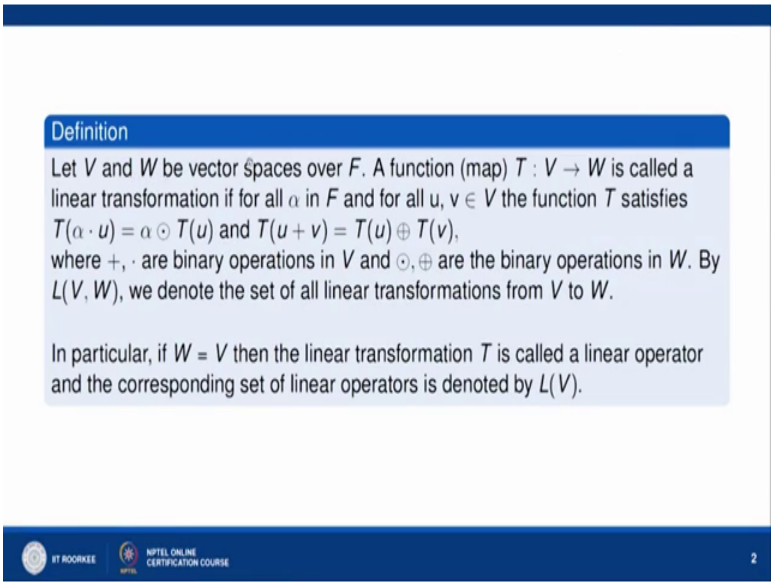
Numerical Linear Algebra
Dr. D. N. Pandey
Department of Mathematics
India Institute of Technology, Roorkee

Lecture - 09
Linear Transformation – I

Hello friends, welcome to this lecture in this lecture, we will discuss the concept known as linear transformation. So, how we define linear transform? Let us discuss here. So, here definition of linear transform goes like this.

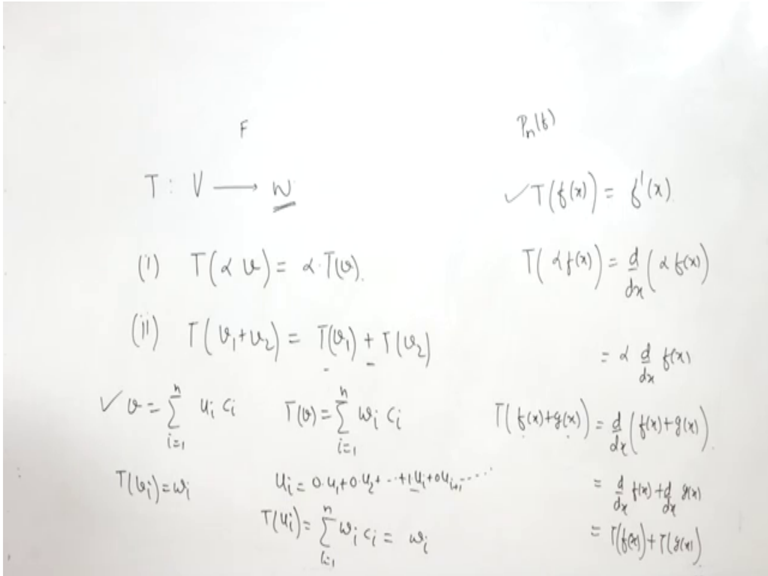
(Refer Slide Time: 00:39)



So, we take two vectors spaces V and W over a common scalar field; let us say F, we may take F as a real number, here now we define a map as a function T from V To W, and we call this map as linear transform; if for all alpha in a scalar field, and for all u v in a vector space V; the function T satisfy the following property, that T of alpha times u is equal to alpha operating on T of u, and T of u plus V is equal to T of u plus T of V.

Here this dot and plus are binary operation in V, and this and this are binary operation in W. So, and we consider all linear map define over V to W, by this space L V comma W and this denote the set of all linear transformation from V to W.

