Numerical Linear Algebra
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Lecture - 08
Bases and Dimensions – II

Hello friends. Welcome to the lecture here. In this lecture, we will discuss something about coordinates and some properties of dimensions of subspaces. If you recall, in previous class, we have discussed certain results on basis and dimension of a vector space. Today we just continue the discussion and discuss the motion of coordinates and coordinate vectors. So, let us define; what is coordinate here. So, definition goes like this that let S be a nontrivial subspace of V and beta consists of v 1, v 2, v k be an order basis for S.

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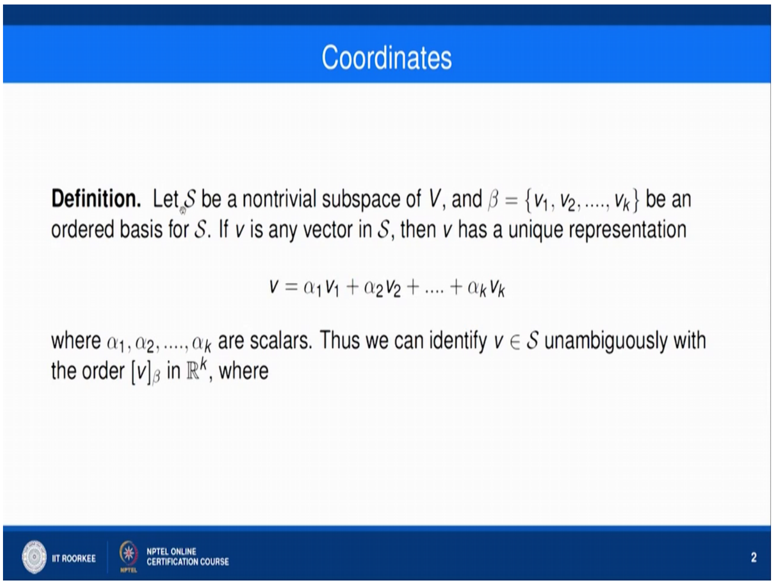

Order basis means that not only the elements are specified but their order is also specified. So, it means that the first vector is v 1 and the second vector is v 2 only, it cannot happen like we have v 2, v 1 or so. If we take v 2, v 1 and v k, then it will be another order basis. So, order basis means not only the elements are specified, but their order is also given, then if v is any vector in S, then already know that we can have a unique representation of v in terms of basis element of S. So, it means, here we can write v as alpha 1 v 1 plus alpha 2 v 2 plus alpha k v k where alpha is are scalars coming from the scalar field.

And here this, we can do for any vector in S. So, it means that if we take any other vector, then also we can find out these kind of scalars. So, corresponding to any every vector we have this kind of set of scalars coming in. So, we can identify for every vector in S with the order v beta in R k what is v beta here?

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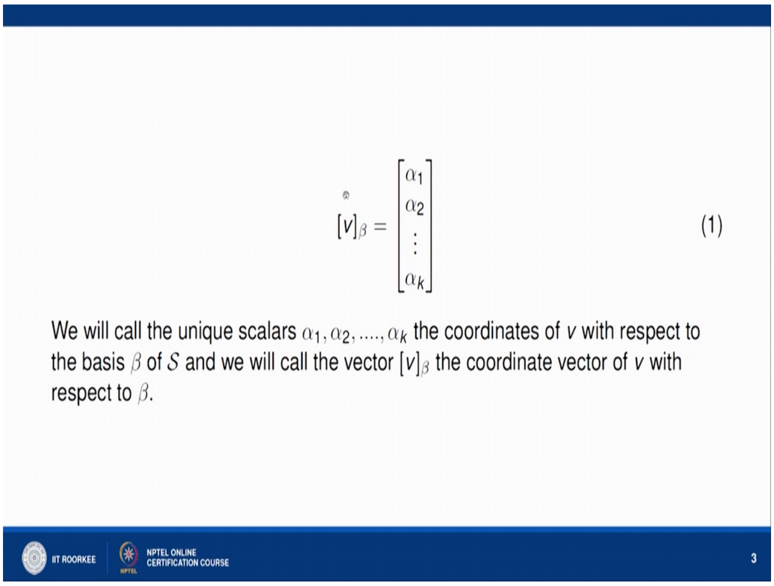

V beta is defined as vector here; alpha 1 to alpha k is a column vector and we say that this alpha 1 to alpha k are unique scalar corresponding to this v and it is known as coordinate of v with respect to the basis beta of S and we call this column vector v beta as the coordinate vector of v with respect to beta just let us come to here.

First thing I want to mention about given this S a subspace of given vector space v.

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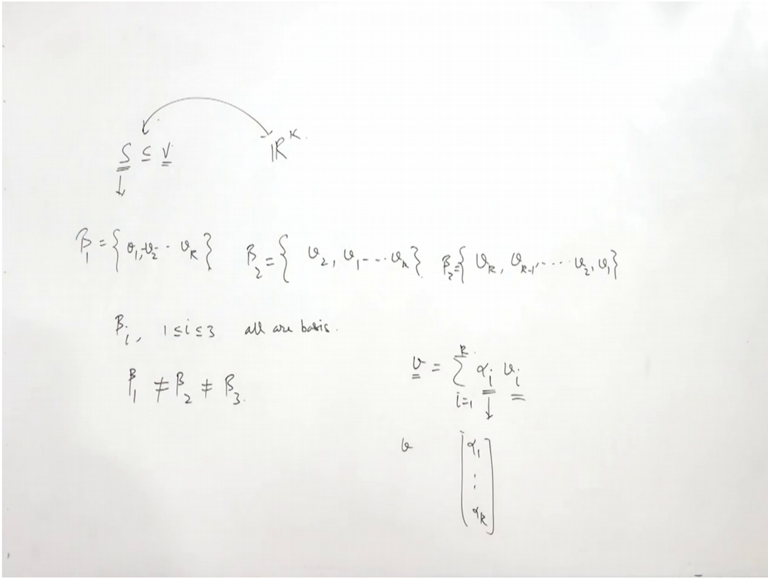

So, corresponding to S, we have defined what is known as order basis. So, order basis is v 1 to say v k; v 1, v 2 and so on. Now order means if you define like say v 2, v 1 to v k and we may define nth order like say v k; v k minus 1 and so on v 2 and v 1. So, if you look at as a set, all these are equal as a set; all these are equal in the sense that elements of the sets are same here v 1, v 2, v k; here v 1, v 2, v k and everything.

But if you look at the order here is a first element is v 1, second element is v 2 and so on, but here first element is not v 1, it is v 2. So, it means that though all these will represent a basis for S, but as all basis if you call this is beta 1, we call it beta 2 and beta 3, then we can say that beta i from 1 to 3, all are basis all are basis, but beta 1 is not equal to beta 2 and similarly beta 2 is equal to beta 3. So, though they are all basis, but they are not same as an order basis. So, beta 1 is an order basis, beta 2 is another order basis, beta 3 is another order basis.

So, this coordinate is basically what you take any element here and you write this element in terms of; say in terms of in terms of v i to v k. So, you can write this as alpha i and v i is from 1 to k. So, these scalars are unique corresponding to this v. So, once your basis is fix, then corresponding to any v, we have a unique set of scalars coming in. So, it means that corresponding to v, we can always assign alpha 1 to alpha k. So, this column vector alpha 1 to alpha k is known as coordinate vector and these scalars are known as coordinates of v corresponding to this order basis v 1 is that ok.

