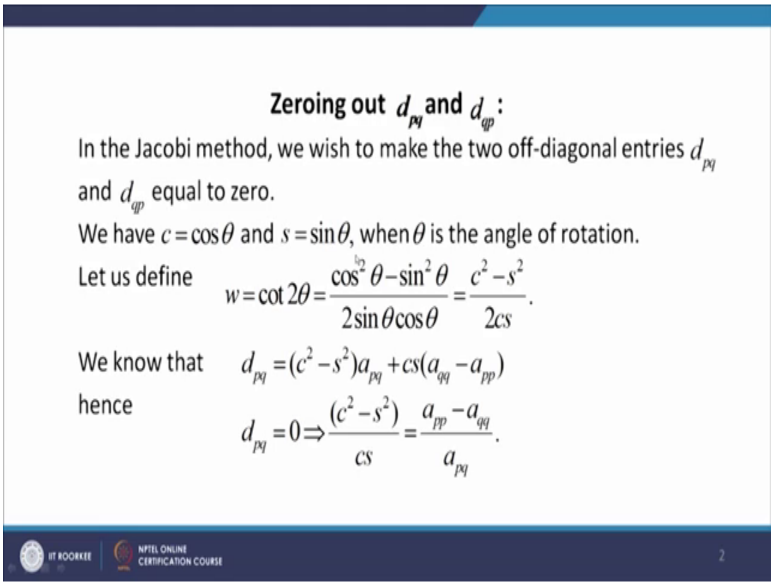
**Numerical Linear Algebra**
**Dr. P. N. Agrawal**
**Department of Mathematics**
**Indian Institute of Technology, Roorkee**

**Lecture – 60**
**Jacobi Method- II**

Hello friends, welcome to my lecture second lecture on Jacobi method.

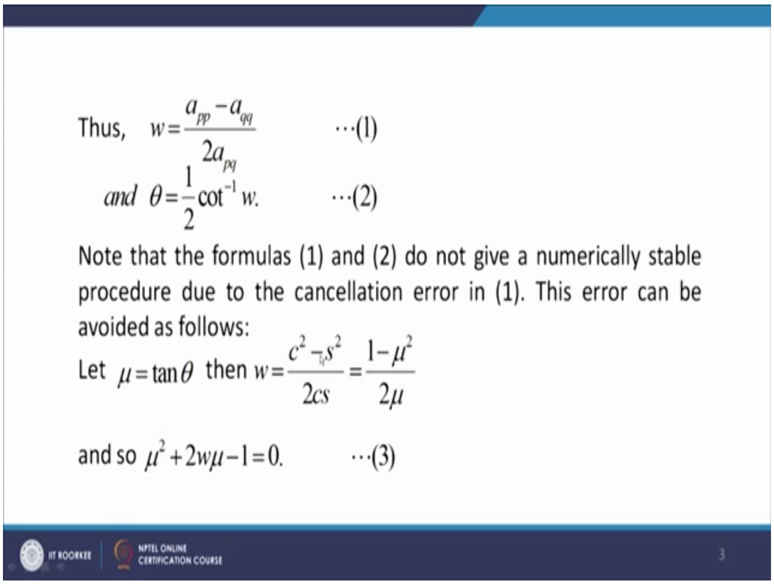
(Refer Slide Time: 00:24)




In the Jacobi method, we wish to make the 2 off diagonal entries, dpq and dqp equal to 0. So, here we have taken c equal to cos theta, and s is equal to sin theta; so where theta is the angle of rotation. Let us define a w equal to cos cot 2 theta which is equal to cos square theta minus sin square theta upon 2 sin theta cos theta ok, cot 2 theta is cos theta cos 2 theta upon sin 2 theta and which is cos square theta minus sin square theta upon 2 cos 2 sin theta cos theta. And so, this is c square minus s squared divided by 2 cs.

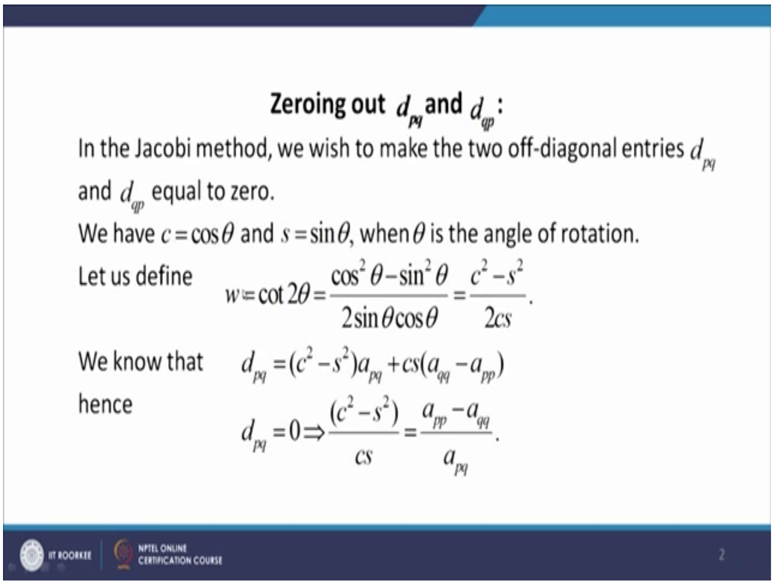
And we have seen in the previous lecture, that dpq the p pth row, and qth column entry in the matrix D ok, is equal to c square minus s square into pa pq plus cs times a qq minus a pp. So, if you put D pq equal to 0, then what do you get here c square minus s square divided by c s, but c square minus s divided by c s is equal to a pp minus a qq divided by a pq. So, let us use this relation to arrive at the value of w ok.

