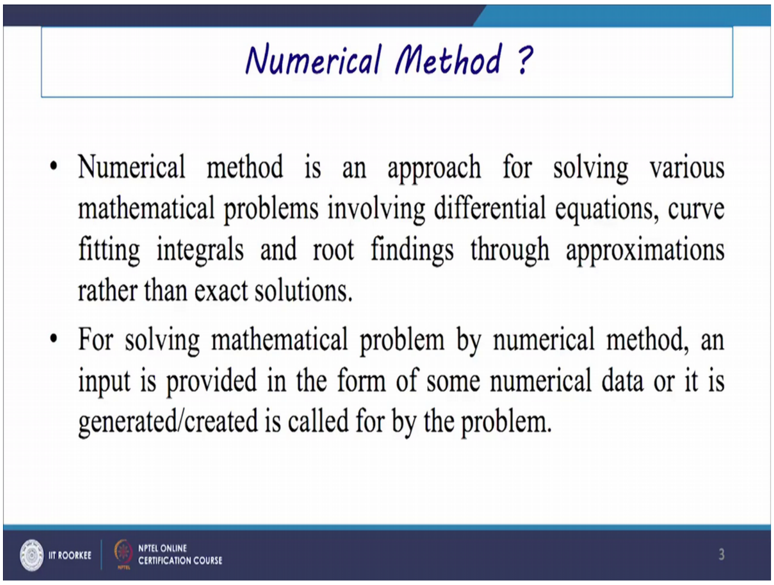

Numerical method is an approach for solving various mathematical problems involving differential equations, curve fitting integrals, root findings through approximations rather than exact solutions.

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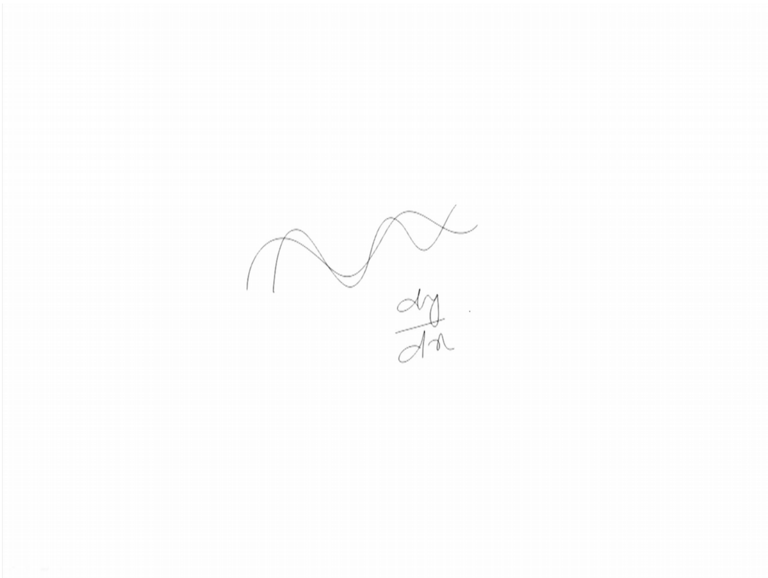

Especially if you will just see here like different class of problems if you will just consider suppose if we are just considering like a curve suppose and how we can just fit this curve with a best approximated polynomial suppose or if you will just consider a differential equation suppose dy by dx. And if suppose it involves some of this complicated like boundary conditions, may be complications means we can just say that its domain is a very irregular shape or we can just consider like if this boundary conditions maybe involves some non-linear boundary conditions here.

So, if you will just consider such types of problems then how we can just a treat that boundary conditions or how we can just solve these differential equations in certain domains that can be a possible through this numerical method sense. Especially if you will have like non-linear equations to find like a roots especially if you will just go for like basic numerical methods. So, several methods are used for the solution technique like we are just a saying that is as the transcendental equations especially we are just using in that case like iterated method or like Newton Raphson method or Bisection method or Regula Falsi method. So, different class of methods as special people are using to get the solution if this is system of 4 equations or if the equation is a highly non-linear in sense.

So, especially to deal such type of a mathematical problem by numerical method an input is provided in the form of some numerical data or it is generated, created or is called for by the problem. Especially if we are just going for this numerical solution process, so we have to supply some data and that data will be carried out inside the system through maybe by use of a hand computation or by use of computers we can just use this data inside the system through various arithmetic operations satisfying the some of the logical operations and finally, we can just obtain certain class of like a outputs there as the results. And if these results are just satisfied with the physical phenomena of or the physical behavior of this equation or the problem then we are just saying that this is the solution of the problem.

So, to deals such type of problems especially we are just using numerical methods. So, then the question arises that why numerical method.

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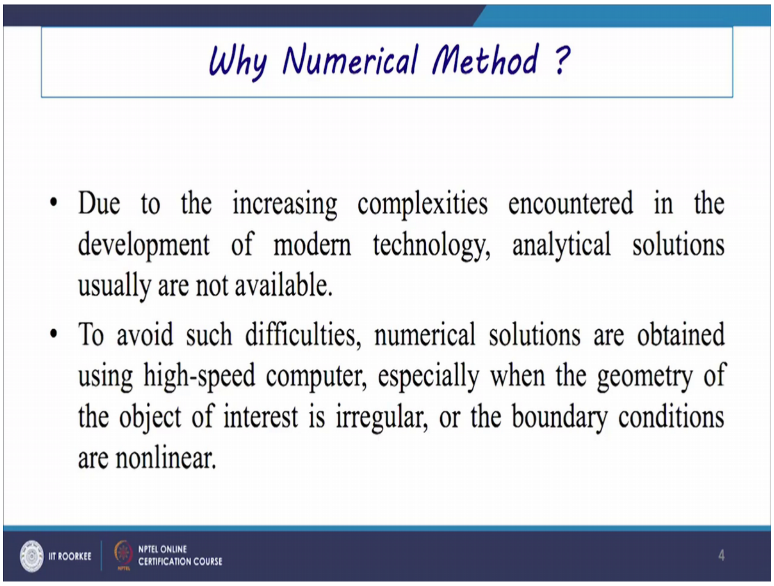

Due to this increasing complexities encountered in the development of a modern technology analytical solutions usually are not possible.