So, here we can consider this as this is kind of one to one correspondence between this vector S and the scalar field. Let us say that here, we are assuming scalars are coming from real number. So, we can say that there is one to one correspondence between S to R, k corresponding to any vector v, here we have a column vector like this alpha 1 to alpha k which is a member of R k. Similarly corresponding to any element in R k, we have a vector assigned in S. So, it is kind of one to one correspondence between this S and this R k, right.

So, now let us move further and this is theorem one which says that let beta which is consisting of these k element v 1 to v k be an ordered basis for a subspaces S of R n, then for all scalars alpha 1 to alpha n and all vectors w 1 to w m.

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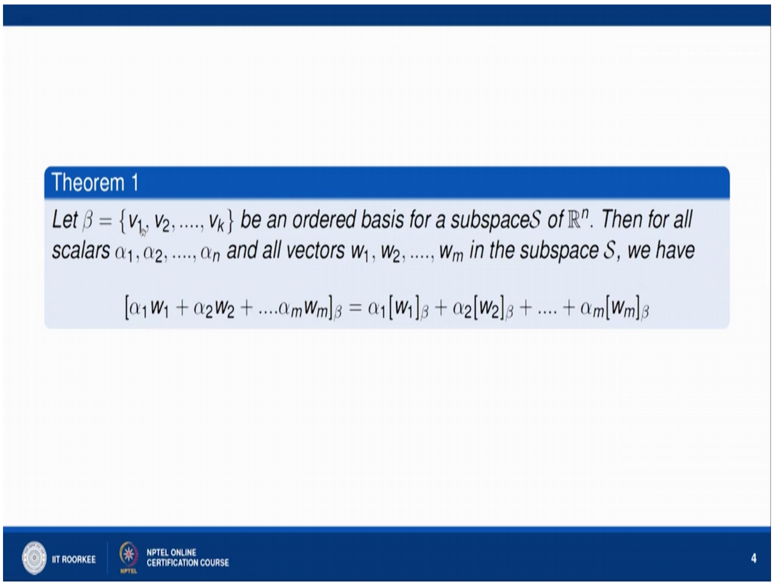

In the subspace S, we have this result that coordinate vectors of this linear combination alpha 1 w 1 plus alpha 2 w 2 plus alpha m w m corresponding to with respect to the basis beta is nothing, but alpha 1 coordinate vector of w 1 plus alpha 2 coordinate vector of w w 2 and so on.

So, this is a kind of very important result and the proof of this theorem is quite easy; but just tries to prove this theorem, just for m equal to 2. So, it means that we want to show that if we have w 1 and w 2 element of S then alpha 1 w 1 plus alpha 2 w 2 will also be the element of S.

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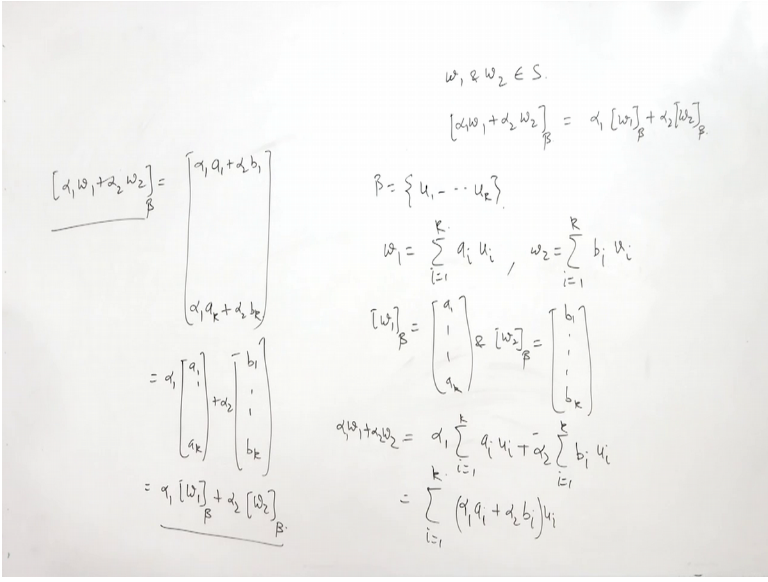

And if we consider the coordinates of this, then it is nothing, but alpha 1 coordinates of w 1 with respect to beta plus alpha 2 coordinates of w 2 with respect to beta. So, this I am just proving for m equal to 2, you can extend it for any integer m.

So, here we know that beta is what beta is a basis say let us say that this is u 1 to u k. So, we can write coordinate of w 1; w 1 is suppose your let us say a i and u i is from 1 to k. Similarly, you can write down w 2 summation i equal to 1 to k say b i u i. So, coordinates of w 1 with respect to beta is nothing, but a 1 to say a k and coordinates of w 2 with respect to beta is nothing, but b 1 to b k.

Now, if you look at what is alpha w 1 plus beta w 2, sorry, alpha 1 alpha 2? So, this is alpha 1 alpha 2. So, if you look at alpha 1 and alpha 2, then it is what alpha 1 w 1 plus alpha 2 w 2 is equal to alpha 1 w 1 is given by a i u i is from 1 to k plus alpha 2 sum and 2 are equal to 1 to k b i u i and if you simplify this, this is nothing i equal to 1 to k, you alpha 1 a i plus alpha 2 b i and into u i. So, again by the definition of coordinates the coordinates of alpha 1 w 1 plus alpha 2 w 2 with respect to beta is given by alpha 1 a 1 plus alpha 2 b 1 and so on here alpha 1 a k plus alpha 2 b k.

Now, here this we can simplify here as alpha 1 a 1 to a k plus alpha 2 b 1 to b k. So, here this is nothing, but alpha 1, this is nothing, but coordinates of w 1 with respect to beta plus alpha 2 this is nothing, but coordinates of w 2 with respect to beta. So, here this is equal to; so, this is a proof for m equal to 2. Now you can apply mathematical induction to prove the given theorem.

So, let us move to second result, we says that if we have a basis beta v 1 to v k be a basis for a subspace S of a vector space v and let u which consist of m element u 1 to u m be a subset of S.

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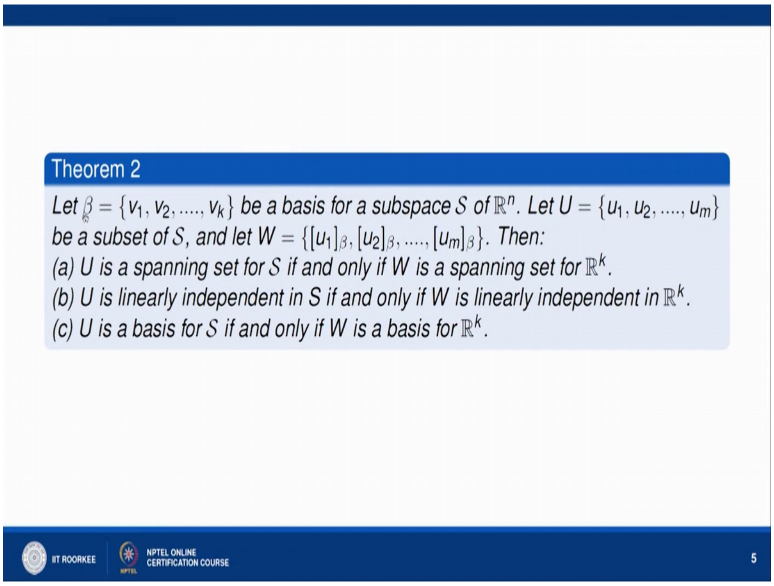

So, we have a subspace and corresponding to subspace, we have a basis for the subspace and let us consider a subset of S say U and the corresponding set W which is say consisting of element u 1 beta 1 u 2 beta u m beta, these are coordinate vectors of u 1 to u m with respect to the basis beta.