(Refer Slide Time: 01:48)



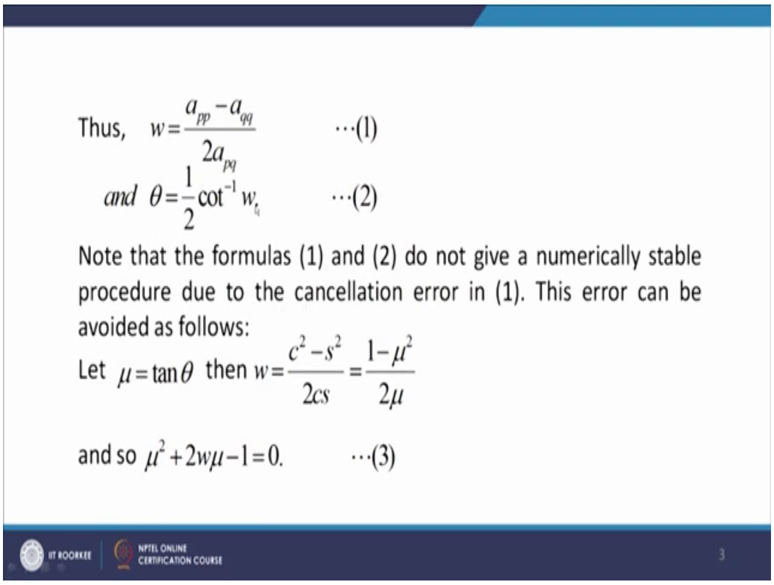

So, w can be written as a pp minus a qq divided by 2 a pq ok.

(Refer Slide Time: 01:58)




And w is equal to cot 2 theta, w is equal to cot 2 theta so, theta equal to 1 by 2 cot inverse w.

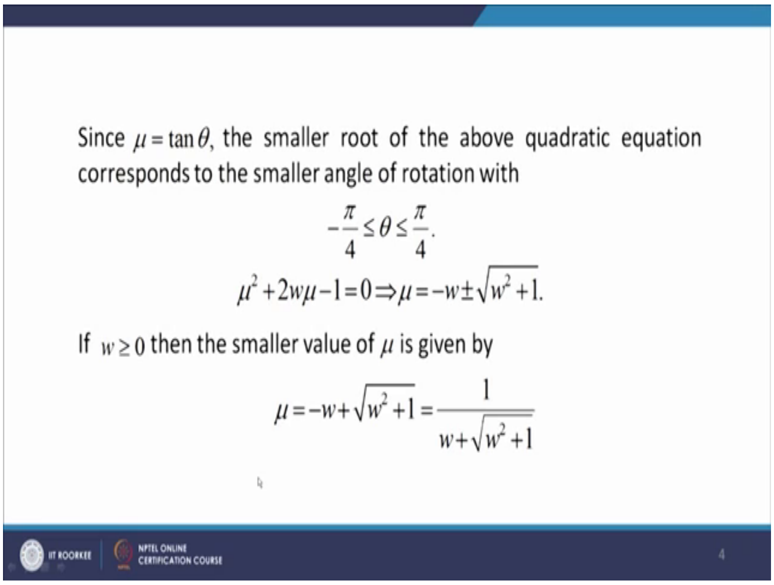
(Refer Slide Time: 02:00)




Now, the formula 1 and 2 do not give a numerically stable procedure ok. Due to the cancellation error, in the equation one here you can see a pp and aqa ua a qq here, there is a cancellation error and so, to avoid the cancellation error because it may lead to very large error in these in the subsequent iterations.

So, to avoid this cancellation error, we what we do is let us define mu equal to tan theta, ok. Mu equal to tan theta means sin theta over cos theta. So, sin theta over cos theta will be equal to s over c. So, mu is equal to s over c, and therefore, w equal to c square minus s square upon 2 cs we can write as 1 minus mu square upon 2 mu. And so, we can write this equation w equal to 1 minus mu square upon 2 mu s mu squared plus 2 w mu minus 1 equal to 0.

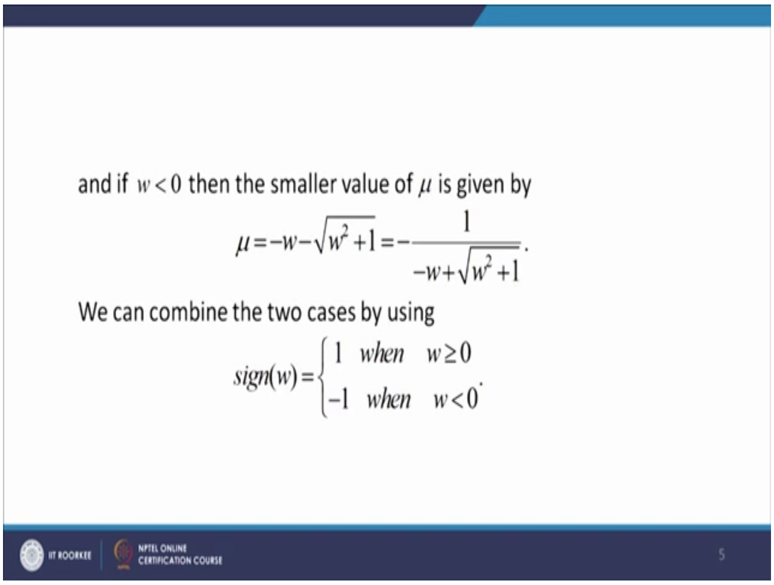
(Refer Slide Time: 03:05)




Now, we can find the 2 roots of this equation mu square plus 2 w mu minus 1 equal to 0, the roots are mu equal to minus w plus minus under root w square plus 1. Now since mu is equal to tan theta, the smaller root of the above quadratic equation. A smaller root of this quadratic equation will correspond to the a smaller angle of rotation. So, minus pi by 4 less than or equal to theta less than or equal to pi by 4, and this will give you mu equal to. So, this equation gives mu equal to minus w plus minus under root w square plus 1.