(Refer Slide Time: 01:52)

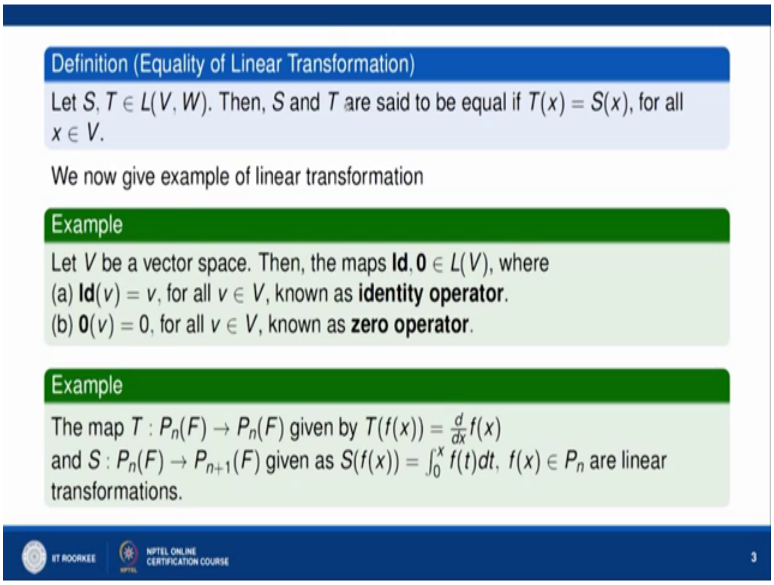


So, basically what is going here is we have a vector space V, we have a vector space W, over the common vector field F, we can take F as R we can take any map T from V to W, we satisfy the following two properties that T of alpha V is equal to alpha T of V here, and second here the T of V 1 plus V 2 is equal to T of V 1 plus T of V 2.

So, here without any ambiguity I am assuming that here operation between alpha and V is operation defined over V, here addition is an operation defined over V, and here operation between alpha and T of V is a operation over W, and here this plus is an operation in W.

Now, here if we take both the vector space as same, if we take W is equal to V, then we call linear transformation T as linear operator and the corresponding set of linear operators is defined as L V to V is denoted by L of V. So, just take certain example, but before that let us consider one definition of equality of linear transform.

(Refer Slide Time: 03:17)



Which says that if we take 2 linear map is defined from V to w, then we call this S and T are same, if T F x is equal to S F x for every element in vector space V. So, in this way we can define the equality of linear transform, that map of T the map T and S gives the same value for each element of vector space V.

Now, let us take certain example of linear transform, first example is identity operator and 0 operator. So, identity operator is what it is define from V to V, and it maps V to the same element V here, and we call this operator as identity operator, and 0 operator it means it takes every element of vector space V to 0 element of V.

So, here we call this operator as 0 operator, these are trivial operators you can say another example of linear transformation is given by differential operator, and integral operator, what is differential operator and integral operator, T define from set of all polynomials of degree at most n and it is given by T of f of x is equal to d by dx of f of x.

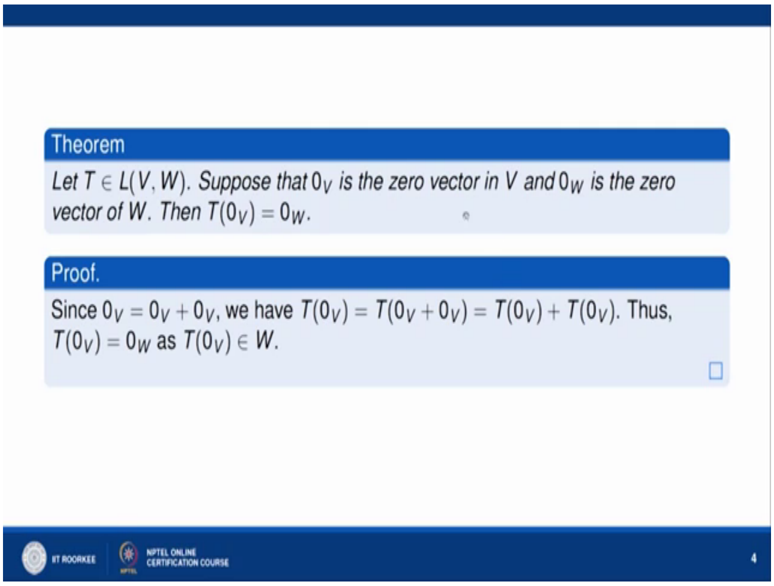
And integral of transformation S is given by, P n F to P n plus 1 F, where P n F is set of all polynomials whose degree is at most n, and P n plus 1 F is the set of all polynomials whose degree is at most n plus 1, and it is defined as S of f of x equal to 0 to x f of t d t, and f x is coming from P n, and these 2 are linear transformation. I am just taking just one example here let us take this differential operator.