Then followings are assumed. So, first part says that if U is a spanning set for S. So, it means that if this subset of S is spanning set for this vectors of space S if and only if W is a spanning set for R k. Second part if U is linearly independent subset in S if and only if W is linearly independent in R k. Third part if U is a basis for S, if and only if W is a basis for R k. So, let us try to prove this theorem. So, first let us consider here, we have say S vectors of vector subspace of a vector space v and corresponding to this S, we have a basis beta.

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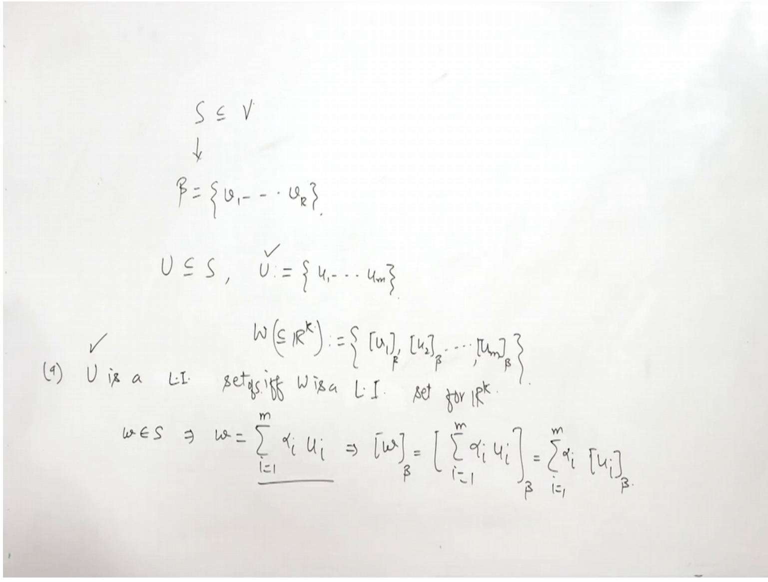

So, we can call this as order basis beta. So, this is your say u 1 to say u k, you can take any say v 1 to v k because. So, let us consider this as some v 1 to v k, then you consider a subset of S that consisting U is defined as some u 1 to u m and you define W a subset of R k. It is defined as coordinate vectors corresponding to these u i corresponding to these beta. So, here it is u 1 with respect to beta u 2 with respect to beta and so on u m with respect to beta. So, basically this is collection of coordinate vectors of this v u.

Now, claim is that if this U is spanning set for S then W is a spanning set for R k. So, first claim is what if U is spanning set, if and only if W is a spanning set for R k. Let us say how it is true. So, if U is spanning set of S. So, if U is spanning set of S means if you take any element of S, then it can be written as linear combination of elements of U.

So, let us take W which is an element of S here, this can be written as summation; some alpha i u i is from 1 to m. Now if you recall the previous theorem; now previous theorem says that if we find out say coordinates of this then you can write here as that coordinates of w with respect to beta is nothing, but coordinates of summation alpha i u i is from 1 to m with respect to beta.

Now, previous theorem says that this can be written as linear combination of summation alpha i is from one to m coordinates of u i with respect to beta. So, it means that if W can be written as a linear combination of u i, then the corresponding coordinate vector can be written as linear combination of this u i beta.

Now here this W is an arbitrary element of this S and here we can say that. Similarly this w beta is an arbitrary element of R k and this can be written as linear combination of this. So, this proves that if U is a spanning set, then W is a spanning set for R k. Similarly we can prove that if W is a spanning set for R k, then the corresponding U is a spanning set of S; this part I am leaving it for you.

Now, second is what second is the that b if u is a L I set in place of spanning set; let us say that if it is an L I set of S if and only if W is L I set. So, for to prove this; this is a b part let me write it b part.

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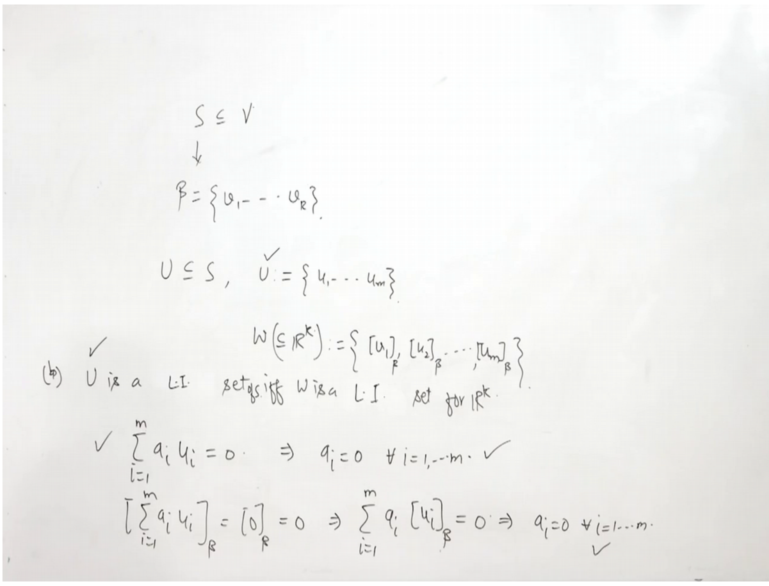

So, to show that U is L I set that is take this linear combination summation a i u i, i is from 1 to m and equate equal to 0. Now since we are assuming that u is a L I set, it means that this equation has a only trivial solution.

So, this implies that this implies that all a is equal to 0 for all i 1 to m. Now again utilize the previous theorem and find out say coordinate of this. So, coordinate of this linear combination a i u i, i is from 1 to m with respect to beta is nothing, but the coordinates of 0 with respect to this beta right. Now, again since beta is a basis and so, 0 will have a only trivial representation. So, this I can write it like this and utilizing the previous theorem, this can be written as summation a i, i is from 1 to m coordinates of u i with respect to beta and it is equal to 0.

And this statement that summation i equal to one to m a i u i equal to 0 implies that all a is are 0 is equivalent to this statement that i is this linear combination i equal to one to m a i coordinates of u i is equal to 0 implies that all a i S are 0 for i equal one to m. So, this statement and this statement are equivalent. Why it is equivalent because we have this and simply utilize the previous theorem and this statement can be written as this and which is nothing, but this. So, it means this statement says that u is are linearly independent and this statement says that this coordinate vector corresponding to u i are all linearly independent.

So, this prove the second part that u is a L I set of S if and only if W is a L I set for R k and if we combine the first and second part; the first part says that if U is spanning set of S, if and only if W is a spanning set of for R k and second part says that if U is linearly independent in S if and only if W is linearly independent in R k; if we combine, then this; then if you take U as a basis means U is a spanning set and U is a L I set, then the first 2 parts gives that this implies that W is a basis for R k. So, c part follows from the part a and b. So, U is a basis for S if and only if w is a basis for R k. So, this also shows that there is a 1 to one correspondence between S and R k.

So, now moving on to next thing, now in this we try to define the dimensions of subspaces.

