Numerical Linear Algebra

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Lecture - 07
Bases and Dimensions-I

Hello, friends welcome to the lecture and in this lecture we will discuss some concept ah basis and dimension, and if you recall in previous ah lecture we have discuss what do we mean by a basis. So, there we started with a vector space say V define on a Say field F.

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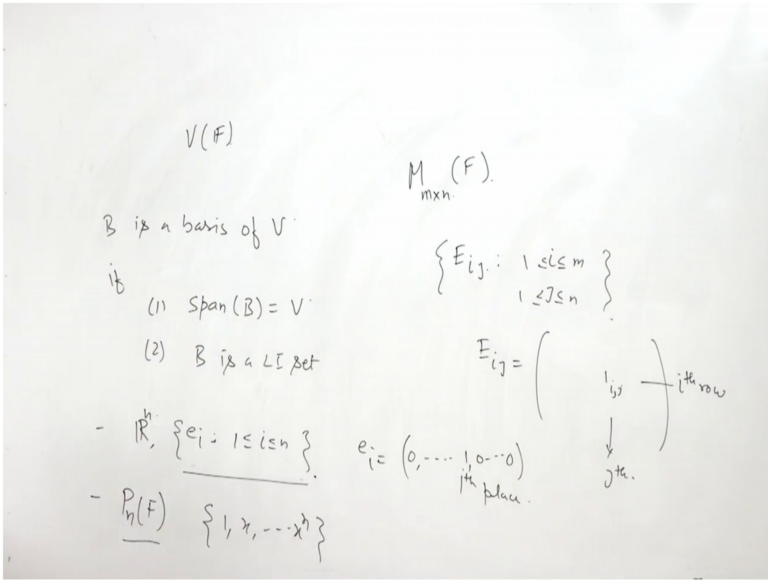

And we say that B is a basis, of say V if 2 things satisfy first thing is the a span of B whole is whole of V, and second thing is that B is a l i set and as an example, we have discussed certain cases. First example, we have seen as set of all R n basically set of all n cross 1 vectors. So, R n and here we have seen that this E i i equal to i is from 1 to n 1 less than i less than n is a basis here E i is nothing, but E i is equal to 0 say 1.

So, here we have at ith place we have 1 so here we have this is ith place, ith place. So, we have seen that that in R n this set is a basis and it is known as standard basis for R n, similarly we have seen another example of set of all polynomials of power of degree not more than F, and here we have seen that set 1 x x n, the set forms a basis for set of all polynomials whose degree is up to n and here we have shown that this act as a basis and.

It is also known as stand basis for this at P n F and we also seen the vector space of matrices of size m cross n defined over this F, and here we have shown that E i j i is from 1 to m and j is from 1 to n, forms a basis for this vector space. So, here E i j is basically what E i j is a matrix whose only nonzero entry is 1 at the position which is i th i j th position. So, at i j th position it is 1 rest all 0, so i j th position means i th row and j th column.

So, this we have discussed in last class now in continuation of this institution, let us consider a next theorem which says that V if let V be a vector space.

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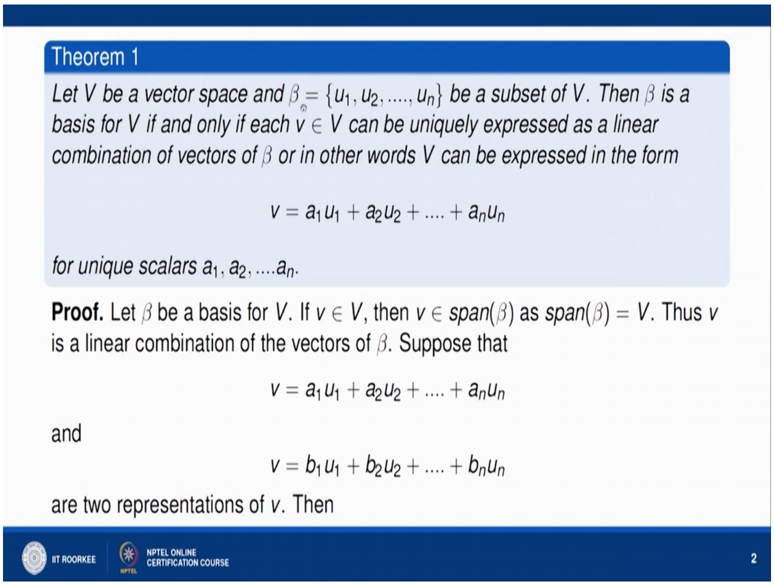

And beta be a subset of V beta consists of these element u 1 to u n, then beta is a basis for V if and only if for each V ah vector in vector space V, we have a unique representation of V or we can say that each vector in V can be uniquely expressed as a linear combination of vectors of beta or in other words V can be expressed in the form v equal to a 1 u 1 plus a 2 u 2 plus a n u n for unique scalars a 1 to a n.