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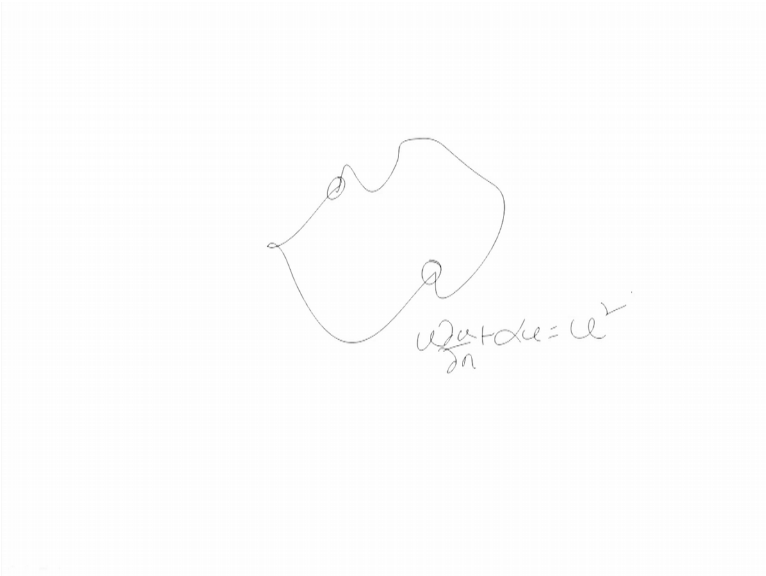

Suppose especially if you will just consider a domain which is like, so this class of a domain. Especially this nonlinearity in nature if you will just see this corners especially it is just existing or if we are just considering this boundary condition that is in the form of like u del u by del n plus like alpha u this equals to suppose u square here.

So, such type of equations if you will just consider here it is very difficult to find the solution in such cases. So, to deal such type of situations especially we are just using some numerical approximation to first to linearize this non-linear equations, then using some of this finite difference method or like finite volume method or finite element method we can just obtain the solutions. Especially I am just considering here this is a finite difference approach since it is easy to understand and it is like easy to handle and it is easy to understandable to the audiences.

So, if you will just go for like a different class of problems. So, then we can just visualize that how we can just move in a different form here. So, why we are just using this numerical method is that sometimes if we can just find this numerical solutions are obtained using high speed computer especially when the geometry of the object is of like irregular shapes or boundary conditions are highly non-linear in nature, that I have just explained and we will just go to the next step that how this ordinary differential equations represented here.

So, especially if you will just see this ordinary differential equation. So, this ordinary differential equation can be expressed either in the form of like x y dy by dx d square y by dx square if it is like nth order differential equation. So, we can just write that as a function of x y dy by dx d square y by dx square d cube y by dxs cube of 2 d to the power n, y by dx to the power n this equals to 0.

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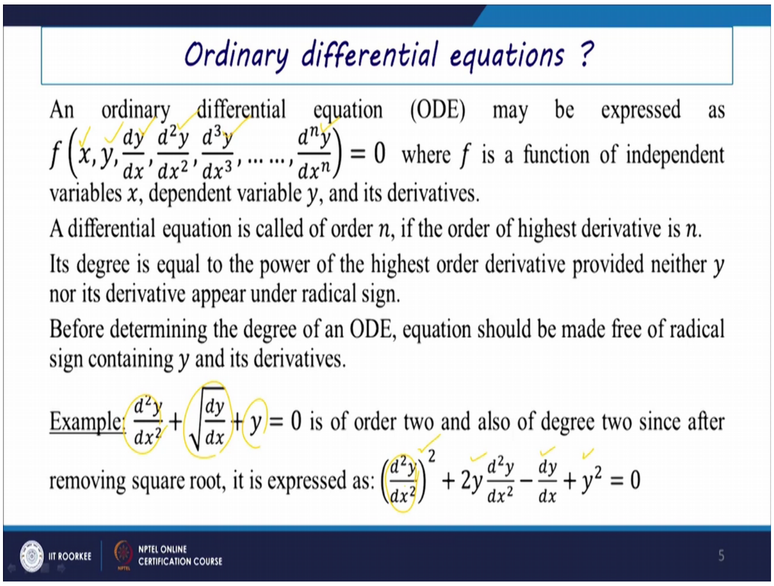

Especially whatever this equation if it is just represented in either in a multiplied form or in addition form or in a like division form, especially we can just equate to 0 if you will just take all the right hand side terms to the left hand side. And if we can just express these differential equations with this dependent variable and independent variable especially if we are just writing this function as in the form of like if you will just see here. So, that is like suppose x is there, y is there, dy by dx is there, d square y by dx square is there, d cube y by dx is dx's cube is there, d to the power n y by dx to the power n is also there.

So, here we can just say that x is the like independent variable and y is the dependent variable here and especially this function f is the function of we can just say it is a function which is a dependent on the independent variable x and dependent variable y and its derivatives especially we are just writing x y dy by dx. Sometimes we can just find that this differential equation can be expressed in the product form also like we can just say that y dy by dx or we can just say some times x dy by dx. So, that is why this f is called the function of independent variables x and dependent variable y with its derivative terms here.

Then when we can just say that this is of order n or n minus 1 or it is of order 1 especially if the order of the highest derivative is n here then we can say that this differential equation is of order n here. And its degree is equal to the power of the highest order derivative provided neither y nor its derivatives appear under radical sign here. Sometimes we can just find that is the differential equations especially it is written in the square sense or it is in a like 1 by 3 power sense or 1 by 2 power sense or some fractional powers or some of this like algebraic operations it can be involved in this ordinary differential equations

So, first to determine this degree of any ordinary differential equation we should have to express in a free radical form or we can just say that if the equations should be free of radical sign containing y and its derivatives. So, if we are just making free from the radicals especially the highest order terms degree will be the degree of this ordinary differential equation. So, that is why it is just written it is its degree is equal to the power of a highest order derivative provided neither y nor its derivative f here under radical sign here. Especially if you will just consider a differential equation that is in the form of like d square y by dx square here plus square root of dy by dx plus y this equals to 0 suppose, first we have to make this differential equation to radical free.

So, radical free means say especially we will just take the square of this terms or we can just take square of this whole equation first then we can just express it has a d square y by dx square whole square plus 2 y d square y by dx square minus dy by dx plus y square this equals to 0 here. So, once it is radical free we can just see that highest order term is here that is d square y by dx square and it is a degree we can just consider that as a 2 is the degree of this differential equation here. So, once this order and degree it is known to us, then we can just go for this like initial value and boundary value where we can just implement these values to get the solution of this equation.