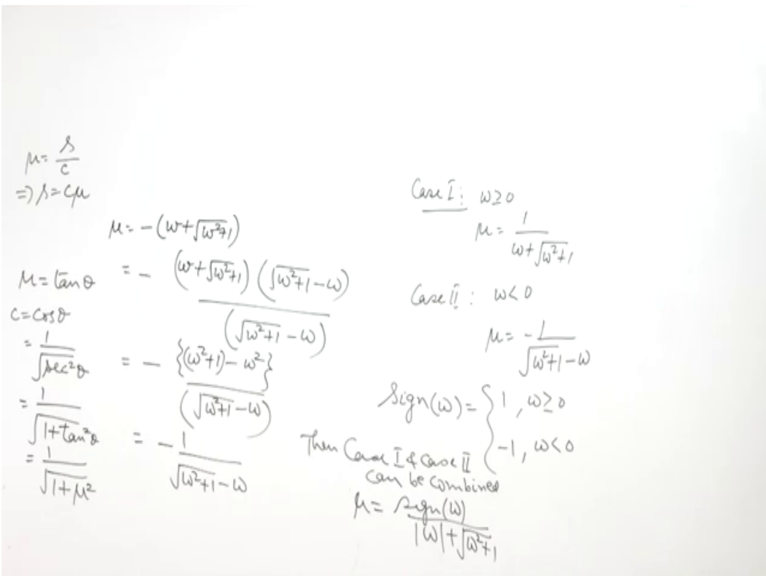
Now, if w is greater than or equal to 0, then the smaller value of w will be equal to mu equal to minus w plus under root w square plus 1 and which we can write also as 1 over w plus under root w square plus 1, by multiplying by w plus under root w square plus 1 in the numerator and denominator. We can write mu like this. So, mu is equal to 1 over w plus under root w square plus 1 it is the case when w is greater than or equal to 0.

(Refer Slide Time: 04:13)




If in case w is less than 0, then the smaller value of w smaller value of mu is smaller value if mu is if w is less than 0, then the smaller value of mu will be given by minus w minus under root w square plus 1. So, mu is minus w minus under root w square plus 1, and this you can write as mu equal to mu equal to minus w plus under root w square plus 1, ok. So, we have this written as minus w plus under root w square plus 1, and we multiplied by under root w square plus 1 minus w and divided by the same value, ok.

(Refer Slide Time: 05:05)

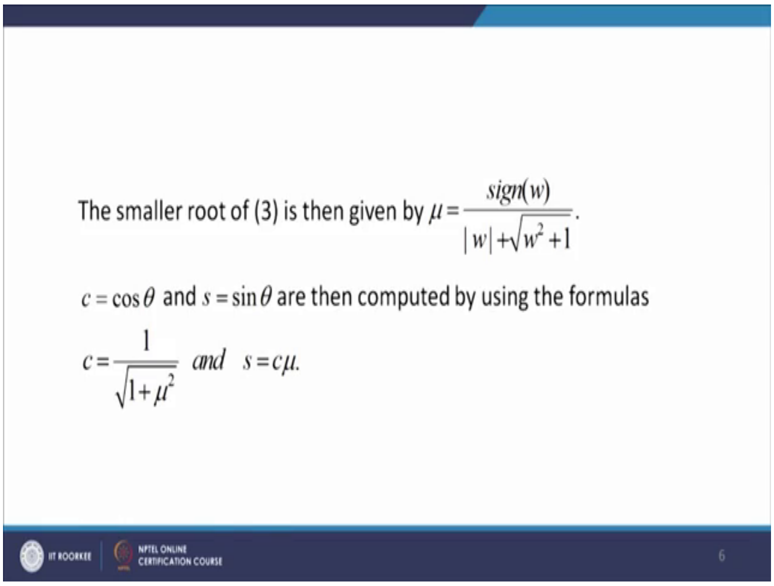


So, what we will have minus we will get the w square plus 1, minus w square divided by under root w square plus 1 minus w. And this will be equal to minus 1 upon under root w square plus 1 minus w. So, this is the value in the case w less than 0.

Now, so, we have 2 cases. Case one is w greater than or equal to 0, then we have mu equal to 1 over mu equal to 1 over w plus under root w square plus 1, or in the case 2, we have w less than 0, then mu is equal to minus 1 upon under root w square plus 1 minus w. So, we can combine the 2 cases by defining the sign function let us define sign function sign w equal to one when w is greater than or equal to 0 and minus 1 when w is less than 0.

So, if we define like this, then case one and case 2 can be combined, we can combine the 2 cases.

(Refer Slide Time: 06:54)

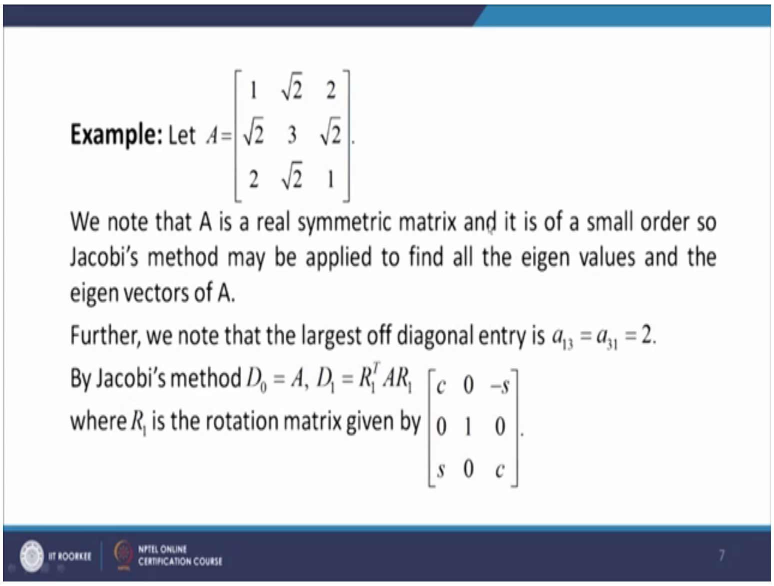



What we do is we can we can then write mu equal to sign w divided by mode w plus under root w squared plus 1. You can see if w is greater than or equal to 0, sign w is one mode w becomes w. So, we get 1 over w plus w square plus 1 and when w is less than 0, sign w is minus 1 and mode of w becomes minus w. So, we get minus 1 upon under root w square plus 1 minus w. So, we can combine both the cases.