So, it means that if we take any vector in vector space V, then corresponding to this vector we have unique scalars a 1 to a n says that we can be written as a linear combination of these a i u i. So, this we wanted to prove and this is if and only if result. So, first let us prove here that let beta be a basis and then we try to find out that we have a unique representation. So, just assume that beta is a basis and take any element in vector space V, then V belongs to a span of beta because a span of since beta is a basis then a span of beta is whole of V and V is a element of capital V. So, it means that V has to be a element of a span of beta

So, it means that V can be written as linear combination of the vectors of beta now let us assume that we may have more than 1 representation it means that V can be written as summation a i u i i is from 1 to n, and we can also write V in terms of b i u i from i is 1 to n, now we want to claim that these 2 representation is nothing, but the same representation it means that E i is same as b m.

So,, so let us take these 2 representation and we subtract these 2 representation and we can have when we subtract here V minus V is simply 0 and here in the right hand side what we have a 1 minus B 1 times u 1 plus a 2 minus B 2 times u 2 and so on ah.

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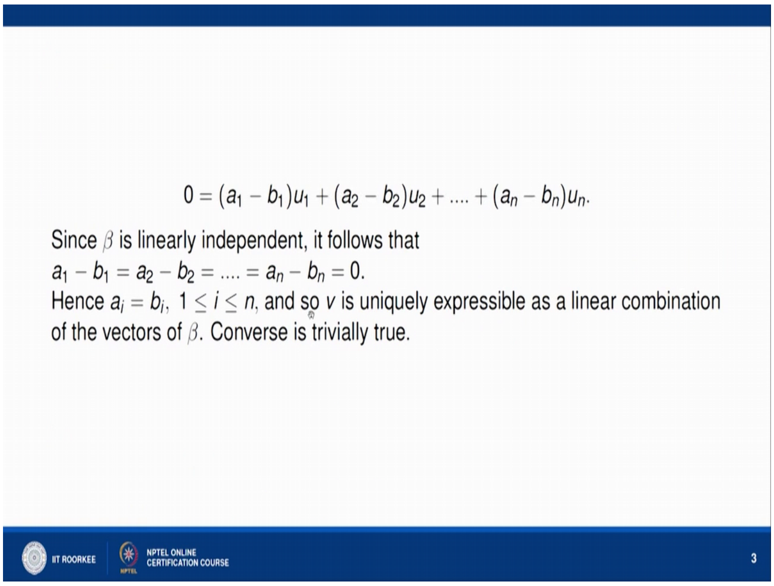

And we can get this kind of equation that 0 can be written as a 1 minus b 1 into u 1 plus a 2 minus b 2 into u 2 and so on. Now this implies that 0 has ah representation in terms of u 1 to u n, now since u 1 to u n forms a basis it means that u 1 to u ends are linearly independent vectors.

So, this implies that that 0 must have only a trivial representation it means what that the coefficient of u is nothing, but 0. So, if coefficient of u is 0 means a 1 is equal to a 1 minus v 1 is equal to 0 a 2 minus v 2 is equal to 0 similarly a n minus b n is equal to 0, and this simply these and equations will give you that a i equal to b i for all i from 1 to n.

So, it means that all constants a i is equal to b i. So, it means that these 2 representation is nothing, but same representation. So, it means that if we have a basis then any vector can be represented in terms of basis vectors, in a unique manner now let us move the converse part. So, converse part is quite trivial, but let us have a small hint here, so, what we want to prove here that if we can any vector V can be written as unique representation in terms of elements of beta then beta has to be a basis here, so now we want to prove that if any vector V.

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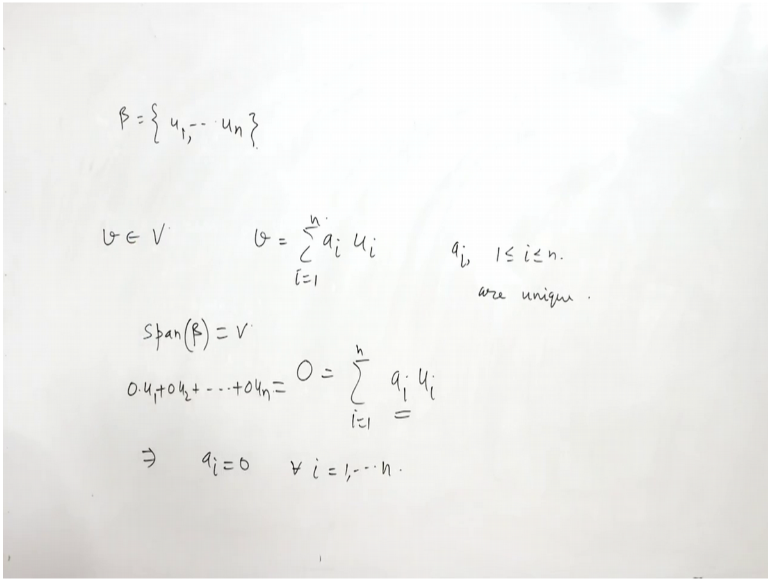

In vector space V has unique representation in terms of elements of this set beta here then this set beta is nothing, but basis for this vector space V. So, here V has a unique representation means V can be written as summation a i u i i is from 1 to n and these a i's are unique a i i from 1 to n 1 less than i less than n all these are unique scalars.