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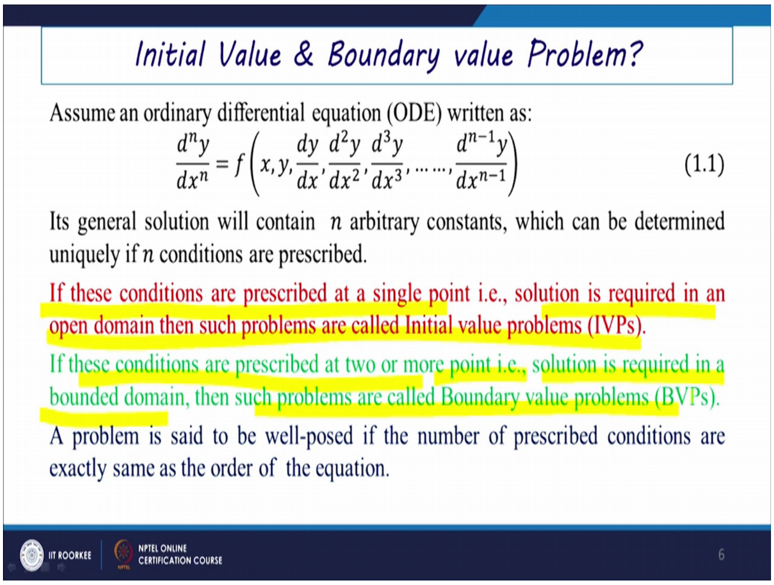

So, if we want to find the solution of this differential equations especially, if we will just integrate these differential equations then we can just find some constants there; to find the values of these constants especially we have to put either this initial condition or the boundary condition to get it free from these constants or independent solutions there.

So, especially if you will just consider this ordinary differential equation that is in the form of like nth order differential equation as a to the power n y by dx to the power n as like f of x y dy by dx d square y by dx square d cube y by dx's cube of 2 suppose d n minus one y by dx to the power n minus 1.

So, its general solution will contain n arbitrary constants which can be determined uniquely if n conditions will be provided. So, especially to find n conditions especially if you will just consider a initial value problem at a point if all these values has been given like why is should I have to be provided why does is to be provided and likely its nth order derivative term should be provided at a single point we can just say that this is the initial value problem or in other sense, we can just say that initially if the value is provided to obtain the solution in open form then we can just say that it is a initial value problem. And if the conditions are provided like a more than one point that is called the boundary conditions here especially two or more points if you will just consider and the solution procedure whatever the solutions we will just get it should be satisfied with that points. So, that is why it is called especially the boundary conditions.

So, I have just highlighted here if you will just see that one. So, especially if I am just considering this one like this section here. Like if these conditions are prescribed at a single point that is a solution is a required in an open domain then such problems are called initial value problems. And especially if these conditions are prescribed at two or more point here the solution is required in bounded domain especially then such problems are called boundary value problems.

A problem is said to be well posed if the number of prescribed conditions are exactly same as the order of the equation here this means that if whenever we are just to finding the solution of a nth order differential equation here especially we can just find n arbitrary constants or we can just say that it will involve whatever these methods you can just apply it over there. Like if you will just use like a simple linear differential equations you can just a find that like second order differential equation we can just a find a like a particular integral and a complimentary function, this will a involve like second order differential equation means two constants it will be involved. To find like a two constants there over we need like a two conditions there over to find the solutions especially that is called well posed problem, that is the number of conditions are exactly the same as the order of the equation. So, easily we can just find a independent solution there over.

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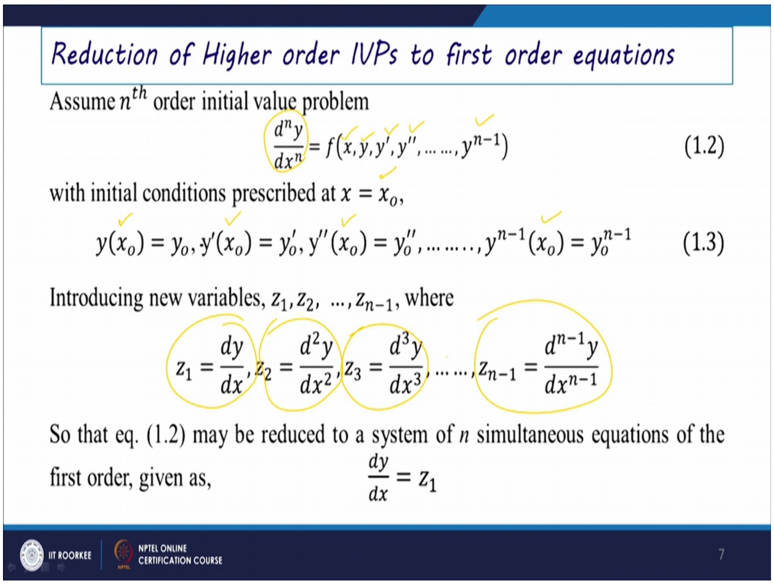

So, then if suppose a higher order differential equation is provided to us then to often the solution we can sometimes reduce into a degree 1 equation also. So, like if this problem is defined in the form if you will just see. So, that is as like a nth order term as a d to the power n y by d x to the power n here and all this dependent and independent variables if it is written as like x is the independent variable here, y is the dependent variable remaining derivative terms y dash y double dash and y to the power n minus 1 here. And the initial condition are prescribed suppose x equals to x naught then at that point only first point y of x 0 is defined and y dash of x 0 is defined, y double dash of x 0 is defined, and y to the power n minus 1 x 0 is defined here.

So, if we will just introduce like a variable suppose z 1 here z 1 in the form of like dy by dx here. And if you will just express z 2 as in the form of d square y by dx square here z 3 in the form of d cube y by dx square and then n minus 1 as d to the power n minus y by dx to the power n minus 1 here. So, that we can just write z 1 equals to dy by dx and especially if you will just take the first differentiation of z 1 here, so dz 1 by dx especially this can be written as z 2 here.