And further we have c equal to cos theta, let us say we call that mu is equal to tan theta. We define mu equal to tan theta so, c is equal to cos theta, let us let us write the value of c in terms of mu. I can write it as 1 over under root sec square theta ok. So, this is nothing but 1 over under root 1 plus tan square theta. And which is equal to 1 over under root 1 plus mu square ok. So, c is equal to 1 over under root 1 plus mu square, and we have s is equal to mu was equal to s over c. Mu was equal to sin theta over cos theta, we denoted sin theta by s cos theta by c. So, mu equal to s by c or we can say that s is equal to c mu. So, we have the 2 formulas c equal to 1 over square root 1 plus mu square and s is equal to c mu.

So, sin theta and cos theta are computed that is c and s are computed by using these 2 formulas, c equal to 1 over root 1 plus mu square s is equal to c mu, and mu is computed by this formula.

(Refer Slide Time: 09:05)



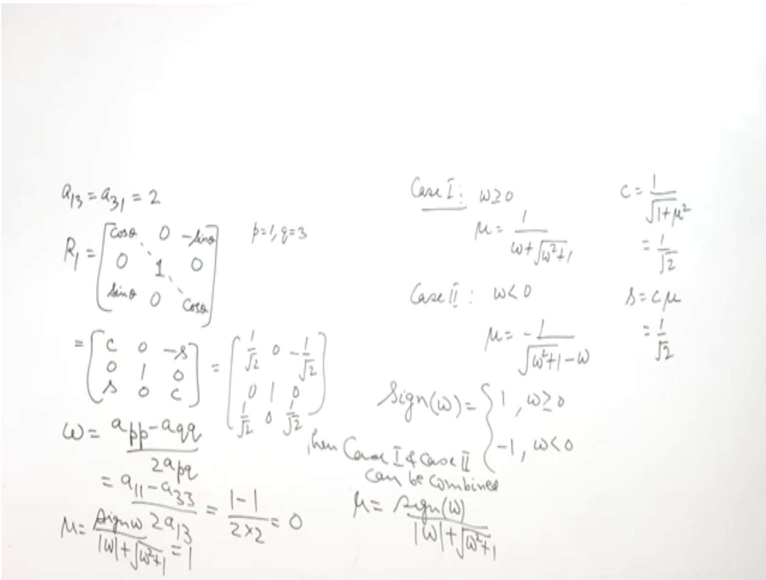

So, let us now take one take some examples, suppose we have s is equal to this A equal to this 3 by 3 real symmetric matrix, ok, 1 root 2 2 root 2 3 root 2 2 root 2 1. So, first of all we have to see that this order of this matrix is small. So, the Jacobi method can be applied to this real symmetric matrix, to get all the eigenvalues and eigenvectors of this matrix.

Now, then we look for the largest off diagonal matrix in magnitude, and we see that off diagonal entries are root 2 2 and root 2 ok. These 2 are simply these 2 are by isometry they are a ap ap q equal to a qp. So, if you see off diagonal entries then this metri entry which is A in the first row and third column, this is the numerically largest off diagonal entry.

So, let us say that the largest off diagonal entries a 1 3 equal to a 3 1, we are denoting the entries of a by a ij. So, a 1 3 is 2 and a 3 1 is also 2, ok. So, a 1 3 equal to a 3 1 equal to t is equal 2. Now by Jacobi's method D naught is equal to A, and D 1 is equal to R 1 transpose AR 1. Now let us write the matrix R 1.

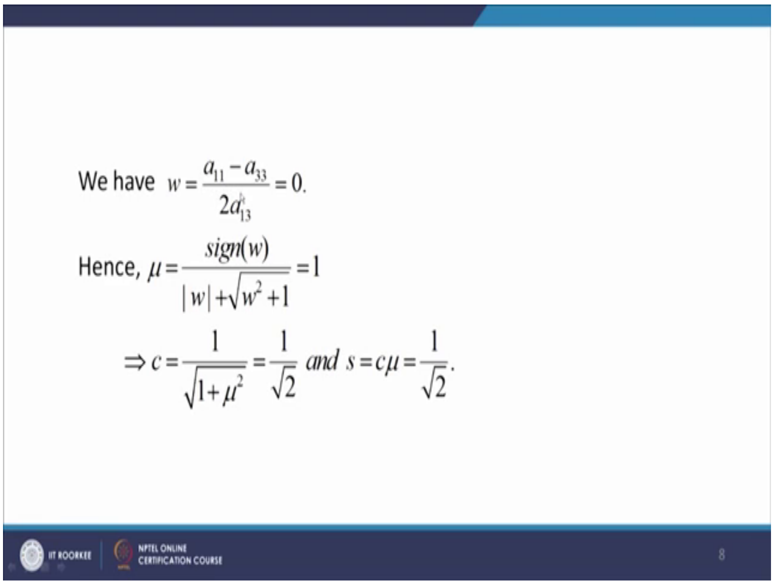
Let us see how we write the matrix R 1. So, a 1 3 is equal to a 3 1 is equal to 2. This is numerically largest off diagonal entry in the given matrix. So, we have we will write the first given matrix R 1, ok. So, what we do? In the one 3 position, in the one 3 position we take minus sin theta. And in a 3 will one position we take sin theta ok. In the if this is pq-th position in the pq-th position, we take minus sin theta in the qp-th position we take sin theta and in the same row same column, in this first row first column this is first row third column ok, here this is first column third row.

(Refer Slide Time: 11:29)



So, this is cos theta and this also be take as cos theta. And then this is column, but this column so, column in the other entries in the column are taken as 1, and the remaining off diagonal entries are taken as 0. So, this is how we write R 1 matrix. So, this can be written as c 0 minus s 0 1 0 s 0 c so, this is our R 1 matrix.