Now to using this we want to show that this set beta is a basis for this, now since it is already given that any vector V can be written as unique in a unique manner in terms of vectors of beta it means that a span of beta is full of v. So, it means that any vector can be written as inner combination of ah elements of beta it means that a span of beta is whole of V, the only thing we want to prove is that this beta is a linearly independent set.

So, for that we start with 0 and we try to find out say 0 as a linear combination of u is. So, here let us say that 0 can be written as sum a i u i i is from 1 to n, we want to show that if this all a is are 0 then we can show that u is are linearly independent. So, this can be easily achieved by saying that this 0 has another representation, representation in terms of u u is that 0 times v 1 plus 0 times u 2 and so on.

Now, we already know that every vector has unique representation means this ah representation is nothing ah different from this representation. It means that if we equate the corresponding coefficient corresponding coefficient has to be 0 0. So, a i has to be 0 for all i ah equal to 1 to see n. So, here this implies that that if we take summation a i u i equal to 0, then all a i has to be 0 which shows that all u is are linearly independent is that.

So, this shows that if every vector V of a vector space V, can be uniquely represented in terms of elements of this set beta here, then beta has to be a basis of this vector space V this is very, very important result in terms of ah in, in terms of forming a basis that if we have a finite generating set. Then ah our vector space V can be generated by a finite set and ah, we can find out a basis from this finite set which is a generating set.

So, here we try to find out that a, a, a, a generating set a finite generating set S can be reduced to find out a basis of this vector space V. So, let us look at the proof of this, so here let us consider the trivial case, so let us say that if S is empty set.

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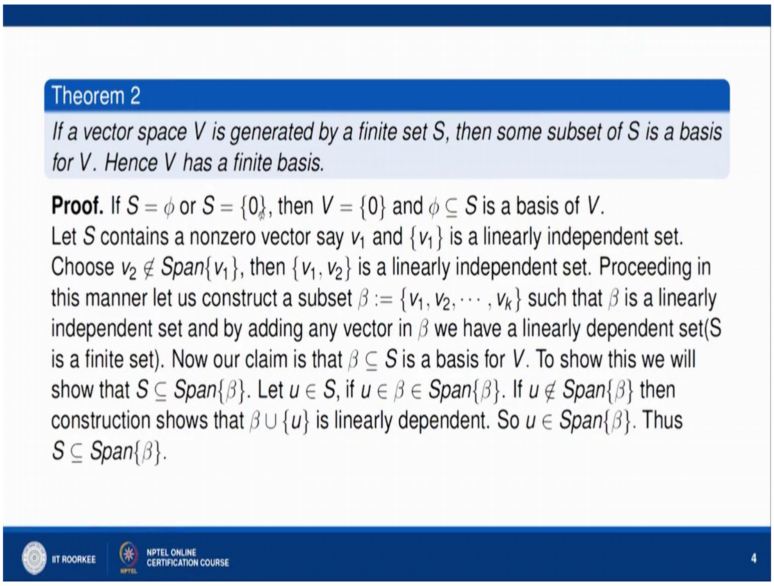

Or singleton 0 in this case a span of S is nothing, but 0. So, it means that since ah V is generated by this is finite set S. So, it means that in this case V is nothing, but a trivial vector space and in this case if V is a trivial vector space then we already seen that in this case our species ah set is nothing, but empty set. So, we can set that embady empty set, which is a subset of this S. In fact, empty set is a subset of each set, so we can say that empty set is a basis of V. So, in case of when S is empty set or singleton 0 set, then we can say that basis of V can be written as nothing, but this empty set.

Now, let us move to nonzero ah non-trivial case, so let us say that let S contains a nonzero vector say V 1, now since V 1 is nonzero, we already know that V 1 is linearly independent set that is clear from the remark given in previous lecture. So, first remark if you remember, then now let us extend this singleton set to the basis of vector space V. So, now let us choose another vector say V 2 of V now this V 2 we are choosing in a way says that it is not belonging to the span of this singleton V 1.

So, it means that V 2 can mean cannot be written as linear combination of this V 1, then way we have already seen that in this case this V 1 union V 2 is a linearly independent set, that is to will from the proof from a theorem which we have proved in a previous class that if we start with a linearly independent set say S. And if we take any element which which is not in S, then S union V is l d if and only if V belongs to span of S.

So, here using that result we can show that if we choose V 2 which is not belonging to span of V 1, then V 1 union V 2 is a l i set. So, ah using this procedure we can proceed and we can find out a set subset beta of set S, consisting V 1 to V k such that beta is linearly independent set. And by adding any vector in beta we have a linearly dependent set, and this we can ah always achieve because S is a finite set.

So, it means that it can go up to this set S now here ah here I am assuming, that this set S is not a l i set, because if S is a l i set it is already a generating set then S is already a basis, we can say that S is already a basis then here we need not to go for them. So, here we are assuming that this S is not a l i set, so it means at s is l d set. So, it means that S is l d set S is a finite set then we can always find out a subset beta of this head says says that if we add any more vector in beta, we have a linearly dependent set this we can get from the previous theorem which we have discussed in previous class.