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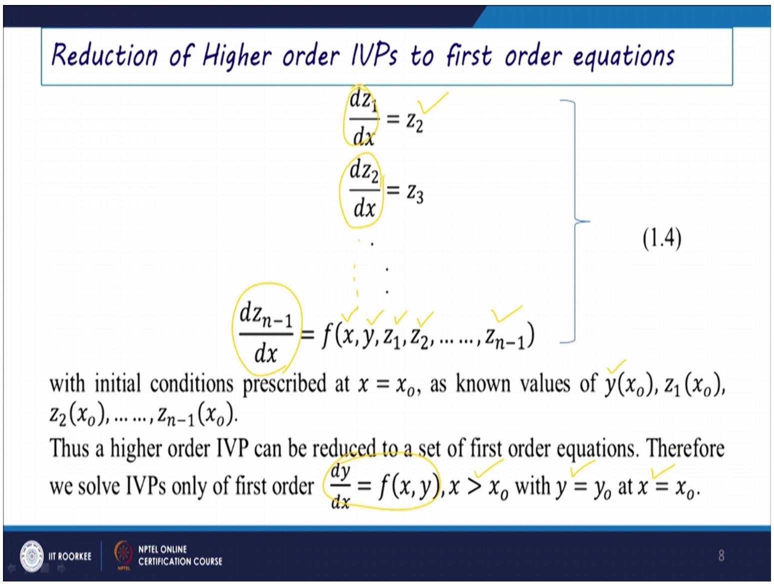

And similarly z 3 can be written as simple first order differentiation of z 2 there itself and likewise if you will just proceed here we can just often d to the d of z n minus 1 by dx this as f of x y z 1 z 2 up to z n minus 1.

Simply we can just say that within the prescribed initial condition at x equals 2 x naught we can just evaluate all the values at the single point there itself only, since x 0 is the initial value which has been provided for all the derivative terms involving the first dependent variable that is all y there itself. Thus we can just say that a higher order initial value problem can be reduced to a set of first order equations here. Therefore, we can just solve this initial value problems only of a first order that is in the form of like dy by dx this is f of x y here where x greater than x 0, with y equals to y 0, at x equals to x 0 here.

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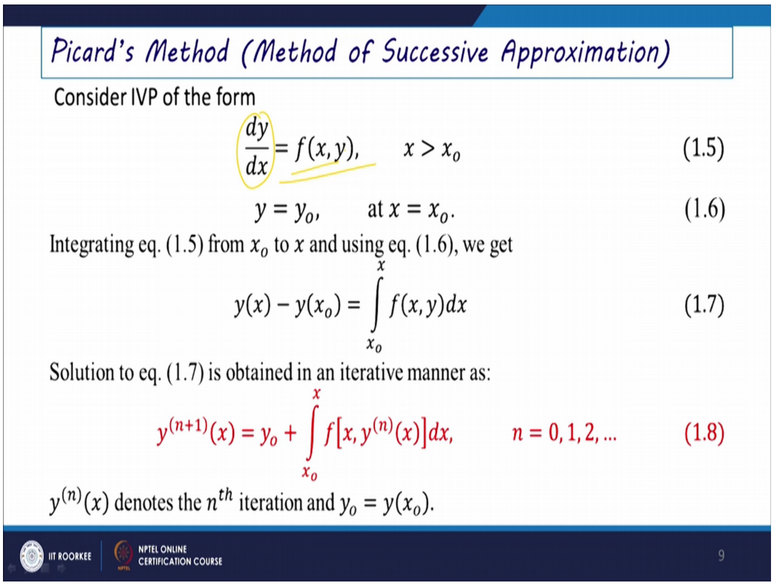

So, if you will just go further like how we can just use any numerical method for such class of like initial value problems. So, first method is we have considered here that is as Picard's method here that is especially called method of successive approximations.

So, in that case if you will just consider a initial value problem that is in the form of like a dy by dx that is f of x y here and x greater than 0 suppose and y equals to y naught at x equals to x 0 that is basically the initial condition it has been provided here. So, especially if you will just consider this equation and if you will just take the integration of this term here that is like dy by dx that is f of x y, where x is greater than x naught and y of x naught is given that is y naught here especially we can just integrate this equation that is in the form of a dy by dx here. This is f of x y here especially we can just write as dy this is f of x y dx here.

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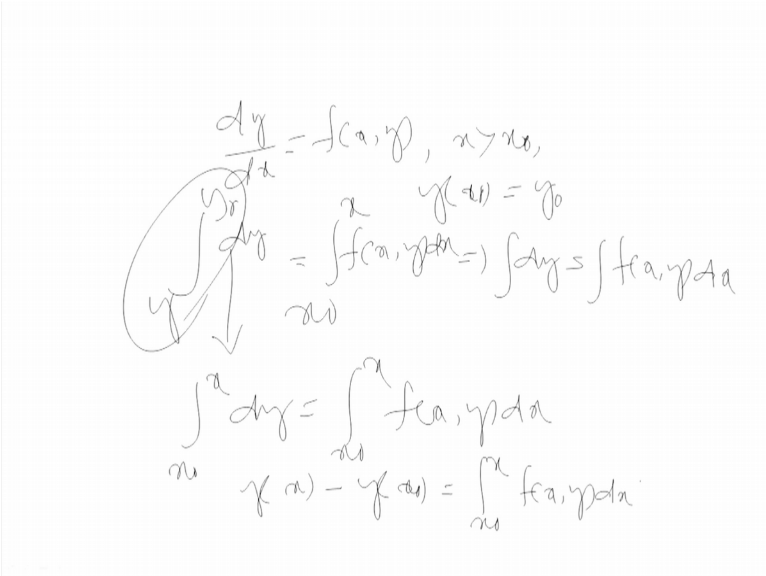

And sometimes we can just write this one as like this is nothing, but a dy by dx is like dx here, and your range that is just a defined as in the form of like y 2 y naught here and this is just defined x 0 to x here and if you will just define this integration in a sense; that means, that we are just integrating this range, y 2 y 0 whenever we are just finding this integration from x 0 to x here only. So, that is why we can just write this integration range as in the form of integration x 0 to x here, dy is x 0 to x f of x y dx here which can be written in the form of like y of x minus y of x naught this is integration x 0 to x f of x y dx here.