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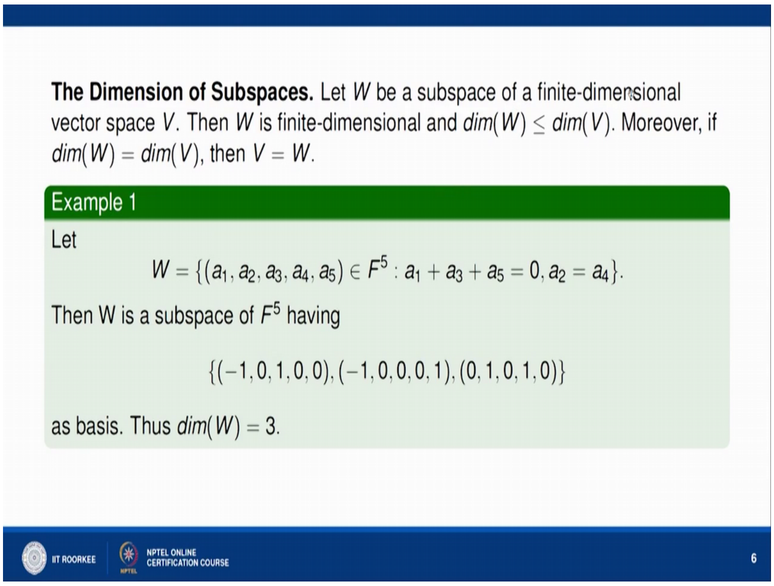

So, here let us consider let W be a subspace of a finite dimension vector space V, then W is finite dimension and dimension of W is less than or equal to dimension of V and if dimension of W is same as dimension of V, then we can say that then V is nothing, but W. So, it means that when if we have dimension same, then the subspace is nothing, but this whole space that is V. So, let us have a small justification of this.

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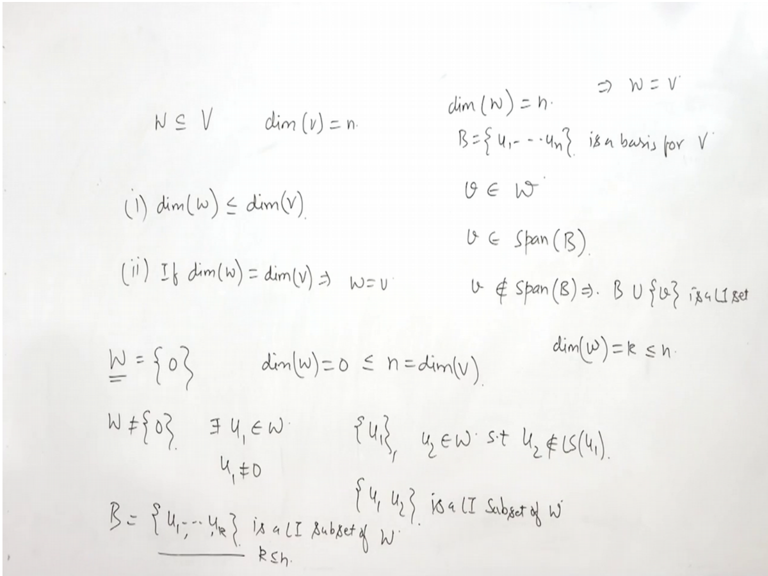

So, here we have a vector space V whose dimension is defined, let us say dimension of V is some finite, let us call this n and let us say that w is a subspace of v, then our claim is that dimension of w is less than or equal to dimension of v, this is our first claim and second claim is that that if dimension of w is same as a dimension of v then this implies that w is nothing, but v. So, how it follows that suppose w is a subspace of v, then it may be a trivial subspace it may be non trivial subspace.

So, first let us say that w is a trivial subspace that is it contains only 0 vector. So, w is a 0 vector, then the basis for this trivial space is nothing, but empty and you can say that dimension of w in this case is 0 which is less than or equal to n which is dimension of v, right. So, for trivial subspace it is trivially true. Now let us say that w is a not a trivial subspace. So, it means that they exists a non trivial element call it u 1 in this w. So, u 1 is non-zero vector. So, now, if u 1 is non-zero vector, then we already know that this u 1 this set this single set is a linearly independent set.

So, now if it is a linearly independent set, then we try to extend this singled n set to the basis of vector subspace w here. So, for this let us consider another element u 2 and w such that u 2 is not in a linear span of u 1, right. So, here if we take u 2 which is not in span of u 1, then we already know that in this case u 1 u 2 addition of u addition of u 2 in this linearly independent set makes a bigger set which is also a linearly independent set.

So, it means that now u 1 u 2 is a linearly independent subset of this w. So, this is a linearly independent subset of up w right now in this way we can continue and let us say that we have u 1 to say u k which is a is a L I set L I subset of w and this we can achieve by since dimension of v is equal to n. So, it means that we already know the result that dimension v equal to n means the number of basis element is going to be n. So, it means that the maximum number of element in a linearly independent set is going to be n. So, it means that we can go up to say k less than or equal to n. So, it k may be equal to n or k is something lesser than n now my claim is that this set u 1 to u k call it say B is a basis for the subspace w.

So, since it is already a linearly independent subset of w, what I need to prove is that that B is a spanning set for this w now how we can prove that if we take any element w here right. So, let us say that let us take u let us take an element v in w here. Now my claim is that v can be written as linear combination of this or you can say that v belongs to span of B if not then if we suppose it is not true it means that v is not in span of B. So, this implies that B union the single term v is a L I set, but this is not true because how we have constructed this B we have constructed this B that if we take any other element include in this B then it is going to be a l d set. So, it means that we can go up to this k provided that if we take any other element here then B is going to be l d set.

So, this implies that we cannot have any element v such that which is not belong to span of b. So, it means that v has to be a span of B, it means that B is a spanning set of this S is that. So, B is a spanning set of w and B is already a L I set L I subset of w. So, it means that B is going to be B is going to be basis for this vector space w. So, it means that what is a dimension of this w. So, dimension of w is going to be k which is nothing, but less than or equal to n, right.

So, this proves that dimension of w is less than or equal to dimension of v. Now if we say that the dimension of w is same as dimension of v. So, it means that if I assume that dimension of w is equal to n it means that this basis B consist of how many elements this will consist of u 1 to u n.

Now my claim is that we already know that that if we have a linearly independent set having exactly the number of dimension of v then this is going to be a basis for the whole space v. So, it means that if we have this then B is a basis for basis for whole space vector whole vector space v. So, in this case B generate w and B generate v this implies that both space w is equal to v. So, w and v both have same basis it means that vector space w is same as vector space v

So, that gives a justification of this fact that if we have w be a subspace of a finite dimension vector space v then w is finite dimension and dimension of w is less than or equal to dimension of v and if dimension of w is equal to dimension of v then both the vector then both the vector spaces are same. So, let us consider one example of subspace. So, let us consider this w which is subspace of f 5 f 5 means f here is scalar field we can take R we can take R 5 of or we can take c 5 or any scalar field. So, the condition on w is that a 1 plus a 3 plus a 5 equal to 0 and a 2 equal to a four.

Now, our claim is that this is a subspace checking that w is a subspaces quite easy you have to consider that if you take any 2 element say u and v then alpha u plus beta u beta v is going to be an element of w. So, that process is already trivial now next thing we want to show is the dimension of this. So, to find out say dimension of this subspace we need to find out say basis of one basis for this vector subspace w. So, here we look at the condition here a 1 plus a 3 plus a 5 equal to 0 and a 2 equal to a 4 we can consider this set as a basis.

So, here what is this set minus 1 comma 0 comma 1 comma 0 comma 0 and second basis element is minus 1 comma 0 comma 0 comma 0 comma 1 and third element is 0, 1, 0, 1, 0 and if we look at the this is L I set and we can check that this is also a spanning set for this w. So, if we have a set which is L I and a spanning set then this will serve as a basis for this vector space w and we can say that the number of basis element is 3. So, we can say that w dimension of w is nothing, but 3.

So, just let us see how we can find out this basis. So, if you look at this is the vector subspace which is defined as a 1 plus a 3 plus a 5 equal to 0 and a 2 equal to a 4.