So, differential operator is defined as what differential operator is defined as t operating on F of x and it is going to V f dash of x, to show that it at linear operator we try to check these two properties. So, let us take alpha of f, and alpha of f is what d of d of x of alpha of f of x now using the property of calculus you can say it is nothing but alpha of d by dx of f of x, which says that first property is true similarly we can consider t of f of x plus g of x, where f of x and g x is coming in from P n F.

So, these are polynomials of degree at most n. So, this can be written as d by dx of f of x plus g of x, and by say property of calculus, you can say that this is nothing but d by dx f of x plus d by d x g of x, and which shows that this nothing but t of f x and plus t of g of x.

So, it means that this operator this transformation define the; satisfy the properties listed here. So, it means that this T is a linear transformation. So, similarly you can prove that S which is an integral transformation n is also a linear transformation.

(Refer Slide Time: 06:49)

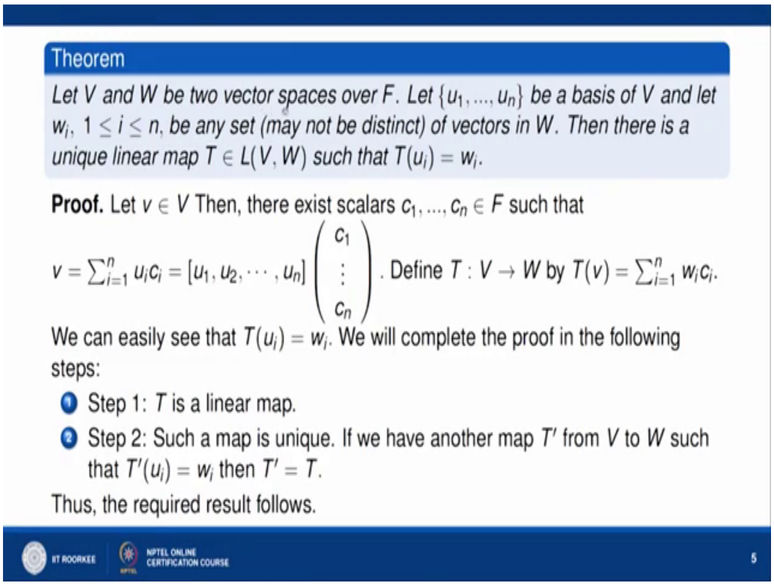


Now, let us take a very simple property of linear transformation, it says that if we have a linear transformation defined from V to d, then it send the 0 element of V to 0 element of W, it means that image of 0 vector of V, will map to 0 vector of W; this has a very simple proof. So, how we can go? So, using linearity we can prove this that T of 0 V is equal to 0 W. So, here by the property of vector space 0 of V can be written as 0 of V plus 0 of V.

So, and then operate T on this; so, T of 0 of V equal to T of 0 V plus 0 of V, now here we can apply the linearity which says that it is nothing but T of 0 V plus T of 0 V.

So, it means T of 0 is equal to 2 times T of 0 V, this implies that T of 0 V is nothing but 0, and this 0 is a element in vector space W. So, it means T of 0 V is equal to 0 of W. So, this is a simple application of linearity property.