Now, our claim is that this set beta which is subset of this S is a basis for V. So, how we can prove we already know that this beta is l i by construction, what we need to prove here is that beta is span the whole of V. So, we need to prove that span of beta is whole of V, so to show that ah we already know that the span of S is whole of V. So, if we can show that S is contained in a span of beta then we can say that span of S is contained in span of beta and so can say that V is nothing, but a span of beta.

So, to show that S contained in a span of beta let us take a element of this capital S. So, let us say this u belongs to this S now if u belongs to beta, means if u is already in this set beta, then u will automatically belongs to span of beta. So, nothing ah we need not to prove anything, but if u is not belonging to a span of beta. So, u belongs to S now if this u is already in this beta then it will also belong to a span of beta.

So, ah you can say that S belongs to a span of beta, but if we can find out 1 u which is not in a span of beta. So, it means that u is not in a span of beta, then ah by previous theorem we can say that u union beta is a l i set, but our construction shows that beta union u is a l i l d linearly dependent as we have already constructed our beta in a way, that if we add any more vector it will be a l d set, but by previous theorem we can say that if u does not belongs to span of beta then ah by u taking u along with this beta will give you a a linearly independent set.

So, here we can get a contradiction and this contradiction we are getting because we are assuming that u does not belong to a span of beta. So, u is belongs to a span of beta and we can say that if u belongs to span of beta means S belongs to span of beta it means that a S span of beta is nothing, but whole of space vector space V. So, it means that this beta is a subset of this S, which is a basis for this vector space V.

So, this complete the proof of this theorem and again reiterating the importance of this theorem too here, that if we start with a finite generating set S then every finite, this finite generating set can be reduced to a find out a basis of this vector space V. So, now, let us move to again a very important theorem that is a replacement theorem which says that let V be a vector space.

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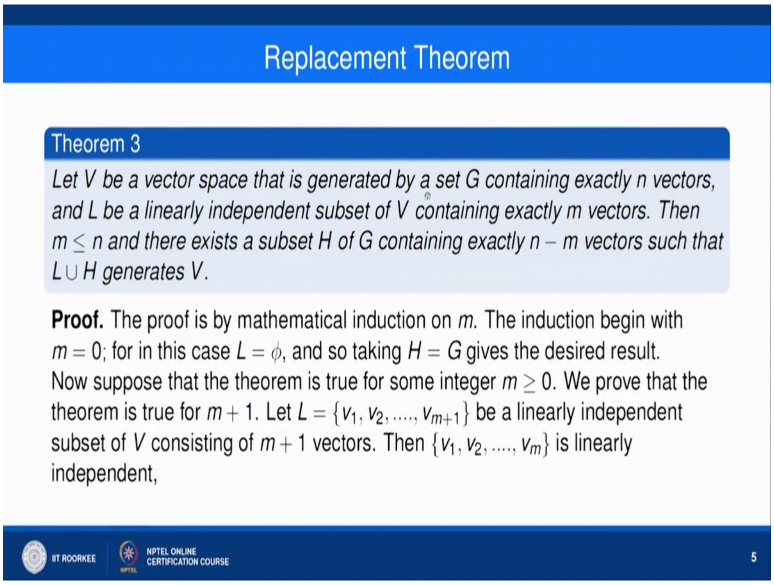

That is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V, containing exactly m vectors then m has to be less than or equal to n what is m here m is number of vectors in linearly independent subset of V. And n is what n is the number of element in a spanning set then m has to be less than or equal to n and, there exist a subset H of G containing exactly n minus m vectors, such that L union that subset H of G generate the whole of V.

Now, this can be understood in a way ah, if you look at the previous theorem previous theorem says that a finite generating set can be reduced to a basis. Now this theorem says that we start with a linearly independent subset of V and we can extend it to the basis of a vector space V. So, previous theorem says the construction of basis from a generating set, and this theorem says the construction extension of set a linearly independent set to a basis of a vector space V both are very, very important.

So, the only thing is that from where we are starting, if we are starting from generating set look at the previous theorem and if we are starting from linearly independent ah set then we can look at this a replacement theorem.

So, let us try to prove this theorem the proof is by mathematical induction on m. So, the induction begin with the say m equal to 0, and if we consider the m equal to 0 means the number of element in this L i set is 0. So, in this case we can say that L is nothing, but empty set. So, if L is empty set said we know that empty set is basically linearly independent set, and span of L is nothing, but 0. So, in this case your subset H of generating set G is nothing, but same. So, here we can take H as whole of G. So, it means that phi union with this G will generate whole of V, because it is already given that G is the generating set of this vector V, so which gives the desired result.

Now, let us assume that that theorem is true for some integer m which is positive, we try to prove that theorem is also true for next integer that is m plus 1. So, let us assume this that let L is a linearly independent set, subset of V consisting of m plus 1 vectors, let us say a call these vectors as v 1 to v m plus 1. So, if v 1 to v m plus 1 this if this set is linearly independent, then any subset of this linearly independent subset is again linearly independent that we have already seen.