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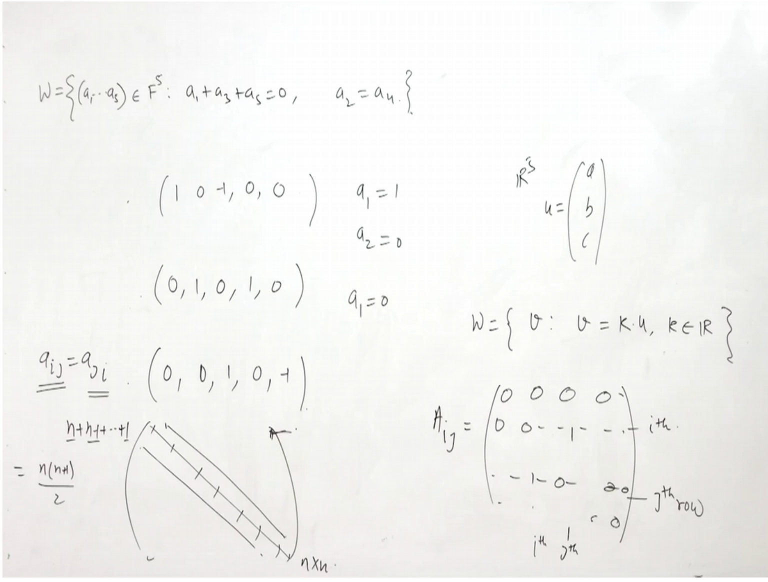

So, this we can write it; we can say that; what are the basis for this. So, what we can say that let us say that a 2 is equal to a 4. So, we are just trying to find out say basis element here. So, if we take a 1 let us say that a 1 is since we have 5 unknown and we have only 2 equations. So, we can say that we can have 3 free vectors. So, let us say that a 1 is my free vector. So, let us say that a 1 is one then you can take a 3 a 2 as 0 a 3 you can take as minus 1 and a 4 is simply again.

Since a 4 is nothing, but a 2 and a 5 is nothing, but 0 here; so, a 1, a 2, a 3, a 4, a 5, so, this is 1 vector. So, here I am considering this a 1 is 1 and a 2 as 0. So, this is a free vector, here, if we consider another one. So, you can consider this as 0, you can consider a 2 as 1. So, here we can say all 0 1 here 0 here. So, this is another, here I am assuming that a 1 is equal to 0. So, here a 2 is equal to a 4, I am assuming. Now we can consider another basis element.

So, here we can take a 1 as 0 a 3, I can consider as 1 and sorry a 2 is 0, a 3 is 1 here, a 4 is again 0 because a 4 is nothing, but d 2 and to satisfy this relation I need to take a 5 as minus of a 3 is that is minus 1. So, we can say that one 2 3 these are 3 basis vectors 3 linearly independent vectors you can check that this 3 will form a basis here. So, that I am leaving it to you. So, by choosing suitably suitable values of a 2 a 1 or a 3 you can consider several basis for this vector subspace w.

So, now moving to another example we say that the subspace of R 3 are given as if we take 0 dimension.

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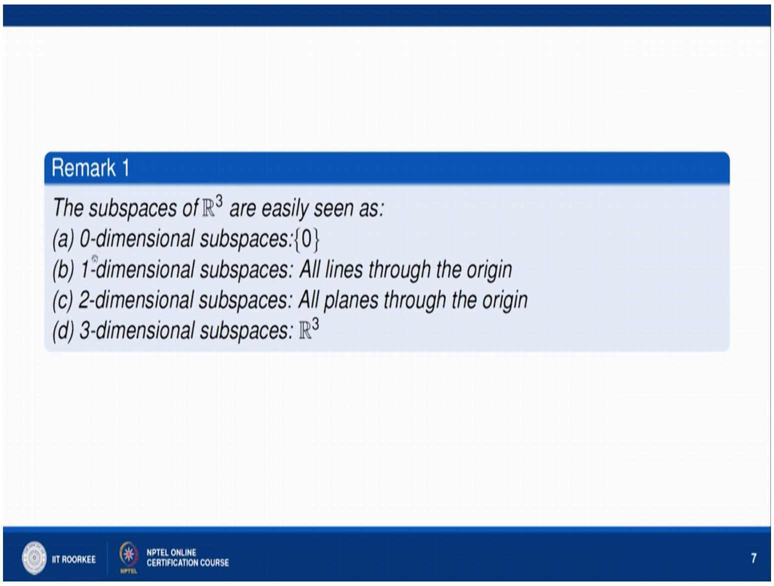

Then it is nothing, but origin if it is one dimension, it is all lines through the origin, if we have 2 dimension subspace, then it is all planes through the origin and if we take the 3 dimension subspace then it is R 3. So, first part a which is 0 dimension is quite easy, it is nothing, but a trivial vector space, if you look at the last part d, then 3 dimension subspace is R 3 because we already know that dimension of R 3 is 3. So, if we take any 3 dimension subspace of R 3 it has to be same R 3.

Now, let us consider this b and c part, I am considering this b part, I am leaving it to you this c part. So, I want to consider; what is one dimension subspace of R 3. So, if you look at R 3 vectors will be looking like what. So, vector will be suppose a, b, c kind of thing. So, here you take any arbitrary vector here let us say u, then your vector subspace will be what it is having one it is going to have a basis consisting of only one vector. So, you can take any non-zero vector and that can that may generate a 1 dimension space.

So, here you can say that w has a generated by all v such that v is written as constant times of this u right where k is coming from this R right. So, here this w will give you a 1 subspace of R 3 and if you look at what is this? This is nothing, but a line passing through your origin. So, if you change your k your line will also change, but it will pass through origin similarly you can prove that the part c that 2 dimension subspace of R 3 is nothing, but all planes through the origin.

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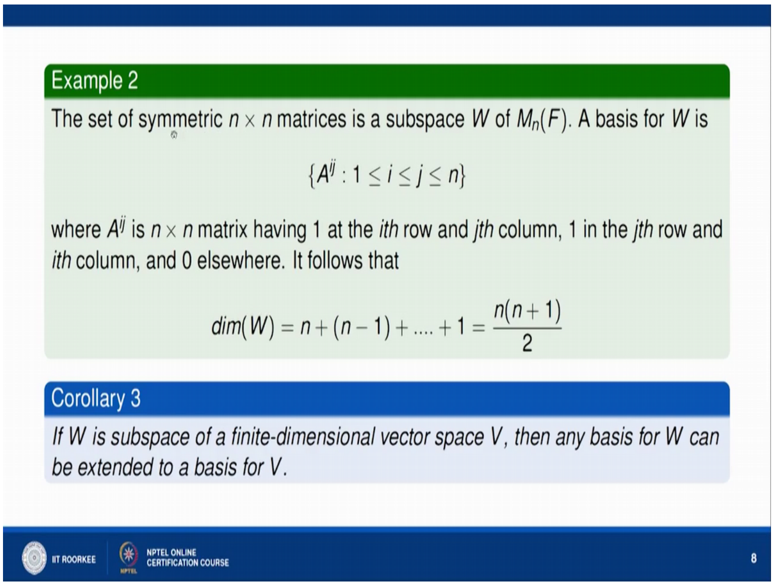

Now, moving to next example, this example consider the set of symmetric n cross n matrices as a subspace of n cross n matrix all matrix and here we can say that proving that this is a subspace is a trivial thing which I am leaving it to you only thing I want to consider here is the that the basis for this vector subspace is given by A i j i and j both are running from one to n and here A i j is define like what here A i j is n cross n matrix having 1 at the ith row and the jth column and 1 in the jth row and the ith column and 0 elsewhere.