(Refer Slide Time: 08:17)



So now let us move to next theorem which gives you; which characterize the linear transform, and it says that let V and W be 2 vector spaces over F. Let u 1 to u n be a basis of V and let W i, i is from 1 to n be any set and this set may not be set of distinct element, it may have only singleton element of vectors in W, then there is a unique linear map T belongs to linear map from V to W such that T of V e V i is defined as W i. So, it means what this theorem says that, every linear map can be characterized by image on the image of the basis element.

So, if we know what is the image of basis element, then you can characterize the whole map T here. So, that is what is given here; so, let us first prove this theorem. So, what we need to prove here that we know W i, then we have a unique linear map define in a way that T of V i equal to W of i. So, let us prove this; so for that let V belongs to vector space V, then there exist a scalar c one to c n coming from a scalar field f, such that V can be written as linear combination of basis element of V. So, basis element u i V can written as V as i equal to 1 to n, u i c i and if you look at in the matrix notation this can be written as matrix u one to u n, and column vector c one to c n this can be written like this. So, any way; so, this is another presentation of the same linear combination. So, we can utilize any of these 2.

So, right now we are utilizing this representation that V is equal to i equal to 1 to n u i c i, now with this linear combination and we know that this representation of V is unique. So, with the help of this define T from V to W, such that T of V is equal to summation i equal to 1 to n W i c i, this c i is nothing but these scalars, we already know that these scalars are uniquely associated with the vector V.

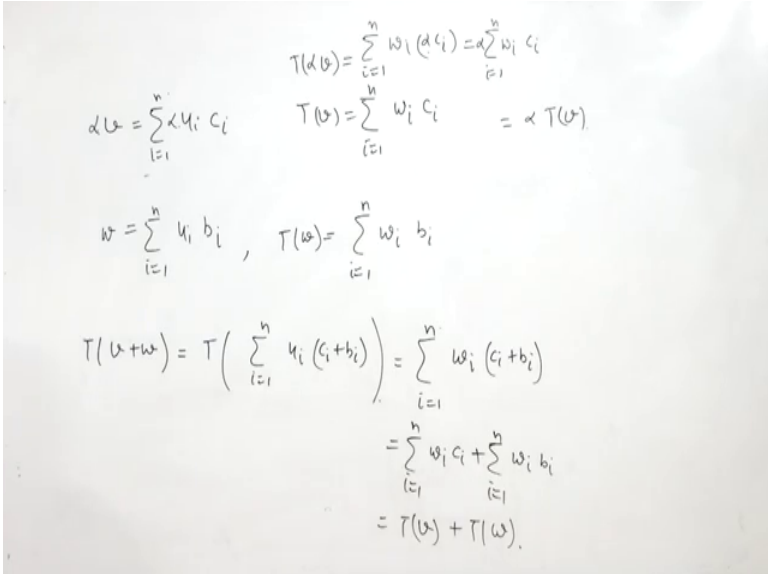
So, we can say that T of V equal to i equal to 1 to n W i c i now once we define this t, we our claim is that it is a linear map and such a map is unique, but before that we try to show that T of v i is equal to W i, now to write T of v i if you look at what is v I, v i can be written us we already know that v is basically what v is your u i c i I is from one to n. So, for this v we have defined T of v as summation i equal to 1 to n W i c i right.

So, this is how we define our linear map then we can write it u i is what, u i is simply a summation i equal to 1 to n u i c i now here c i will be what let me write it here just a minute i can write it u i as 0 times u 1 plus 0 times u 2 and so on, only one into u i rest are all 0 and so on. So, if we follow this then T of u i will be what u i will be only i th position it is one rest all 0.

So, it means that. So, c i So, c one to c i minus 1 is 0, c i plus 1 2 c n n is 0. So, you can write it here as summation i equal to 1 to n W i c i, and using this property you can say that only i th position it is one rest all 0. So, you can say that it is nothing but W i. So, T of u i is going to be W of i. So, this property is true.