So, remove this v m plus 1 then we have v 1 to v m and it is linearly independent, then if you look at the the assumption here then this by assumption we can say that this theorem is true for m. So, it means that this m is less than or equal to n, and for this set we can always find out we can find out a subset H of G containing exactly n minus m vector such that this set union that H generate the whole of V.

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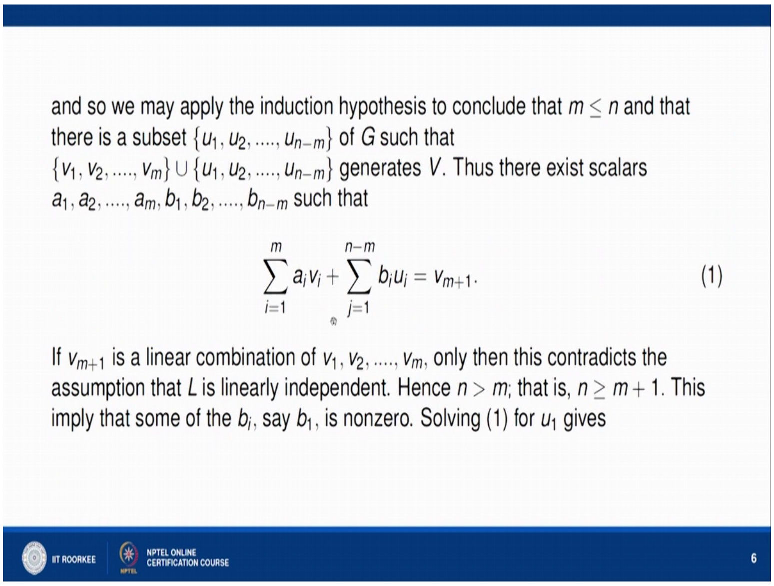

so it means that we may apply the induction hypothesis to conclude that m is less than or equal to n and that there is a subset u 1 to u n and u n minus m of G such that this set union u 1 to u n minus 1 generates the whole of V.

Now, with this we want to ah prove the result for this set L. So, now, since this span the whole of V. So, it in particular we can always find out say constant a 1 to a m b 1 to b n minus 1, such that this vector v m plus 1 can be written as linear combination of v i n u i.

Now here my claim is that there exist at least 1 b i which 1 b j's, which are nonzero because if all b j b 1 to b n minus 1 are 0 it means that v m plus 1 can be written as u linear combination in terms of v i ah, which contradict the fact that l is linearly independent because l is linearly independent means v m plus 1 cannot be written as as a linear combination of v 1 to v m. So, at least 1 of this b 1 to b b 1 to b n minus 1 has to be nonzero, it means that this n minus m has to be positive then only ah some of b j's are non zero.

So, here n is strictly greater than m or we can say that n is greater than or equal to m plus 1 this implies that some of the b i's let us say at least 1 b 1 is non 0 and then we can write down equation 1 in terms of b 1. So, b 1 u 1 we can write it b 1 u 1 in terms of a i v i and v m plus 1.

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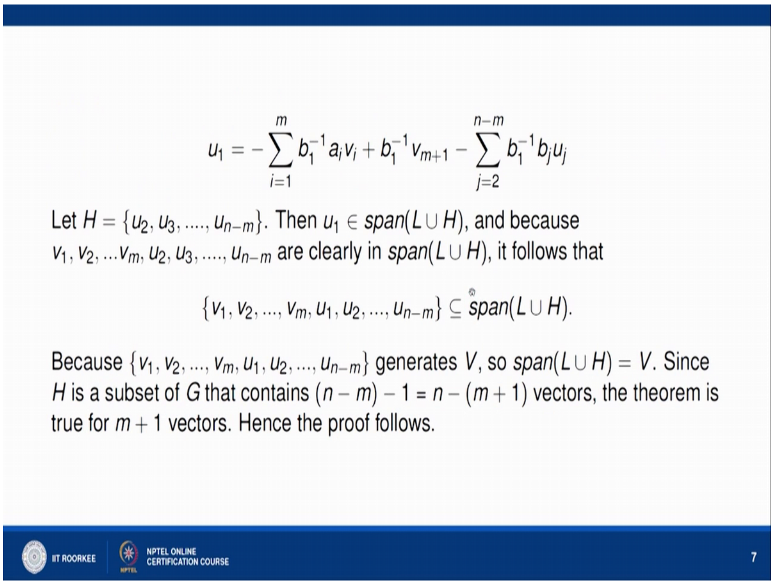

And we can write it u 1 as minus summation i equal to 1 to m b 1 inverse a i v i plus b 1 inverse v m plus 1 minus j equal to 2 to n minus 1 b 1 inverse b j u j

So, it means that u 1 can be written as linear combination of v i v m plus 1 and u j's, and we can say that let us consider h now in this case h is nothing, but remove this u 1 from this set u 1 to u n minus 1 now h is what u 2 to u n minus 1. Then u 1 belongs to span of l union edge l is what l is v 1 to v m plus 1 h is what u 2 to u n minus 1. So, we have shown that u 1 can be written as the linear combination of v i's v m plus 1 and uj. So, we can say that u 1 belongs to a span of l union S and because v 1 to v m u 1 to u n minus m these are already in span of an union H basically these are element of L union H, so this will belongs to a span of H.