(Refer Slide Time: 12:13)

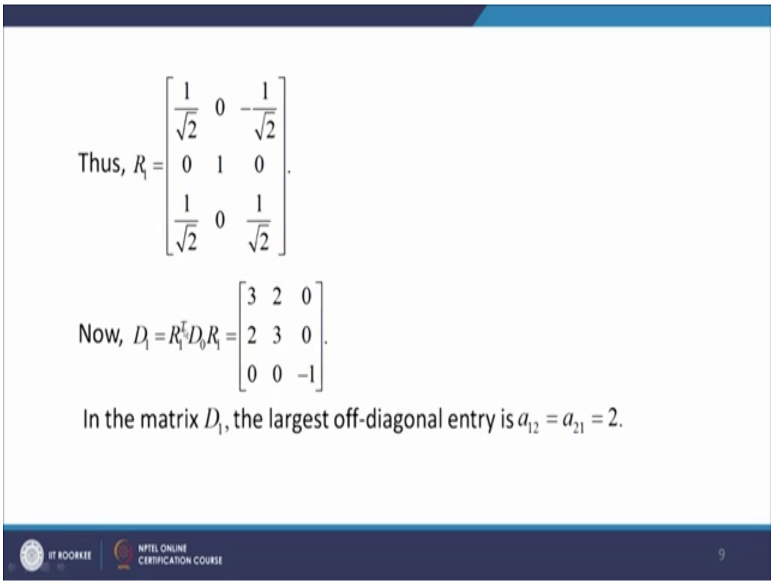



Now, then we determine the value of w is equal to a pp minus a qq divided by w w equal to a pp minus a qq divided by 2 a pq, ok. So, this is p this is q p is equal to 1, q is equal to 3. So, we have a 1 1 minus a 3 3 divided by 2 times a 1 3. Now a 1 1 a 1 1 let us see what is a 1 1 in the given matrix. A 1 1 is equal to 1 a 3 3 is equal to 1. So, we get 1 minus 1 divided by 2 times a 1 3 that is 2 into 2. So, we get 0. So, w is equal to 0 and then mu mu is equal to sign w, mu is equal to be upset that when w is greater than or equal to 0, sign w is equal to 1. So, this is 1, this is 0 this w is 0 so, we get one here. So, we get mu equal to 1.

When mu is equal to 1, c is equal to 1 upon under root 1 plus mu square, ok. So, c is equal to 1 by root 2, s is equal to c into mu. Mu is equal to 1 c is equal to 1 by root 2. So, s is equal to 1 by root 2. And therefore, what do we get here; 1 by root 2 0 minus 1 by root 2 0 1 0 and then 1 by root 2 0 1 by root 2.

So, this is the first given symmetrics, ok. We then find out R 1 transpose AR 1. So, this is R 1 matrix, and then we find D 1, D 1 equal to R 1 transpose D naught R 1 D naught is a matrix a R 1 and what do we get 3 2 0 2 3 0 0 0 minus 1.

(Refer Slide Time: 14:27)

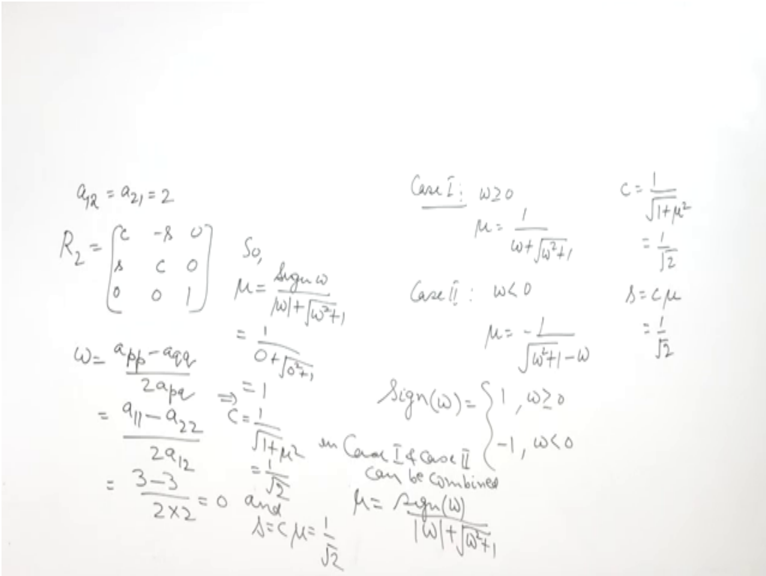



So, the element in the pq-th position, that is first row third column and here the corresponding element in the third row first column, we can see that in the D 1 matrix they have become 0s.

Now, in this matrix D 1 let us find the largest off diagonal entry. And numerically largest entry so, we can see that off diagonal entries are 2 0 0 ok. So, this 2 is the numerically largest entry here in the off diagonal entries. So, and it occurs in the first row and second column. So, a 1 2 is equal to a 2 1 equal to 2.

Now, let us define R 2 let us my find the matrix R 2 so that in D 2 first row and second column entry becomes 0.

(Refer Slide Time: 15:34)




So, we have a 1 2 is equal to a 2 1 is equal to 2, ok. Here a 1 2 in this a 1 2 a 2 1 described the first row and second column entry. In the D 1 matrix a 2 1 described the entry in the D 1 matrix which occurs in the second row and first column.