So, we have we can show that T of u i equal to W i with this representation, now next thing we want to prove here is that that T is a linear map, and second that such a map is unique. So, we have already defined our map T we want to show that this T is a linear map, and T is unique map, such a T is unique to show that linearity here, what we take here let us consider here that we have a representation for v, then we can representation for say W, which is given by summation i equal to 1 to n u i say b i, then the corresponding T of W is given by summation i equal to 1 to n W i b i.

(Refer Slide Time: 12:59)



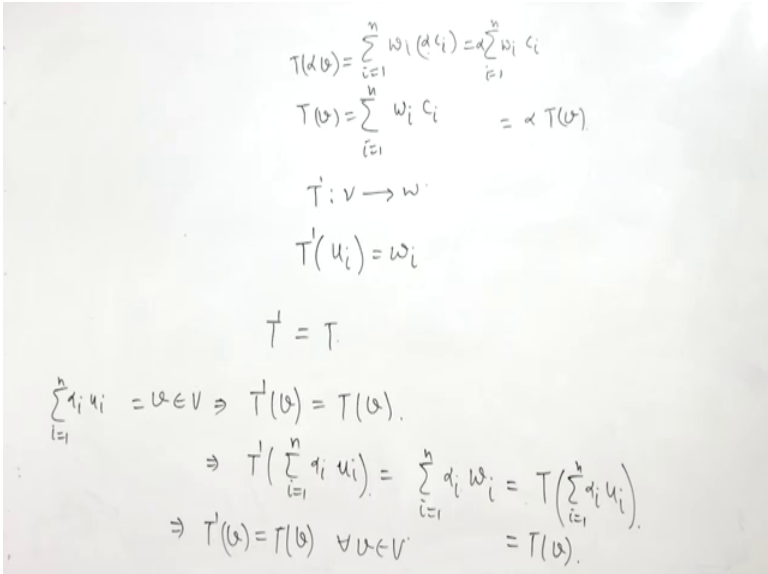
And you can prove that that T of v plus W, now v plus W is basically what T of v plus W is written as summation of i equal to 1 to n u i and c i plus b i, i can write it here, then with this definition i can write this as summation i equal to 1 to n W i and the coefficient here is c i plus b i, basically if you look at what is the relation here how we defined our map here, we take v as summation i equal to 1 one to n u i c i. So, these c is are uniquely represented uniquely associated with this basically these c is are coordinates of v with the respect to u i.

So, using these coordinates we define T of v. So, T of v is basically what W i and these coordinates. So now, if you look at here v plus W the coordinates of v plus W is nothing but c i plus b i. So, with the help of these we can write the image of T of v plus W as i equal to 1 to n W i and coordinates of coordinates of v plus W now this can be written as summation i equal to 1 to n W i c i plus summation i equal to 1 to n W i b i and this is nothing but T of V here, and this is nothing but T of W here.

So, this part is now if you consider the alpha V, alpha V will be nothing but alpha of u i c i and T of alpha V will be given by what T of alpha V is basically, what summation i equal to 1 to n W i now coordinates of alpha V, V will be what coordinate of alpha V is nothing but alpha c i.

So, you can write it here alpha c i. So, this can be written as summation i equal to 1 to n, since alpha is independent of this suffix side. So, you can take it alpha out and this is nothing but W i c of i and this is nothing but alpha times T of V. So, this satisfies the properties of linearity. So, T of alpha V is nothing but alpha times T of V and T of V plus W is nothing but T of V plus T of W. So, it means as such a map is a linear map. So, this map T is a linear map.