So, it follows that v 1 to v m u 1 to u n minus m it also belongs to a span of L union H, since this theorem is true for ah vectors v 1 to v m, then this set generate the whole of v. So, we are using this fact and we are saying that v 1 to v m u 1 to u n minus 1 generates v. So, and this is already contain in span of a L union H. So, it means that v containing a span of L union H. So, we can say that span of L union H is whole of v.

So, since H is subset of G now how many element it contain it contains n minus m minus 1. So, it means it contains n minus m plus 1 vectors and the theorem is true for m plus 1 vectors also, and hence the proof follows.

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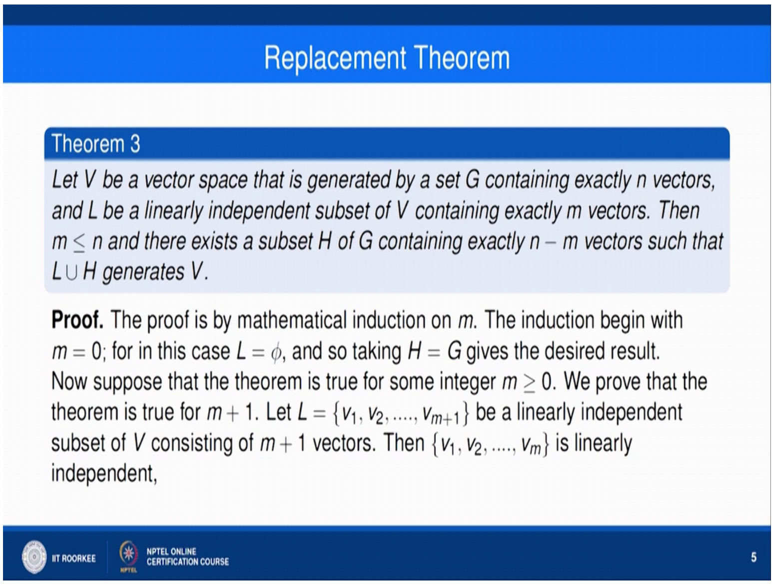

Now, this has a very, very say useful corollaries which we are going to discuss, so first ah corollary is this that.

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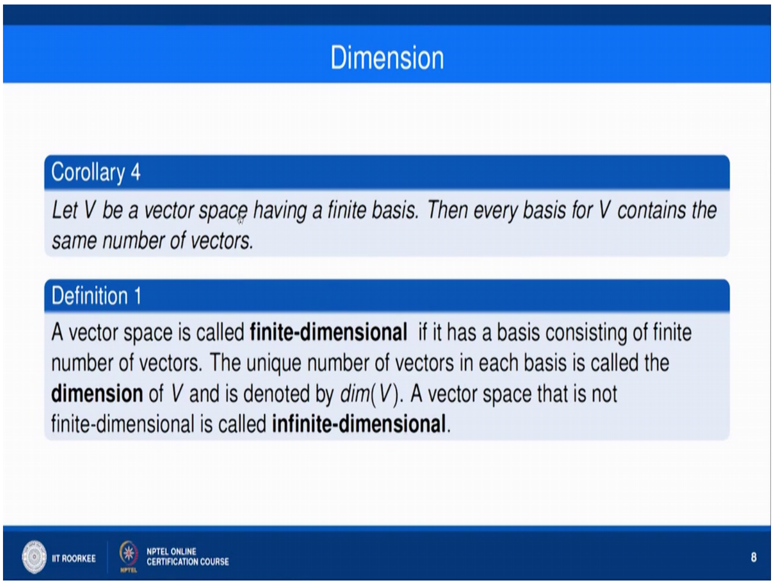

Let V be a vector space having the finite basis then every basis for V contains the same number of vectors let us have a small proof of this. So, what we need to prove here that we have a.

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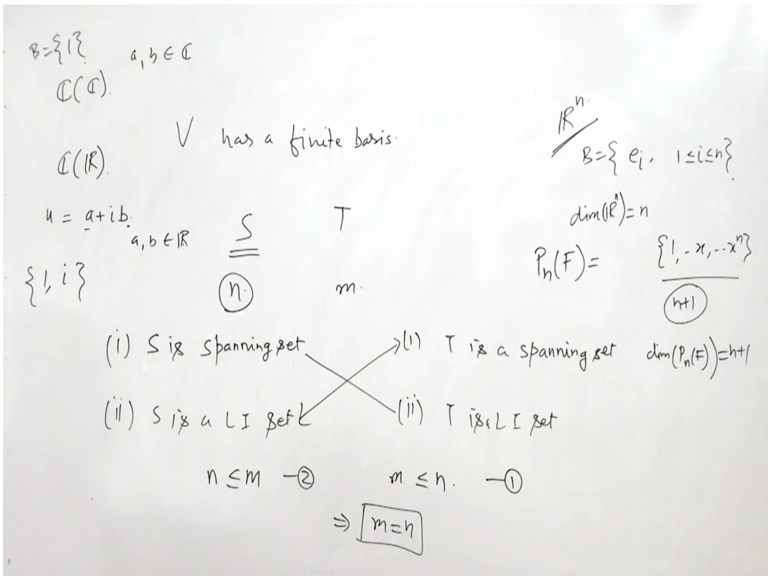

Vector space V and we know that it has a finite basis, has a finite basis, finite basis, what we want to prove here that every basis of this V contain the same number of vectors. So, suppose we have 2 basis say S and T, since it has a finite basis let us consider this finite basis as this S. So, let us take the number of element in this basis is n and consider another basis say T and here the number of element is this m.