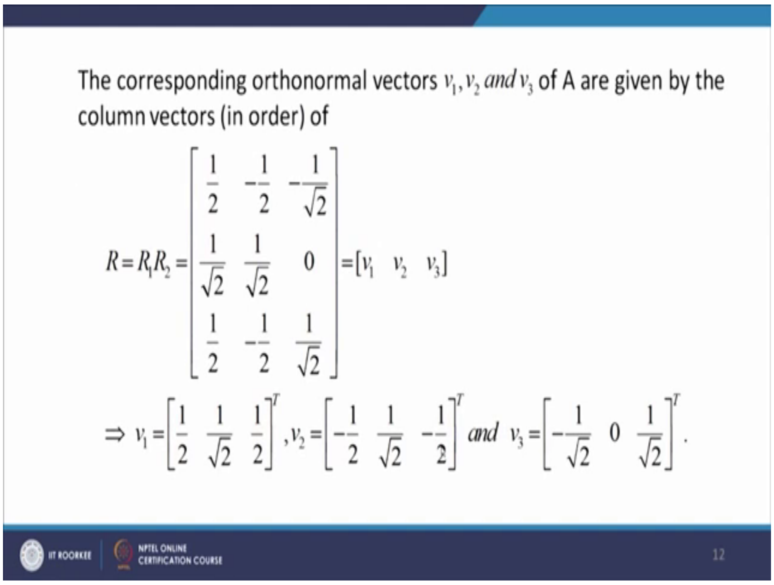
So, what do we do? So, let us write R 2 matrix, R 2 matrix is a in the first row second column. We take minus sin theta; that is, minus s first row second column ok, then second row first column I take s ok. Here I take c here I take c. Then here 0, here 0, here 0, here 0, here 1. These two are last the other one is 1, all other diagonal entries are 1, and all other off diagonal entries are 0 so, this is R 2.

Now let us see what did the value of w w is equal to a pp minus a qq divided by 2 times ap q. In the matrix, D 1 let us see what is a 1 1. In the matrix D 1 a 1 1 is 3 minus a qq, a qq is equal to qq. So, we get a 2 2 minus 2 times a 1 2 ok. A 1 1 is equal to 3 a 2 2 is also equal to 3 in D 1, divided by 2 times 2 so, we get 0 ok.

So, again w is comes out to be 0, and then we find mu. So, mu is equal to again let us recall that mu is equal to sign w divided by mode w plus under root w square plus 1 ok. W is greater than or equal to 0 so, sign w is equal to 1. So, 1 divided by 0 plus under root 0 square plus 1 so, I get one. So, mu is 1, so c is equal to s is equal to c mu, c is equal to 1 by root 2 mu is equal to 1 so, 1 by root 2 ok. So, let us put the values of c and minus c and s in the matrix R 2, and we get R 2 equal to 1 by root 2 minus 1 by root 2 0 1 by root 2 minus 1 by 1 by root 2 0 0 0 1. And then we find D 2, D 2 is R 2 transpose D 1 R 2.

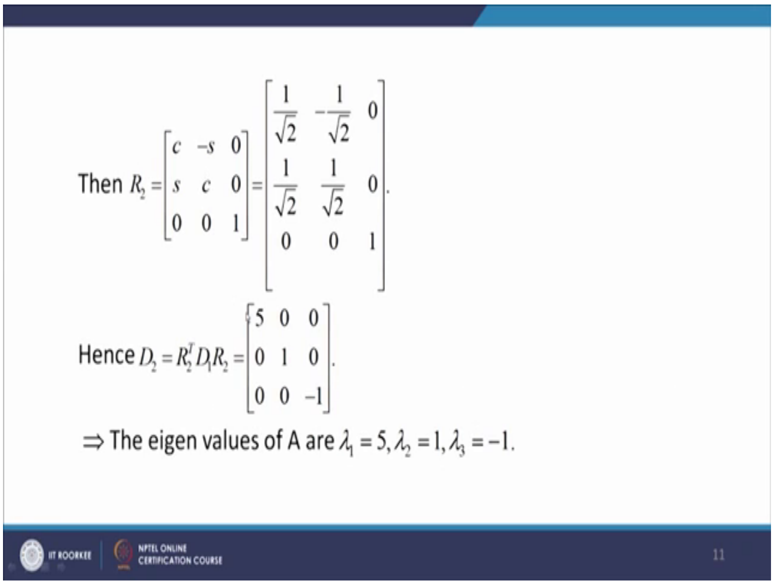
And you can see that in the D 2 matrix the element in the first row second column, and the element in the second row first column have become 0's, ok. And the matrix now that we get, ok, it is a diagonal matrix. In the diagonal, we should we have the eigenvalues of the matrix A ok. So, the eigenvalues of the matrix A are 5 1 and minus 1.

(Refer Slide Time: 19:07)



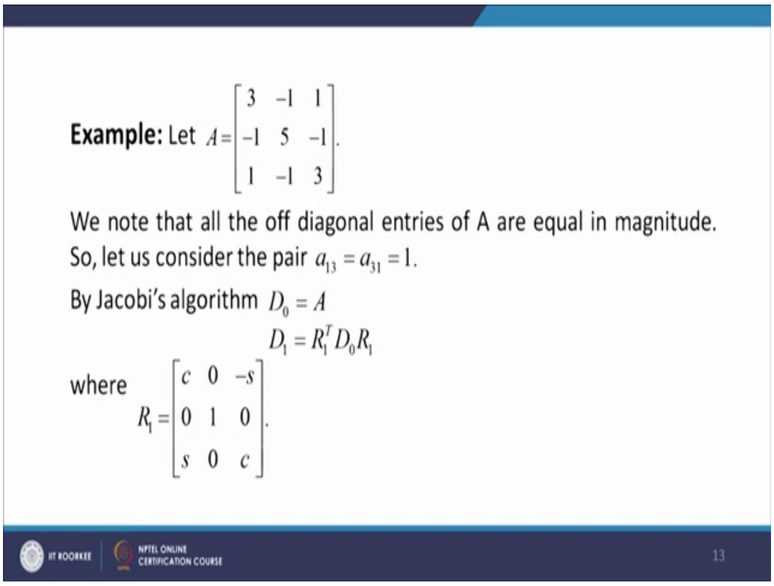

Now, let us find the corresponding eigenvectors. The eigenvectors, orthonormal eigenvectors v 1 v 2 v 3 of A are given by the matrix R which is equal to R 1 into R 2 ok. So, R equal to R when you multiply R 1 and R 2 matrix, you will see that R 1 into R 2 comes out to be this matrix ok; so first column 1 by 2 1 by root 2 and then 1 by 2, ok. This first column is the first orthonormal vector, first vector v 1 corresponding to the eigenvalue lambda 1, lambda 1 means 5, ok.