(Refer Slide Time: 15:52)



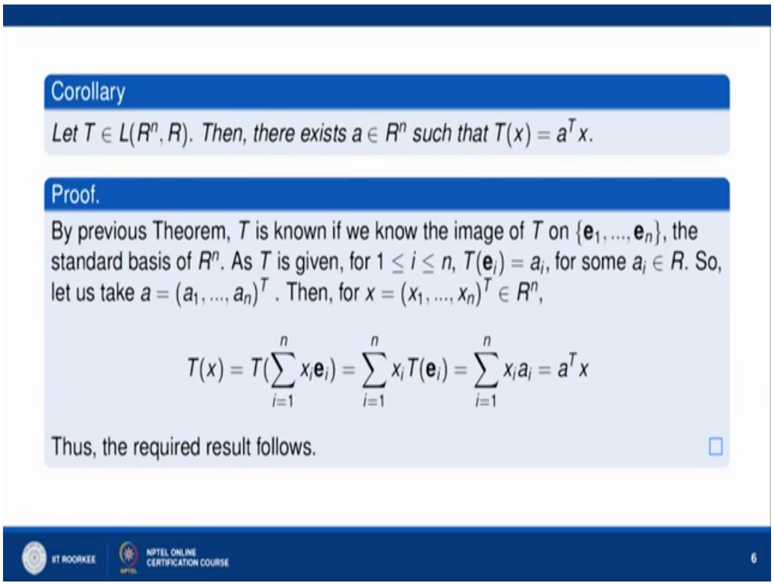
So, to prove the uniqueness let us define another map say T dash from V to W. So, T dash from V to W, such that T dash of u i is equal to W i and we want to claim that your T dash is equal to T. So, by definition of so, to show this that T dash equal to 2 T we want to show that T dash operating on any element of vector space V say V here. So, take V and vector space V you want to show that T dash V is nothing but T of V here.

So, to show this let us move here. So, T dash now V is basically what V is we define as V can be written as linear combination of u i. So, this can be written as i equal to 1 to n let us say that this alpha i u i. So, here since V is an element of V here, we can write it i equal to 1 to n alpha i u i. So, this is the presentation of V now this I can write it as T dash.

So, this I can write it here map T dash as say summation i equal to 1 to n alpha i now W i, now we already know that this is what this is how we write as map of T of V now V is what summation i equal to 1 to n alpha i u I, this is how we define our T of V, T of V is defined as W i and coordinate of V corresponding to u i. So, here if u look at this will be what this will be the image under T provided that alpha i are the coordinates of vector V with respect to u i.

So, this can be written as this is nothing but T of V here. So, it means that we started with T dash V and we are ending at T of V. So, here we can say that that T dash V is nothing but T of V for all V in vector space V. So, this proves the uniqueness of a map having the property that T of u i is equal to W of i.

(Refer Slide Time: 18:33)



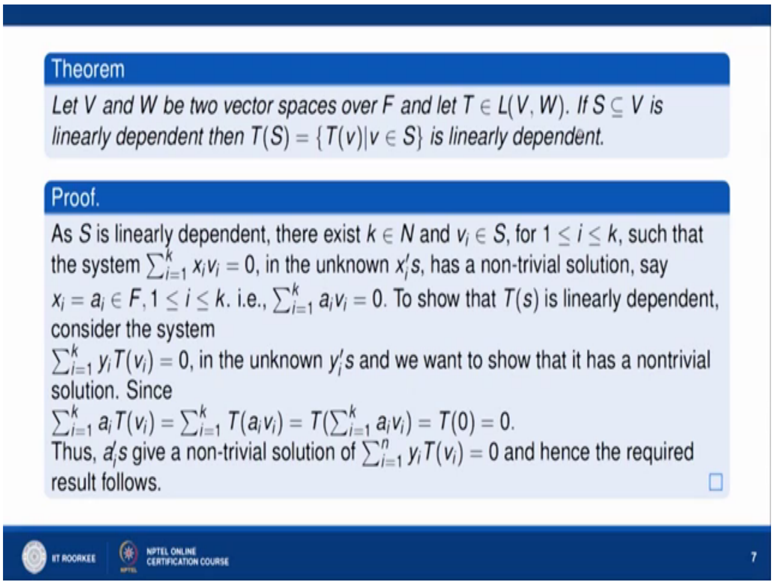
So now let us a take a corollary of this, which say that if we take a linear map from R n to R, then there exist a vector from R n such that T of x can be written as a transpose x. So, in other word we can say that it is kind of riesz representation theorem, which is commonly very important properties in function as is. So, it is nothing but riesz representation theorem of a linear operator in terms of finite dimensional cases.