And if we will just define a particular relationship establishing in a successive manner then we can just interpret this side value is the n plus 1th range here and this side value in the nth range here. And especially if you will just write in a modified sense this one in a complete form especially we can just write this as y to the power n plus 1 x, this can be written as y naught plus x 0 to x here f of x y to the power n x dx here, where n is varying from 0 1 2 up to like way. And especially we can just say that y to the power n denotes the nth iterated value here and n plus 1 nth denote the n plus 1th iterated value here.

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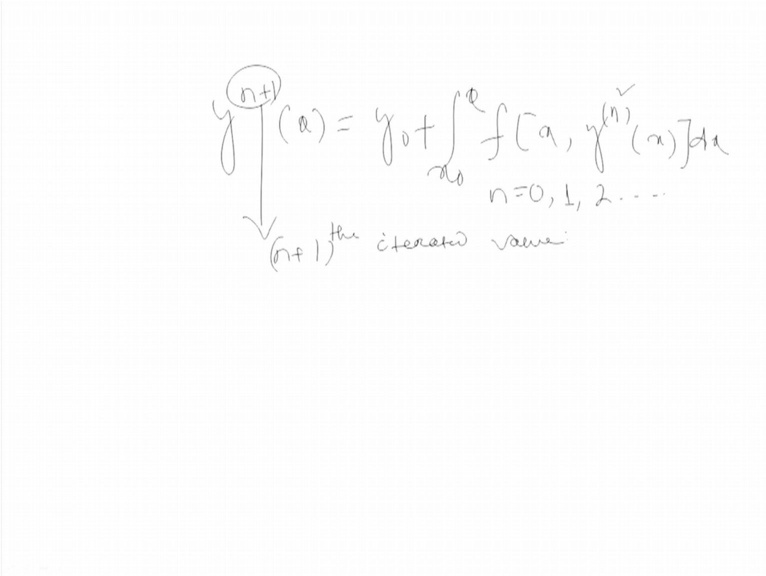

So, especially if we want to move in this method we can just say that first we will just guess this value for y to the power n here which can be used to for the further calculation of y values afterwards there over. So, if we will just go for a particular example based on like a differential equation we can just write this one as in the form of like, find the approximate solution by Picard's method for dy by dx equals to suppose x square minus y here where the initial condition is provided as a y 0 equals to 1 here.

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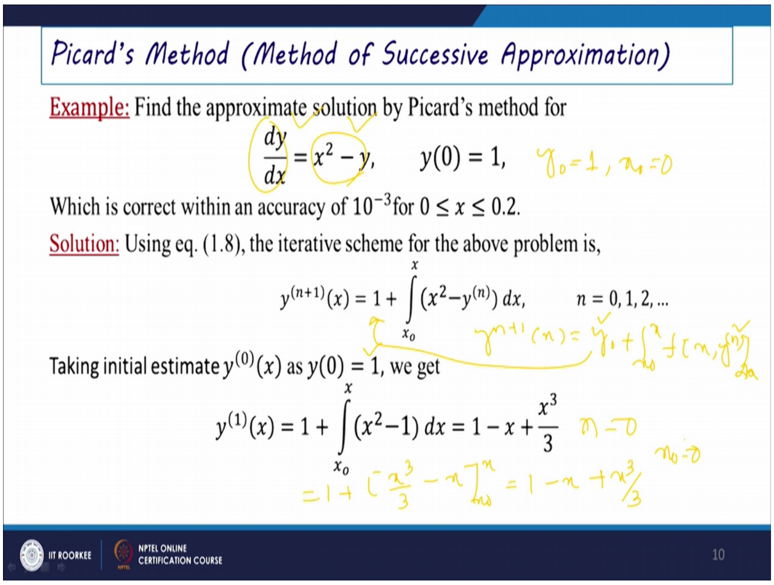

Especially I am just signifying this one this differential equation as a dy by dx that is the first order differential equation we can just say and this f of x y part it is just carrying like a x square minus y part here and the initial condition that is the given as y naught, we can just write here y naught equals to 1 when x naught equals to 0 here.

So, we have to compute this one, this approximate solution of to like a accuracy of 10 to the power minus 3 here. So, for this calculation first we have to consider like x naught equals to 0 here and x range is a given just varying from 0 to 0.2 here. And this formulation especially I have just a discussed in the previous slide that we can just write this and the n plus 1th iterated step as y to the power n plus 1, this can be written as like y n plus 1 of x is y naught plus x naught to x f of like x y power n dx here.

So, that is why if you will just see this problem here then it can be written in the form of like f of x y means that is nothing, but a x square minus y power n that is nothing, but nth iterated value there over and dx is there and y 0 is a specified as a 1 here so that is why this value is just replaced by 1 there itself. And if we can just a take this initial estimate y 0 equals to 1 and integrate this a terms to here over then we can just write the first iterated value whenever n equals to 0 suppose.

Then we can just write y 1 of x this equals to 1 plus x 0 to x this is x square minus y 0 here y 0 means this is nothing, but 1 for this one and into dx this can be written if you will just take this integration here. So, it can be written as like x’s cube by 3 plus 1 this is like minus x here. So, if you will take this ranges, so that we can just write this one as 1 plus x to the power 3 by 3 minus x this is x 0 to x here which can be written as 1 minus x plus x’s cube by 3. And all other the conditions like x 0 equals to 0 so that is why this x involving terms which takes only the 0 values since x 0 equals to 0 here.

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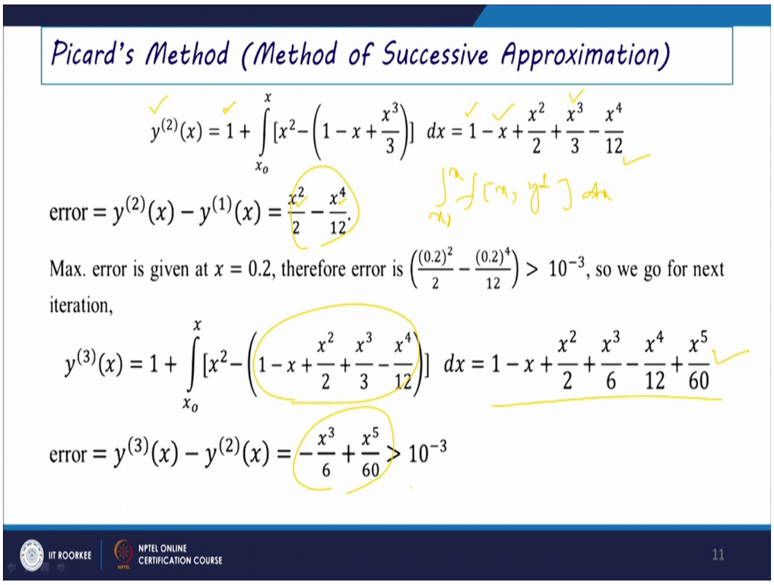

Next we will just go for like a Picard's method of a successive approximation for like second iterate term. So, if you will just consider the second iterated term here. So, we can just write this is as a y 2 of x and this is nothing but we can just write to y naught term and this is x 0 to x and the next approximation we can just say that is like a x 0 to x and we can just write f of x y power of 1 here into dx here. So, y power of 1 we can just say that this is nothing but like we have just obtained 1 minus x plus x's cube by 3 here.