(Refer Slide Time: 19:45)




And then v 2 is minus 1 by 2 1 by root 2 minus 1 by 2. It is the eigenvector v 2 corresponding to the eigenvalue lambda 2, which is one. And then v 3, minus 1 by root 2 0 1 by root 2, this is eigenvector v 3 it corresponds to the eigenvalue minus 1. So, we have found all the eigenvalues of the matrix A and the corresponding eigenvectors for the given real symmetric matrix A.

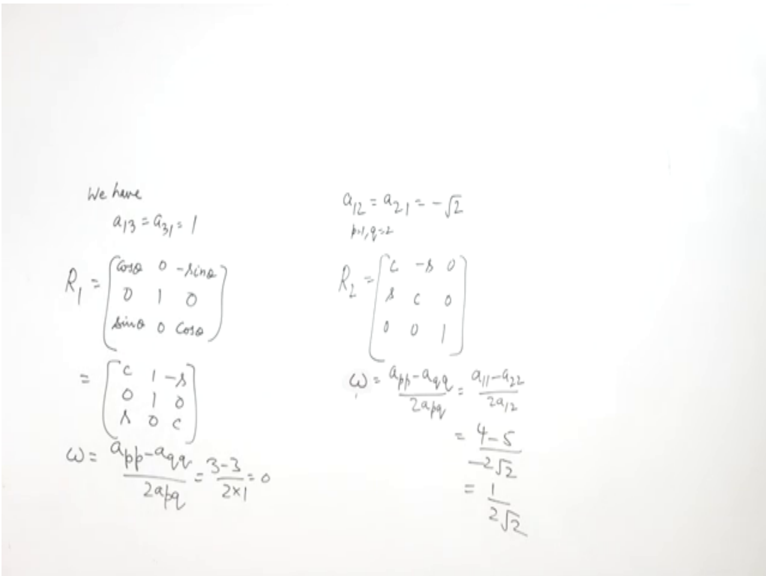
(Refer Slide Time: 20:18)




Let us take one more example to make the things more and more clear. So, we will have take this real symmetric matrix A. 3 minus 1 1 minus 1 5 minus 1 1 minus 1 3, you can see it is a real symmetric matrix. And the order of the matrix is 3. So, it is not I mean it is a small and therefore, Jacobi's method can be applied to find all the eigenvalues and the eigenvectors.

Now, here we noticed one thing that all the off-diagonal entries are numerically same, ok. Minus 1 1 minus 1 minus 1 1 minus 1 all of them numerically they are all equal to 1 ok. So, we can pick any one one off diagonal entry and define the rotation matrix, such that in the matrix D 1 at the corresponding position we have 0. So, what we have done here? I have chosen the element in the first row and third column which is this, a 1 3 as 1, ok. You could choose minus 1 also, that is a 1 2 ok, and a 2 1. So, I have chosen here the pair a 1 3 a 3 1 which are both equal to 1, now by Jacobi's algorithm D naught equal to A. So, D 1 is equal to R 1 transpose D naught R 1 and R 1.

(Refer Slide Time: 21:53)



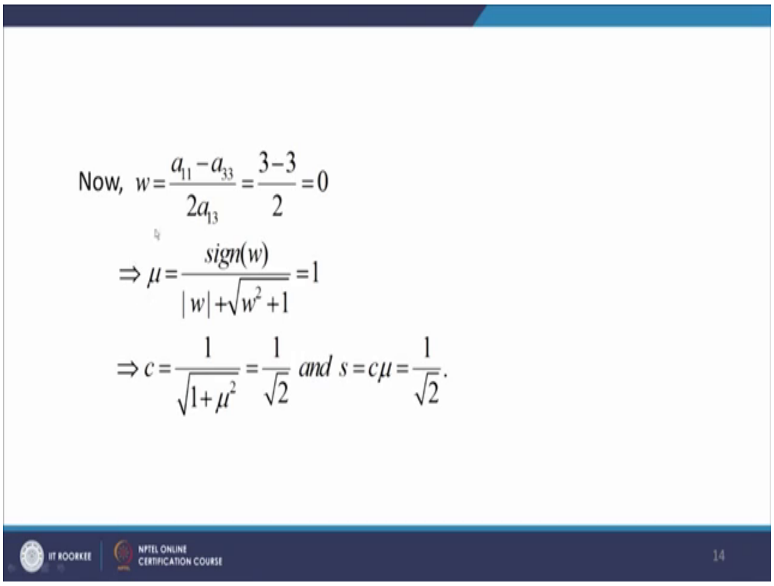
Let us write R 1 as we have written in the previous example. So, we have a 1 3 equal to a 3 1 equal to 1.

So, we want to make, when we write R 1 matrix ok, p is equal to 1 here, q is equal to 3 here, ok. First row third column so that I should write sin theta ok, and then I should write sin theta here, in the third row and first column, and in the first row first column, I write cos theta ok, and in the third row third column, I write cos theta. So, a 1 3 a 3 1 means that this a a 1 1, I mean in the matrix R 1 at the diagonal, first row first column and third row third column. There should be taken as cos theta cos theta, at the first row third column we should take minus sin theta. At the third row first column we should take sin theta. And then this entry should be taken as 1, remaining should be taken as 0's.

I can write it also as c 1 minus s 0 1 0 s 0 c, ok. And then we can find w, w is equal to ap p minus a qq divided by 2 times a pq. So, let us see what are the values off app and a qq. They are both equal to 3 you can see. P is equal to 1, ok, q is equal to 3. So, a 1 1 is 3 a q a 3 3 is equal to 3. So, a pp minus a qq will be 3 minus 3 divided by 2 times 1, ok. So, this is 3 minus 3 and we get 0.