So, here we want to prove this corollary. So, by previous theorem we know that T is known if we know the image of T on a basis element. So, let us take the basis as a standard basis. So, it means that image of basis element is given to us. So, it means that T of e i is defined as a i. So, it means a i is known to us. So, with a help of this a i let us form a vector a having this a one to a n as component.

So, a is given to us then take any element in R n. So, element x equal to x one to x n transpose t. So, let us try to define T of x. So, T of x so, here x can be written as summation of x i e i I is from one to n, now using the linearity here you can take write the as i equal to 1 to n x i T of e i. So, T of e i is already known as a i; so, this is can be written as i equal to 1 to n x i of a I, and this is this we can write i as a matrix product of a transpose x, a transpose is what a transpose is column vector a one to a n and x is a row column vector x one to x n here. So, this summation i equal to 1 to n x i a i can be written as a transpose x.

So, it means that T of x given as a transpose x, thus the required result follows. So, it means that whenever we have T as a linear operator from R n to r, linear map from R n to R then there exist a vector a in R n such that T of x can be written as a transpose x here.

(Refer Slide Time: 20:59)



So, let us consider another important result which we which says that, if we have linear map then it maps linearly dependent map to linearly dependent set to linearly independent linearly dependent set. So, statement of this theorem goes like this, that we have a 2 vector space V and W defined on a same scalar field F and let T be a linear map from V to W, and if S which is a sub set of V is linearly dependent, then the image of this linearly dependent set is going to be linearly dependent.

Now how we define the image of linearly dependent set s, it is basically set of all element T V V is in as. So, let us try to prove here, what we want to prove here that if S linearly dependent then T S is linearly dependent provided that T is a linear map. So, we already know the S is linearly dependent it means that we can find out a positive integer say k, such that and such that V i belongs to S for i is one to k such that summation i equal to 1 to k x i V i equal to 0, in the unknown x i is has a non-a trivial solution.

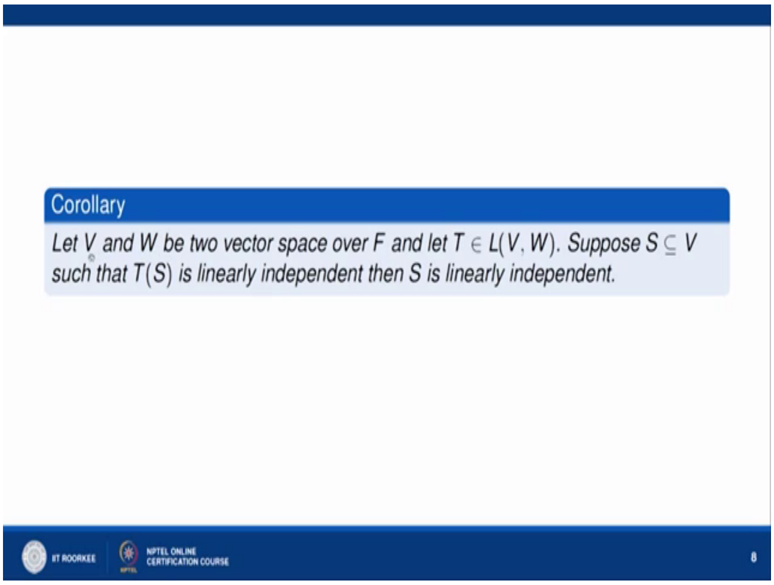
So, this is the definition of linearly dependent set that if S is linearly dependent then we can always find out say some vectors such that linear combination gives you 0, and this has a nontrivial solution now let us call this call the nontrivial solution as x i equal to a i. So, it means that summation i equal to 1 to k a i V i equal to 0 for some non-0 a i here.