So, that is why in k in a place of like a y to the power 1 here I am just putting 1 minus x plus x's cube by 3 here, and if integrating this whole term here that can be represented as like 1 minus x plus x square by 2 plus x's cube by 3 minus x to the power 4 by 12 here. And if we will just go for this a error term here, this means that what about this improvement of the solution is just obtaining after next iterated step if you will just take the difference of our two consecutive steps we can just find the difference that is nothing, but the error in that step only.

So, if you will just consider like y 2 of x minus y 1 of x here. So, it can be written as like if you will just see here last slide the terms are like 1 minus x plus x's cube by 3 here and if you will just a subtract y 2 of x minus y 1 of x we can just write this one as like 1 minus x plus x square by 2 plus x's cube by 3 minus x to the power 4 by 12 minus 1 minus x plus x's cube by 3 they are over. So, this is nothing but x square by 2 minus x square x to the power 4 by 12 here over.

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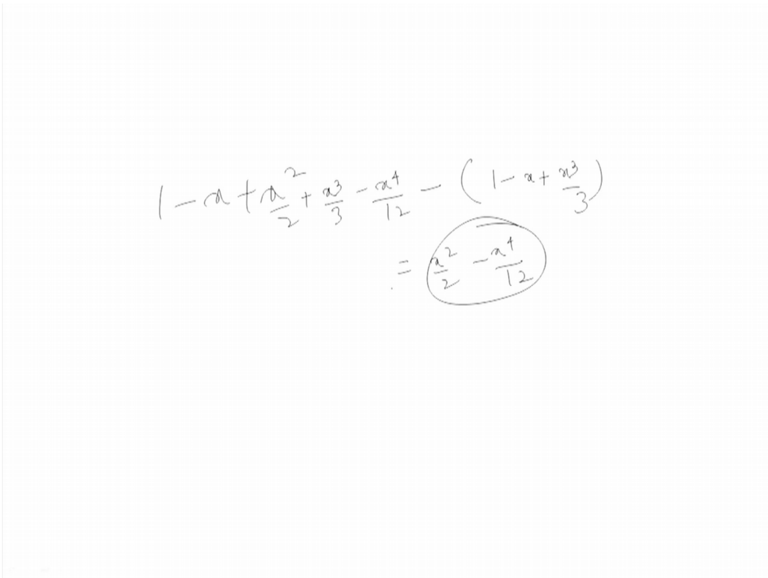

So, that it has been a represented in the form of like here only, x to the power 2 minus x to the power 4 by 12. So, maximum error at x equals to 0.2 we can just often if you will just put this value at x as 0.2 here and x as 0.2 here then we can just find this value is greater than 10 to the power minus 3, but the question is asked that we have to find this error should be less than 10 to the power minus 3 they are over.

So, to find this accuracy of the error we have to further move to the next step. So, when this error will be justified at that step only we can just terminate in we can just say that this is the final answer of this problem. If you will just go for this next iteration we can just write y 3 of x that is nothing, but like y 0 plus x 0 to x and f of x and y to the power of 2 there over and a dx. So, y to the power 2 that is nothing but this term we are just obtaining there over the whole term and if you will just integrate the total term obtaining is in this form here 1 minus x plus x square by 2 plus x's cube by 6 minus x to the power 4 by 12 plus x to the power 5 by 60 here.

And again we will just go for the computation of like error this means that immediate next step if you will just consider like y 3 of x minus y 2 of x here. And if you will just a take these differences since the earlier step we have just got here 1 minus x plus x's square by 2 plus x's cube by 3 minus x to the power 4 by 12 here, and if you will just subtract then we can just obtain rest of the term that is as minus x's cube by 6 plus x to the power 5 by 60 here and which is also greater than 10 to the power minus 3 here. Then we have to go for the further step that is as a y 4 x here, that is nothing but also y 0 plus x 0 to x f of x y to the power 3 into dx.

So, if you will just put that value again then we can just obtain this y for x as1 minus x plus x square by 2 plus x's cube by 6 minus x to the power 4 by 24 plus x to the power 5 by 60 minus x to the power 6 by 360 here.

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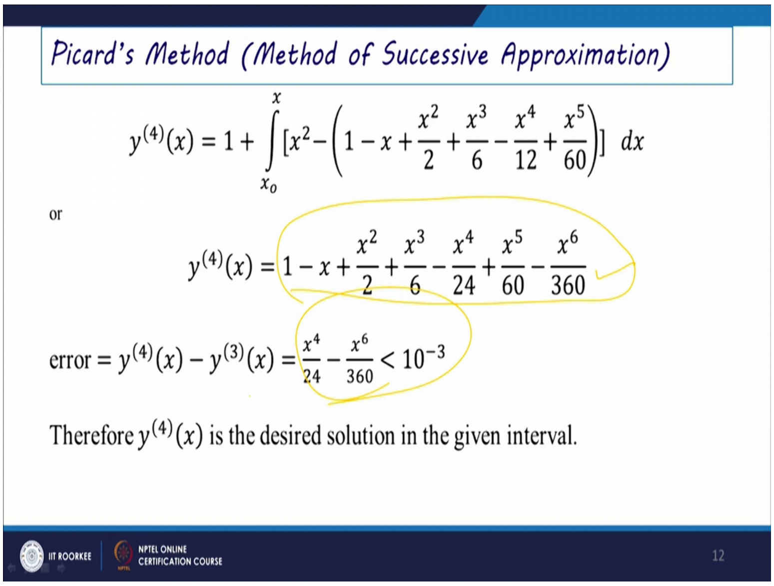

And if you will just go for this error computation at the same point that is as x equals to 0.2 then we can just find that this error is less than 10 to the power minus 3, hence we can just say that y 4 x is the desired solution within that given interval.