So, here since S is a basis it means that S is L I set, and S is a spanning set similarly T is a basis it means T is L I set and T is a spanning set. So, let us consider this as S is a spanning set means spanning set first second thing is that S is l a S is a L I set, similarly we can write down for T. So, that T is a spanning set a spanning set means span of T is whole of V and second is that T is L I set.

Now, please recall the replacement theorem and replacement theorem says that the number of that if we have a vector space V, and a spanning set having ah number less having number n and this n elements here then if we take any L I set, then number of L I independent vectors must be less than number of a spanning elements.

So, it means that here by a replacement theorem if we use these 2 result that if we have a S is a spanning set, means as generate the vector space V and T is a L I set, then the number of ah vectors in T must be less than or equal to number of the element in this spanning set. Now the number of element in L I set is m must be less than or equal to number of the element in this spanning set that is n. So, it means that m has to be less than or equal to n let us call this as equation number 1.

Now, let us consider again the same situation, but this time we use this set of ah things that in this case T is a spanning set, and S is a L I set. Now again use the replacement theorem which says that the number of number of vectors in a L I set, must be less than or equal to number the number of the element in a spanning set. Now here the number of element in L I set is what here the number of element in L I set is n this must be less than or equal to number of the element in a spanning set that is less than equal to.

So, n is less than or equal to m call it 2 now if we look at the both the equation number 1 and 2 this can be possible only when m is equal to n. So, it means that that if V has a finite basis then the number of element in any of the basis must be same, and this shows that that the number of basis element it means that number of basis may not be unique, but the number in each basis has to be unique and this number unique number assigned to this vector space V as a dimension of this vector space V.

So, let us with the help of this let us define the dimension of a vector space V. So, a vector space is called finite dimension if it has a basis consisting of finite number of vectors. And as we pointed out in this corollary 4 the unique number of vectors in each basis is called the dimension of V, and it is denoted by dimension of V.

Now if we do not have finite dimensional vector space then we call this as infinite dimension vector space, examples we have shown seen is R n and here we have seen a basis we have seen as e i 1 less than i less than or equal to n. So, here we can say that the dimension of R n is nothing, but n. So, dimension of R n is nothing, but n similarly dimension of P and F if we have seen that P n F is what P n F is the set of all polynomials whose degree less than or equal to n.

And we have seen that the basis element and basis are what 1 x up to x m. So, here number of element is n plus 1. So, we can say that dimension of P n F is nothing, but n plus 1 is it they are certain more example here ah the trivial example is that.

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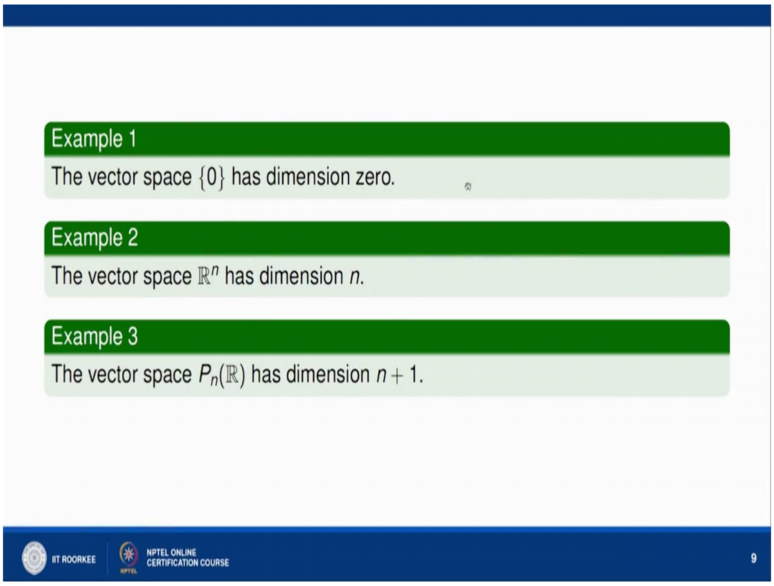

The vector space 0 has dimension 0, so here ah we have defined as the. So, here ah basis set is nothing, but empty set and empty set has no element. So, we can say that vector space 0 has dimension 0 the next example which we just now shown that R n has dimension n. Similarly, the vector space P n R has dimension n plus 1 now P n R means that set of all polynomials whose of degree less than or equal to n, whose coefficients are coming from real number is it ok, so here dimension is n plus 1.