So, my claim is that this set will form a basis for vector subspace of all the symmetric n cross n matrix. So, it is like what you consider this A i j? A i j will be what we already know that it is symmetric matrix. So, it means that if we have say ith row and say jth column, if we have one rest for all 0, then we already know that it is symmetric matrix.

So, the same entry will be in ith column say ith column here and jth row rest for all 0 right and our claim is that the dimension of set of all symmetric matrices is nothing, but this n into n plus 1 by 2; how we can show that if you look at any n cross n matrix here, we already know that this is the diagonal entry. So, the property of symmetric matrix says that the value here A 1 2 is same as A 2 1. So, here we can say that A i j is equal to A j i. So, it means that knowledge of A i j will give you the knowledge of A j i. So, it means that how we can calculate the dimension we just look at all the diagonal entries.

So, how many diagonal entries we have we have n diagonal entries. Now if you look at say just above the diagonal matrix how many element we have we have n minus 1 element and similar it will go up to this last entry that is this and this is nothing, but one because if we know this entries then we know the entries here in sub diagonal.

Similarly, if you know entry here, you know the entry here. So, how many element we required, we required if we can calculate this is nothing, but arithmetic progression series and you can say that it is nothing, but n into n minus 1 by 2. So, dimensions of all the symmetric matrix of size n cross of the size n is given by n n plus 1 by 2. So, this is small correction here it is n plus 1 by 2 is that ok.

Now, we recall small result we says that if W is a subspace of a finite dimension vector space v then any basis of for W can be extended to a basis for v. In fact, we have proved one thing that if we have a say linearly independent subset of v then it can be extended to the basis of the whole space now here we consider the W is a subspace of dimension space then the basis of W is nothing, but linearly independent set and this we can extend to the linearly independent set consisting the element of having the number of dimension of v. So, it means that this corollary can be easily proved with the help of the concept that any linearly independent subset can be extended to the basis of the vector space v.

So, using this let us prove the important result which is given as that let W 1 and W 2 be 2 subspaces of n dimension vector space, then dimension of W 1 plus W 2 is equal to dimension of W 1 plus dimension of W 2 minus dimension of W 1 intersection W 2. So, proof of this theorem will depend on this part W 1 intersection W 2.

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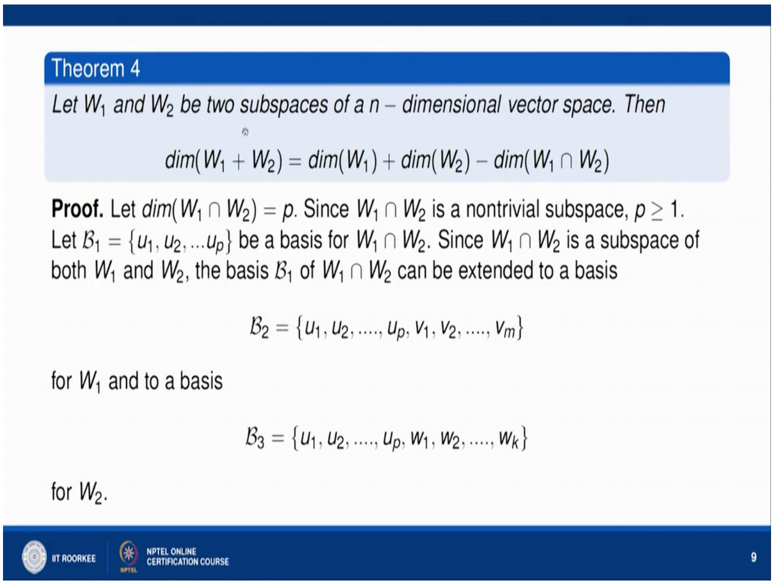

So, let us assume this that W 1 intersection W 2 is basically intersection is nothing, but trivial space trivial vector space or non trivial vector subspace.

So, first I am just going to discuss the non trivial thing that suppose one of this W 1 is nothing, but trivial space then in that case W 1 plus W 2 is nothing, but W 2. So, if W 1 is trivial then W 1 plus W 2 is nothing, but W 2 and this statement is trivially true. Similarly, if we assume that W 2 is a trivial space means it is a single ton 0, then again W 1 plus W 2 is nothing, but W 1. So, again this statement is trivially true. So, I am assuming that that both W 1 and W 2 are not a trivial subspaces. So, it means that both W 1 and W 2 are non trivial subspaces.

Now, again there are 2 possible cases that intersection W 1 and W 2 is trivial subspace or non trivial subspace. So, I am going to present a proof for non trivial case, but the similar proof can be for the case when W 1 intersection W 2 is a trivial vector subspace. So, I am I will hint after the proof of this. So, let us assume that that W 1 intersection W 2 is a non trivial subspace.

So, let us it non trivial subspace means dimension of W 1 intersection W 2 is some p and p is greater than or equal to one. So, when this is a we already that W 1 intersection W 2 is a vector subspace and dimension is p. So, it means that we can find out a basis having p element let us call this as B 1 consisting this p element u 1 to u p and be a basis for W 1 intersection W 2 and we already know that this W 1 intersection W 2 is a subspace of both W 1 and W 2.

So, first let us consider that W 1 intersection W 2 is a subspace of W 1 and recalling this corollary, we says that the basis for subspace can be extended to basis for whole of space v. So, it means that the basis for W 1 intersection W 2 can be extended it to the basis of W 1 call it say B 2. So, B 2 is what we started with basis of W 1 intersection W 2; we extended it to the basis of W 1. Similarly we can extend the basis B 1 to the basis of W 2 which is given by B 3.

So, B 2 is a basis for W 1 and B 3 is a basis for W 2 and with this we claim that the set B which is defined as u 1 to u p v 1 to v m and W 1 to W k is a basis for W 1 plus W 2.

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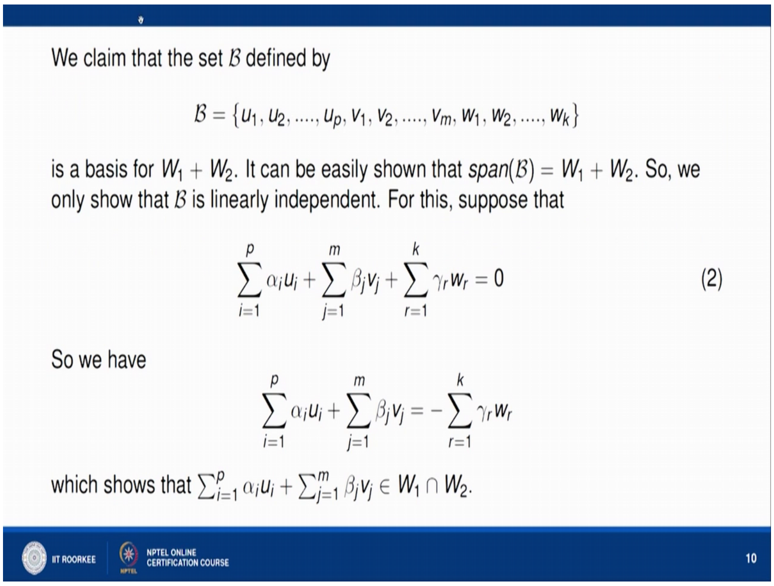

So, here I am assuming that we have a basis B 1 we have a basis B 1 is a basis for w 1 intersection w 2 B 2 is a basis for w 1 B 3 is basis for w k and with this we can form a another basis B having p plus m plus k number of element and we try to claim that it is a basis of w 1 plus w 2.