So, further if you will just move for like a Taylor series method, since I have in the beginning of the lecture I have told that different type of finite difference methods we will first a deal for ordinary differential equation then we will just go for like a higher order equations or partial differential equations. So, in the Taylor series method if you will just consider like a first order differential equation here, this equations can be written in the form of as we have a discussed you here that is in the form of like a dy by dx. That is nothing, but f of x y and initial condition y of x 0 that is nothing, but y 0 here.

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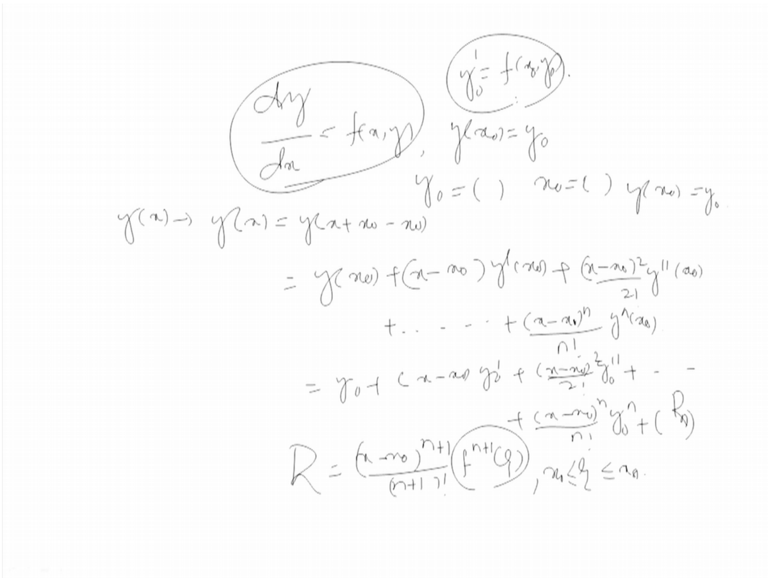

So, if you will just consider this initial condition as a y 0 as certain value and x 0 as certain value here. Then we can just expand like if you will just function is a continuous function in the neighborhood of the initial point if you will just consider then we can just expand this series in Taylor series from there.

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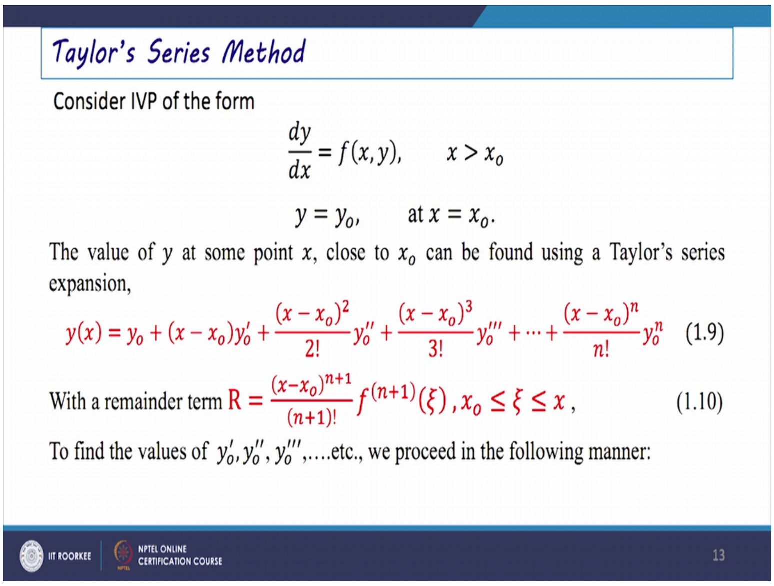

So, first we have to consider that if the function is defined at its initial point or at certain point close to that point this function should be continuous, if the function is continuous at the neighborhood point then we can just expand in Taylor series form. So, if you will just take this Taylor series function of y of x at x equals to x naught, then we can just write y of x is y of x plus x naught minus x naught there and then it can be expanded as y of x 0 plus x minus x 0 y dash of x 0 plus x minus x 0 whole square by factorial 2, y double dash of x 0. So, likewise it will just to be continued like x minus x 0 whole to the power n divided by n factorial y to the power n of x 0 here. Especially we are just denoting here y of x 0 is nothing but y of naught here that is nothing, but y 0 x minus x 0 y 0 dash plus x minus x 0 whole square by factorial 2, y 0 double dash plus up to x minus x 0 whole power n by n factorial y 0 n here.

And especially if you will just consider after the next immediate term of y to the power n term especially this is called our reminder term here and this reminder term especially it is just written R as in the form of x minus x 0 whole power n plus 1 divided by n plus one factorial here then we can just write this as f to the power n plus 1 zeta since y is expressed in the form of like a f of x y here or dy by dx is a represented in the form of x y; f of x y here. So, that is why we can just write this as f to the power n plus one zeta here where zeta should be lies between x 0 to x n.

So, whenever we are interested to find the solutions for like y 0 dash, y 0 double dash and y 0 triple dash etcetera we can just proceed in the following manner here that is nothing but like y of 0 is nothing, but y of x 0 here and y 0 dash that is nothing but f of x 0 y 0 here since y of x is a written as f of x y here.

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$$y_0 = y(x_0)$$

$$y_0' = f(x_0, y_0) \quad \text{since } y(x) = f(x, y).$$

$$y''(x) = \frac{d}{dx}[f(x, y)]$$

$$y''(x_0) = f_x + f_y \frac{dy}{dx} = [f_x + f_y \cdot f]_{(x_0, y_0)}$$

$$y'''(x) = \frac{d}{dx}[f_x + f_y \cdot f]$$

$$= f_{xx} + (y')^2 f_{yy} + 2y' f_{xy} + y'' f_y$$

$$y'''(x_0) = [f_{xx} + (y')^2 f_{yy} + 2y' f_{xy} + y'' f_y]_{(x_0, y_0)}$$

And in the previous slide if you will just see here d y by dx I have just written as f of x y here, especially if you can just express y as a function of y 0 dash it can be written in the form of x 0 and y 0 here.