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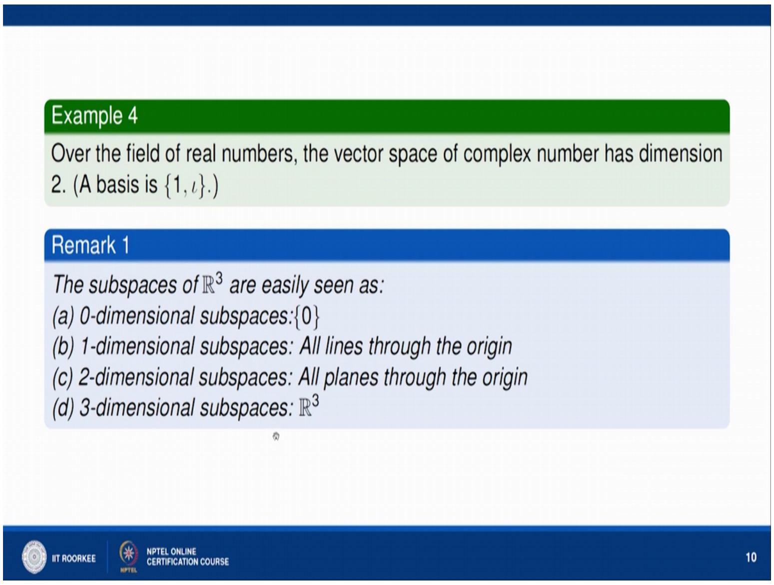

Next let us consider the example 4, which says that consider the vector space complex number and underlying field is real numbers. So, if underlying field is real number then we have basis vectors 1 and i. So, it means that you take any element in say ah C over R. So, C over R means you take any element here. So, let us call this as u here, so you can write and write down a plus i b, now since a and b are coming from where a and B are coming from R right.

So, here basis elements are 1 and i, so we can say that ah this is l i and this can this spend the whole of vector space C. So, we can say this is a basis element. So, here we can say that dimension of C over R is nothing, but 2, but if we consider the vector space C over C. So, it means that now the possibility for a and b that this a and b is coming from c, then here basis element contain only 1 element that is any nonzero element let us take this as 1. So, here we have only singleton set which is spanned the whole of C, and since, since it is a nonzero, so it will be a linearly independent set.

So, here C over C has dimension 1 while C over R has dimension 2. So, ah we can consider this subspace of R 3, and we can say that 0 dimensional subspace is 0 1 dimensional subspace we can consider all lines through the origin 2 dimensions away spaces all placed through the origin and 3 dimensional subspace is nothing, but R 3 will discuss more about this remark in next lecture. So, here we skip we are skipping it and in next lecture we will discuss more about it, and next important ah result is corollary 5 which says that.

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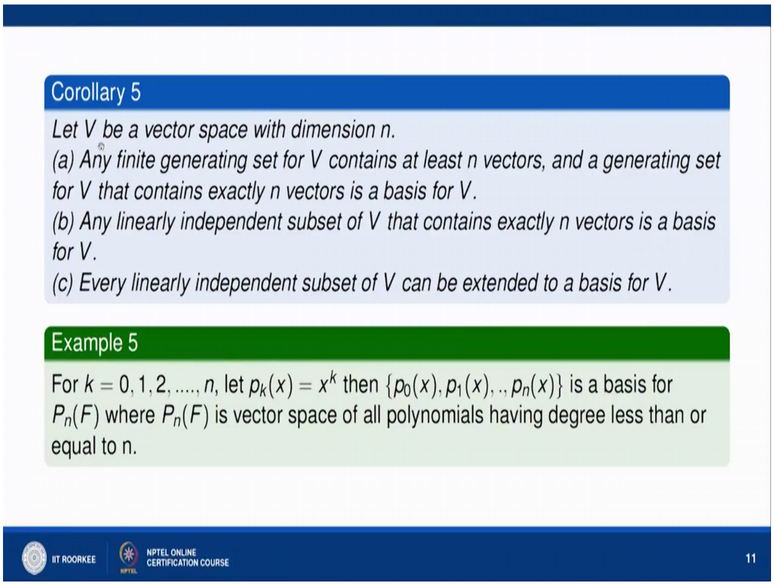

Let V be a vector space with dimension n then any finite generating set for V, which contains at least n vectors and a generating set for V that contain exactly n vectors is a basis. So, it means that if we already know the dimension here then if we can find out a generating set having exactly n vectors that is nothing, but a basis second thing be that any linearly independent subset of V that contains exactly n vectors is a basis for V, and third is that every linearly independent subset of V can be extended to a basis for V.

Now, we can consider as an example of this corollary 5 is this, then since it is L I set having n plus 1 element which is nothing, but the dimension of P n F. So, by second point of corollary 5 we can show that this set is a basis of P n F next lecture we will discuss the concept known as a coordinate of a given vector. So, how this concept of basis will help you to find out the coordinate of a given vector V is it ok. So, we will discuss in next lecture thank you for listening us we will meet in next class.

Thank you.