Now, first thing to prove that is a basis that we have to show the span of B is whole as w 1 plus w 2. So, this is quite easy how we can prove let us look at here. So, here we want to show that this B span the vector space w 1 plus w 2 for that let us take an element of a w 1 plus w 2. So, w can be written as say w 1 plus w 2.

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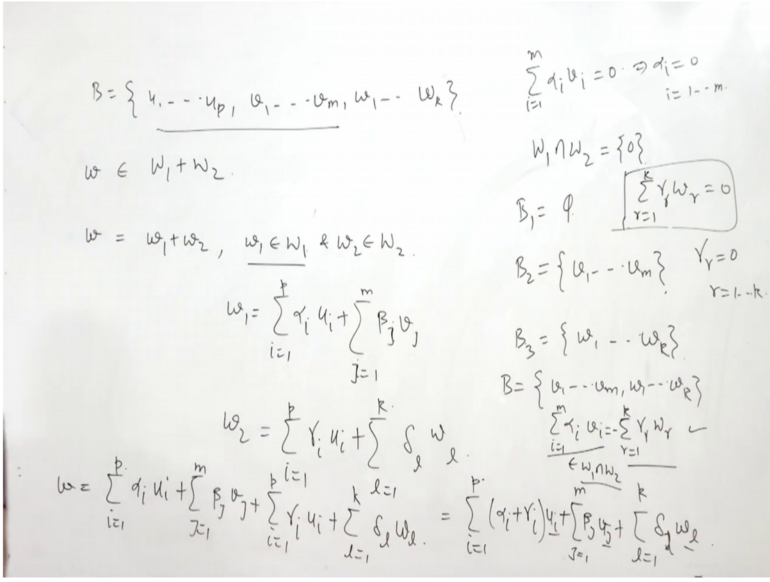

Here w 1 is an element of capital w 1 and w 2 is an element of capital w 2.

Now, since w 1 is a element of w 1. So, it means that w 1 can be written as linear combination of the basis of w 1. Now basis of w 1 is basically this. I think this is already given here. The basis of w 1 is u 1 to u p and v 1 to v m, right. So, we can write w 1 as say alpha i u i, i is from one to p here plus summation beta i; i is beta j j is from one to m v j right. So, w 1 can be written like this similarly since w 2 is an element of capital w 2. So, this can be written as say i equal to one to p say suppose you write it gamma i u i plus summation k equal to one to sorry let me erase it.

So, here we have l equal to one to k and here it is we have delta l and w l. So, using this we can write w S; what w we can write it w 1 plus w 2. So, we can write it i equal to one to p here alpha i u i plus summation j equal to 1 to m beta j v j plus summation i equal one to p gamma i u i plus summation l equal to one to k delta l w l and if we simplify then it can written as summation i equal to one to p.

Now, here coefficient of u i you can write it as alpha i plus gamma i u i plus summation you can write it j equal to one to m beta j u j plus summation l equal to one to k delta l and w l. So, here we can say that this w can be written as linear combination of u i v j and w n. So, that simply shows that this B is spanning set for w 1 plus w 2.

Now, to show that this B is linearly independent; so, we consider a linear combination which gives you 0. So, we try to find out that all these constant alpha i beta j and gamma r are nothing, but 0. So, to prove this what we try to do here we shift this summation r equal to one to k gamma r w r to the right hand side. So, it means that i equal one to p alpha i u i plus j equal 1 to m beta j u v j is written as minus r equal to one to k gamma r w r if you look at this is what this is this simply says that this left hand side element can be written as linear combination of right hand side.

Now, this w is is basically what w i is a basis element of w 2 that is given here that w i is a basis element of w 2. So, it means that this can be written as a element of w two, but if you look at this is what this is linear combination of u i and v j. So, it is the element of w one. So, it means that we can say that this is the element of w 1 and w 2 both. So, it means that summation i equal to one to p alpha i u i plus summation j equal one to m beta j v j is an element of w 1 intersection w 2.

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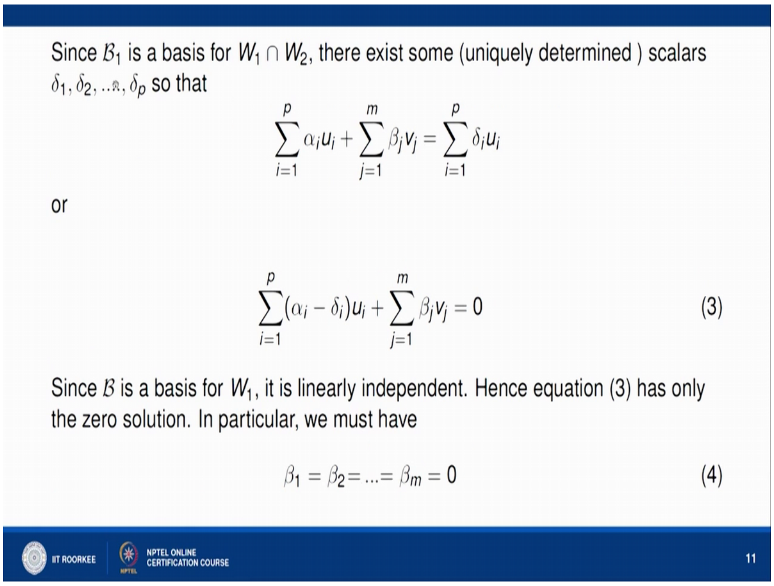

So, using this and we already know that basis of w 1 and w 2; w 1 intersection w 2 is given by b 1. Now using this, we can write down this as linear combination of u i. So, summation i equal to 1 to p alpha i u i plus summation j equal to 1 to m beta j v j can be written as some linear combination of u i. So, we can find out some scalars delta i such that it can be written as i equal to one to p delta i u i when you simplify, you can take it this side and we can simplify that i equal to 1 to p alpha i minus delta i u i plus j equal to 1 to m beta j v j equal to 0.

So, this equation number 3 gives you the representation of 0 in terms of u i and v j, but we already know that u i and v j forms a basis for w 1. So, it means that 0 will have only trivial representation in terms u i and v j it means that your beta j must be 0 and alpha i delta i is equal to 0. So, using that beta j is equal to 0 means we have beta 1 beta 2 equal to beta m all are equal to 0. So, u utilizing this information we can go back to our equation number 2 and we have this relation.

So, let me go back here. So, using 4 in equation number 2; so, all beta j are simply 0. So, we can write i equal to 1 to p alpha i u i plus r equal to one to k gamma r w r equal to 0. So, it means that 0 has a representation in terms of u i and w r, but again this u i and w r forms a basis for w 2. So, 0 must have only trivial representation in terms of u i and w r. So, it means that all alpha is are 0 and gamma r are 0. So, this implies that if this equation is true, then all these constant alpha i beta j and gamma r all are 0.