Now, we want to show that T S is linearly dependent, to show that T S linearly dependent we want to consider the system i equal to 1 to k y i T of V i equal to 0, and we want to find out a nontrivial solution of this equation, and for this let us consider this value i equal to 1 to k a i T of V i and since T is linear, we can write this as the summation i equal to 1 to k T of a i V i.

So, here i am using the property of linearity, and then again we can use the property linearity property and we can write as T of summation i equal to 1 to k a i V I, and we already know this is nothing but 0 here this is already given here that summation i equal to 1 to k a i V i equal to 0, and this is written as T operating on 0. Now T operation on 0 is nothing but 0; so, it means that summation i equal to 1 to k a i T of V i is equal to 0 and not all a is are 0, it means that some of a is are non 0. So, it means that this will give that if we take consider this summation summation i equal to 1 to k y i T of V i equal to 0 then we have a nontrivial solution gives as y i equal to a i. So, it means that this T of V i forms a linearly dependent set.

So, it means that that if we have a linear map then it map linearly dependent set, to linearly dependent set.

(Refer Slide Time: 24:44)



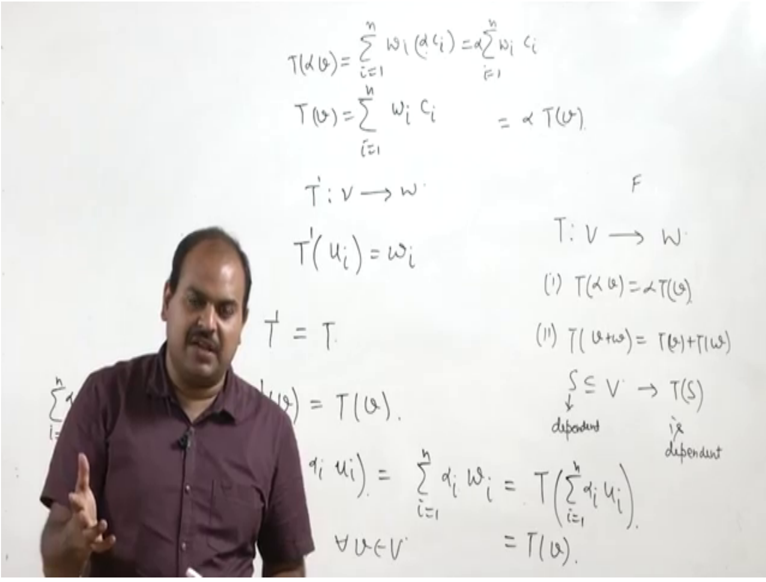
So, an corollary of this which says that if we have a linear map defined on a vector space V to w, then it sends linearly dependent linearly independent set to linearly independent set, it means that if we take any set S of V such that T S is linearly independent then S is linearly independent.

So, how we can prove this? So, this proof follows from the proof of the previous theorem. So, let us say that S is linearly dependent in place of what, we want to prove here we want to prove that if T S is linearly independent then S is linearly independent.

So, let us assume the contrary let us assume that S is linearly dependent in place of independent let us assume that S is linearly dependent, then previous theorem apply here if we look at previous theorem says that T map on S where S is linearly dependent then T of S is linearly dependent, but here what we have assumed here we have already assumed that T of S is linearly independent. So, it means that the assumption that S is linearly dependent is not true S has to be linearly independent.

So, we stop here and before stopping this; what we have discussed in today's lecture?

(Refer Slide Time: 26:10)



We have define linear map from 2 vector space V to w over a common scalar field say f, which satisfy the following 2 properties that T of alpha V is equal to alpha of T of V, and second is that T of V plus W is equal to T of V plus T of W. So, this is how we define linear map linear transformation, we have discussed certain example and some properties of linear map the important property is that if we have S which is the subset of V, as if S is dependent, then T of S is also dependent and a corollary of this that if T of S is linearly independent, then S has to be linearly independent. So, I will stop here and in next class we will discuss more properties of linear transformation.

Thank you very much.