So, based on this hypothesis we can just say that y 0 dash equals to f of x 0 and y 0 here, and if you will just go for like a second order differentiation here we can just write y double dash of x that is nothing but dy dx of f of x y here. And if you will just go for this differentiation of this one, then we can just write this one as like f x plus f y into we can just write dy by dx here that is nothing but f x plus f y f at x naught y naught if we want to find the value y double dash of x naught here.

So, then if you will just proceed like a third order differential equation we can just write y triple dash of x that is nothing but dy dx of f x plus f y into f here. Especially if you will just to differentiate then we can just obtain this one as f xx plus y dash whole square f yy plus 2 y dash f xy plus y double dash, f y this one. And if it is asked to evaluate at suppose y triple dash at x naught we can just say that we can just find f xx y dash whole square f yy plus 2 y dash f xy plus y double dash f y at x naught y naught here.

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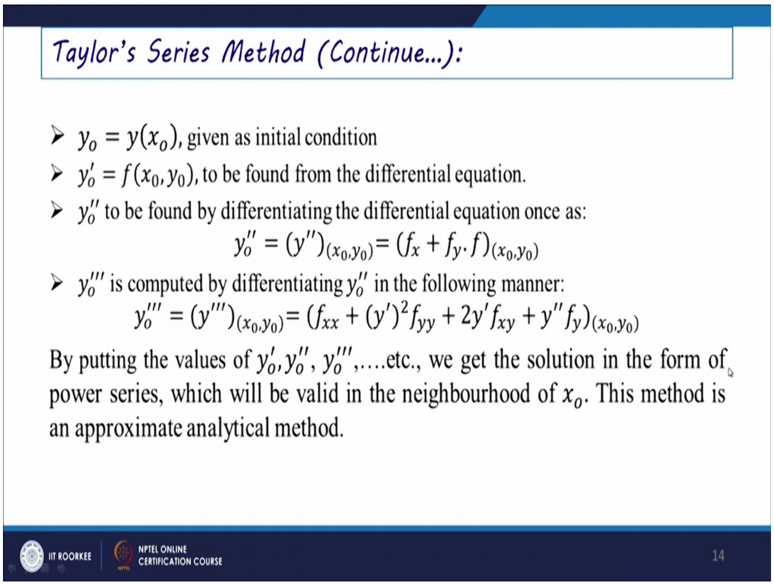

So, to find this values first if you will just use this conditions especially we can obtain the values at that points also at the differential equations at that point also.

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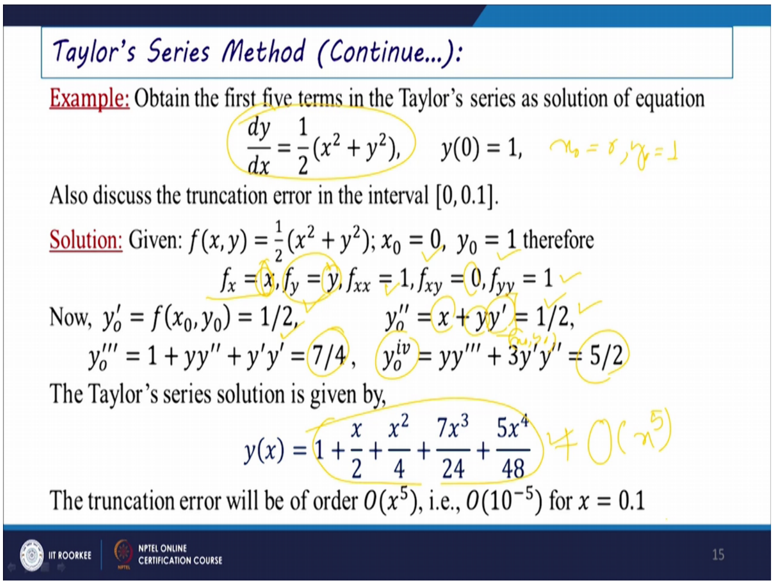

So, for a particular example if you will just consider like obtain the first 5 terms of a Taylor series expansion then we can just write this Taylor series expansion first that is like your terms like y of x equals to y 0 plus x minus x naught y dash of x 0 plus x minus x 0 whole square by factorial 2 y double dash of x 0 plus up to fifth order term if we will just write. Then we can just apply there over that f of x y is since this question it is just to given that obtain the first 5 terms in the Taylor series as a solution of the equation dy by dx equals to half into x square plus y square here where y u 0 equals to 1, especially we can just write here x 0 equals to 0, where y 0 equals to 1 here.

So, if you will just write this equation that f of x y equals to half into x square plus y square here with x 0 equals to 0 and y 0 equals to 1 and for the first order differentiation f x if you will just calculate here f x is a nothing, but d by dx of f of x y; sorry del by del f by del x f of x y that is nothing, but half into 2 x here. So, that is why x is coming over here then if you will just to consider del f by del y here, so half into 2 y. So, y is coming here then f xx one more differentiation if you will just take for x here then we can just get it as 1 here, then if you will just to take f xy, if you will just differentiate this equation with respect to y here then we will just get 0 term. Then one more differentiation with respect to y if you will just consider for this term here then we can just get f yy is 1 here.

Now, if you will just put these values at y 0 dash y 0 dash is a nothing, but f of x 0 y 0 here we can just obtain this term is half here and y 0 double dash that we can just obtain as x plus y y dash here that is nothing, but half here. Since y 0 double dash especially we are just expressing that is f x plus f y into f here so that is why f x is nothing but x here, f y is a nothing but we can just say y here and f means that is nothing, but y dash here and if you will just put all these values at x 0 and y 0 especially we can just obtain this value as half here.

Similarly, for the third order differentiation if you will just go then we can just find that first term it has 1 plus y, y double dash plus y dash into y dash that is nothing, but 7 by 4 here and y forth as x 0 y 0 is nothing but 5 by 2 here. So, then if you will just put all these terms then we can just obtain this Taylor series expansion as 1 plus x y 2 plus x square by 4 plus 7 x's cube by 24 plus 5 x to the power 4 by 48 here and the last term immediate next term we can just consider as the remaining term there or remainder term there over. So, that is why this remainder term if you will just see here that is nothing but order of x to the power 5 here, since the series expansion is going up to 4th order term here over there and for x equals to 0.1 here, for x equals to 0.1 here we can just find the error term for this one.

Thank you for listen this lecture.