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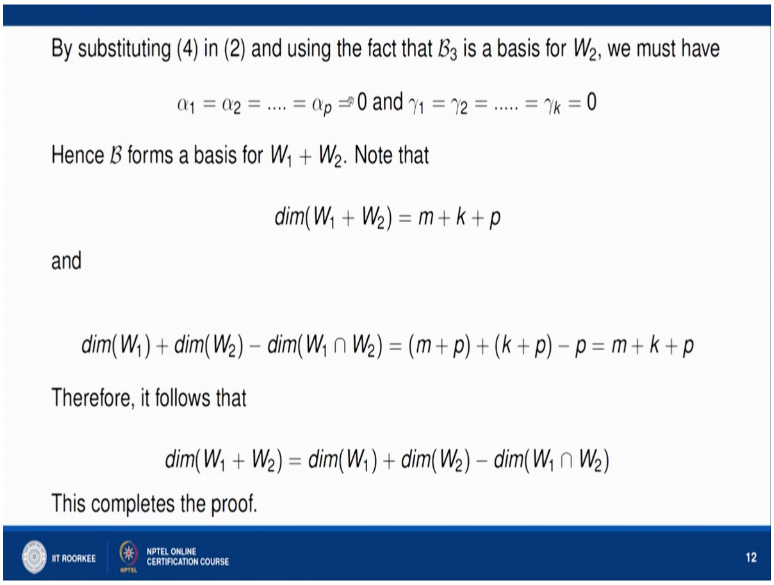

So, that proves that that proves that b forms a basis for W 1 plus W 2 now just look at the dimension of number of element in this b number of element in this B is your m plus k plus p. So, dimension of W 1 plus W 2 is nothing, but number of basis element and a basis. So, this can be written as m plus k plus p. So, this can be written as dimension of W 1 plus dimension of W 2 minus dimension of W 1 intersection W 2 why because dimension of W 1 is what number of basis element W 1 that is m plus p dimension of W 2 is what number of basis element in W 2 that is k plus p and dimension of W 1 intersection W 2 is nothing, but p. So, m plus p plus k plus p minus p is nothing, but m plus k plus p.

So, we can say that dimension of W 1 plus W 2 is equal to dimension of W 1 plus dimension 2 minus dimension of W 1 intersection W 2 this proves the theorem here. Now as I promise that we will also discuss the case when W 1 intersection, W 2 is a trivial subspace. So, it means that here dimension of W 1 intersection W 2 is nothing, but 0.

So, in this case we have a no basis for W 1 intersection W 2. In fact, empty set is a basis for W 1 intersection W 2 and rest will follow in a same way. So, we can say that B 2 which will/is given in this form will now nothing, but B 2 is only consisting B 1 to B m and similarly B 3 can will be W 1 to W k this I am telling when W 1 intersection W 2 is a trivial vector space. So, B 2 will consist only v 1 to v m let me write it here

So, the case that W 1 intersection W 2 is basically 0 single term 0 here then basis. So, B 1 is basically nothing, but empty set B 2 is going to be only say v 1 to say v m and your B 3 is basically W 1 to W k and we try to prove the that basis B; basis B is nothing, but v 1 to v m and W 1 to W k. So, we are going line by line and we try to claim that this B is a basis for W 1 plus W 2. So, that span of B is full of W 1 plus 2 W 2 follows from this right and this you can easily proof the only thing is difficult part is set B is a linearly independent set. So, to prove that it is a linearly independent set you simply write it that summation of alpha i v i; i is from one to m plus summation say r equal to one to k say gamma r W r is equal to 0.

So, this is true. So, this implies that i can write it like this that you can take it the other side and this is this can be written as equal to minus of this right. So, here this is an element of W 1 and this is an element of W two, but this element of W 1 is equal to some linear combination of basis element in W 2. So, it means that this both the elements will belongs to W 1 as well as in W 2.

So, you can say that this element belongs to W one. So, this element belongs to W 1 intersection W 2 and we already know that W 1 intersection W 2 is nothing, but trivial vector subspace that is 0. So, it means that you can write it summation of alpha i v i, i equal to one to m equal to 0.

So, now this implies that all alpha i are 0 because we know that v 1 to v m forms a basis. So, this implies that all your alpha is are 0 i equal to 1 to m. So, utilizing this value here we can use this again and this says that we have summation gamma r W r; r equal to one to k equal to 0, but as soon as we have this result we all recall that B 3 is a basis. So, it means that all gamma r is also equal to 0 for r equal to one to say k. So, here we can also prove that this B is a linearly independent set and span that B is a spanning set for W 1 plus W 2 will be followed in a similar way.

So, it means that in that case also when W 1 intersection W 2 is a trivial space the similar kind of proof can be produced. So, this theorem we can write it like this that is W 1 and W 2 be 2 subspaces of n dimensions of dimensional vector space then dimension of W 1 plus W 2 equal to dimension of W 1 plus dimension of W 2 minus dimension of W 1 intersection W 2.

So, with this we finish the proof of this theorem.

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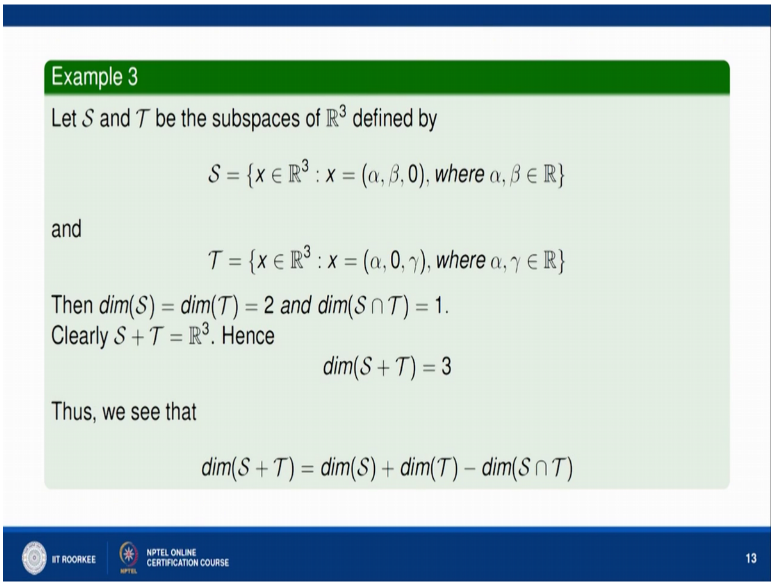

And let us take every small example of this proof this theorem. So, here we consider 2 subspace of R 3. So, one is subspace of R 3 is S where elements of S is given by alpha comma beta comma 0 and T and element of T is given by alpha gamma 0 gamma delta.

So, here we want to verify our theorem. So, if you look at the S vector subspace S, then the dimension of S is given by 2 and in fact, your basis element we can take as 1 0 0 and 0 1 0 that can be given by since we have alpha beta both can be taken as free variables.

So, we take alpha as 1 beta as 0 and we can change role here you can take 0 1 0. So, basis element for S is 1 0 0 and 0 1 0. So, that makes the dimension of S as 2 similarly look at the basis of T basis of T can be given at 1 0 0 and 0 0 1. So, basis element for T will be again having 2 element. So, it means that dimension of T is again 2.

Now, if you look at the intersection of S intersection t. So, intersection of S with T will be what. So, if you look at S; S will have third component 0 and if you look at T that second component is 0. So, if you take any element of S intersection T that must have second element 0 and third element 0. So, only non-zero element in this S intersection T will be alpha comma 0 comma 0 and the basis of S intersection T can be given by 1 0 and 0.

So, here dimension of S intersection T is given by only one because it has only one spanning element that is 1 comma 0 comma 0, I am just giving you standard basis, you may generate any basis for these vector subspaces S T and S intersection T. So, we know dimension of S equal to dimension of T equal to 2 and dimension of S intersection T is equal to 1. So, and we already know that S plus T is nothing, but R 3, if you look at element will be what alpha plus beta plus gamma, right.

So, element of S plus T is nothing, but R 3 elements of R 3. So, dimension of S plus T is going to be 3 and when we have all the data then we can easily see the dimension of S plus T is nothing, but dimension of S that is 2 plus 2 minus one. So, 2 plus 2 minus one is going to be 3 which gives you the dimension of S plus T. So, this verify the proof of the previous theorem. So, here in today’s lecture, we have discuss the coordinate vectors and some properties of dimension of vector subspaces and. So, here I am just going to stop and in next lecture, we will discuss more properties more results of numerical linear algebra.

Thank you very much.