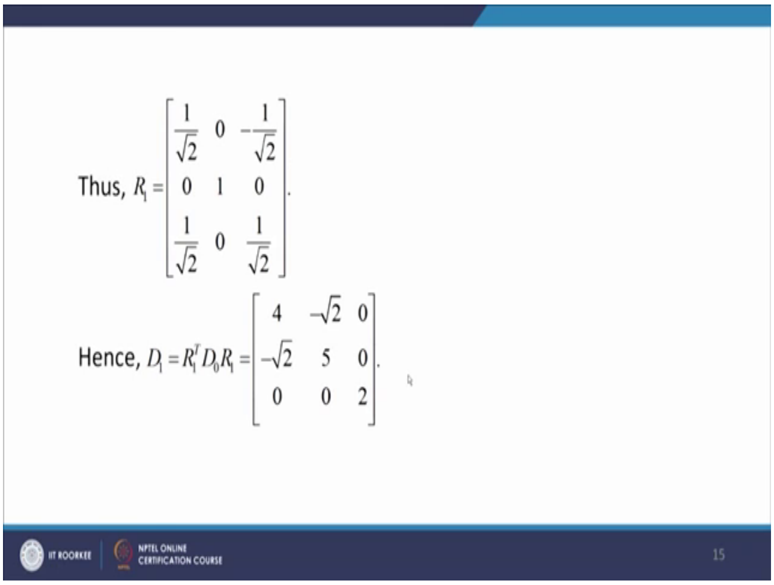
So, what we get is w equal to 0.

(Refer Slide Time: 24:03)




And then mu will be equal to sign w, sign w will be 1 divided by mode w plus under w square plus 1 which is 1. C will be equal to 1 by root 2, s will be equal to 1 by root 2, should be get the matrix R 1, ok.

(Refer Slide Time: 24:15)




By putting the values of c and s; so 1 by root 2 0 minus 1 by root 2 0 1 0 1 by root 2 0 1 by root 2 and then you can find D 1, which is R 1 transpose D naught R 1, D naught is A. So, what we get 4 minus root 2 0 minus root 2 5 0 0 0 2. You can see in the first row third column, we have at this element has become 0, and this element has become 0 in the third row and first column.

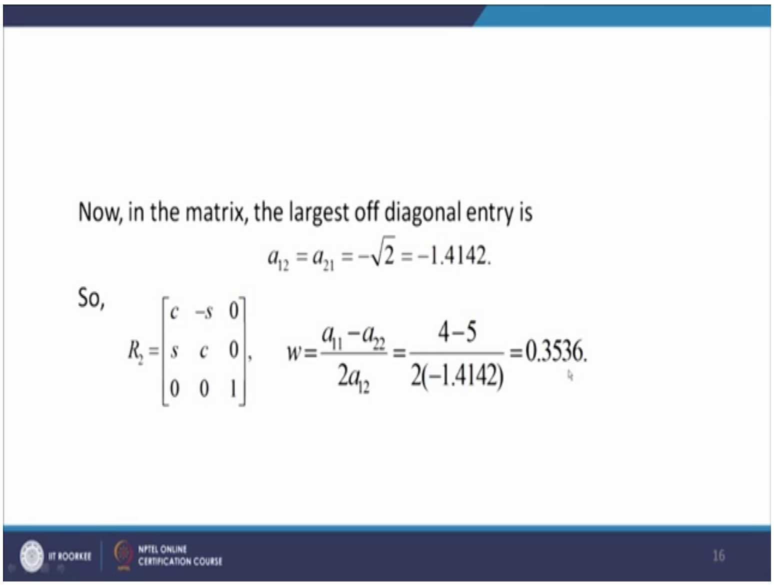
Now, let us again look at the R largest off diagonal entry. So, the off-diagonal entry is minus root 2. So, we have to make we have to define R 2 is so that R 2 transpose D 1 R 2 ok, has the element in the first row and second column as 0. So, let us see what we have a 1 2, equal to a 2 1 equal to minus root 2.

So, this element occurs p equal to 1 q equal to 2, in the first row second column ok. So, R 2 let us write R 2 so, a pp first row first column we should take c ok, and then in the second row second column we should take c ok. And then first row second column, here I should take minus s here I should take s, ok. Here I will take 0, here I will take 0, here 0, here 0, and here I will take one, ok.

So, this how I write R 2, ok, and I find w equal to a pp minus a qq divided by 2 a pq. So, this will be equal to a 1 1 minus a 2 2 divided by 2 a 1 2, ok. A 1 1 minus a 2 2, it is 4 minus 5. 4 minus 5 2 times a 1 2, a 1 2 is minus root 2 ok. So, I get this, and this is what 1 over 2 root 2 we get.

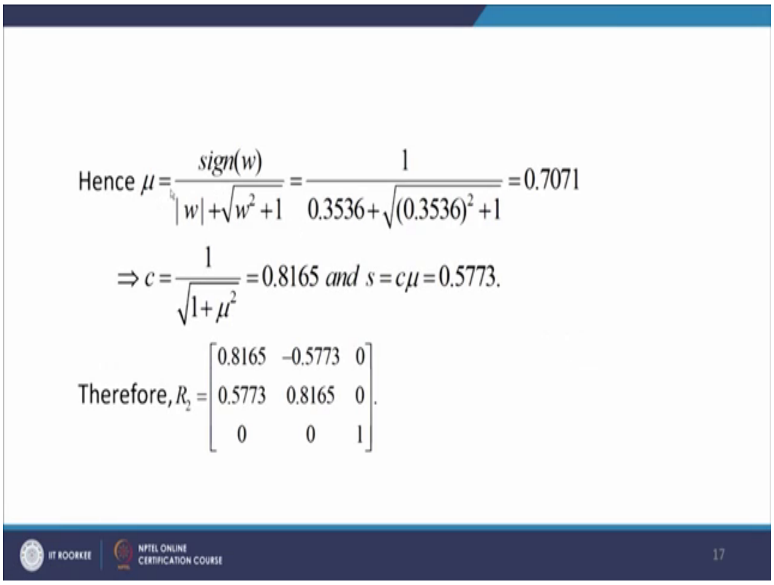
And let us see what we get then.

(Refer Slide Time: 26:33)




So, minus root 2 is if you write it is one pint minus 1.4142 this is your R 2 matrix, w is equal to 4 minus 5 divided by 2 into minus root 2, and what it comes out is that, it is 0.3536.

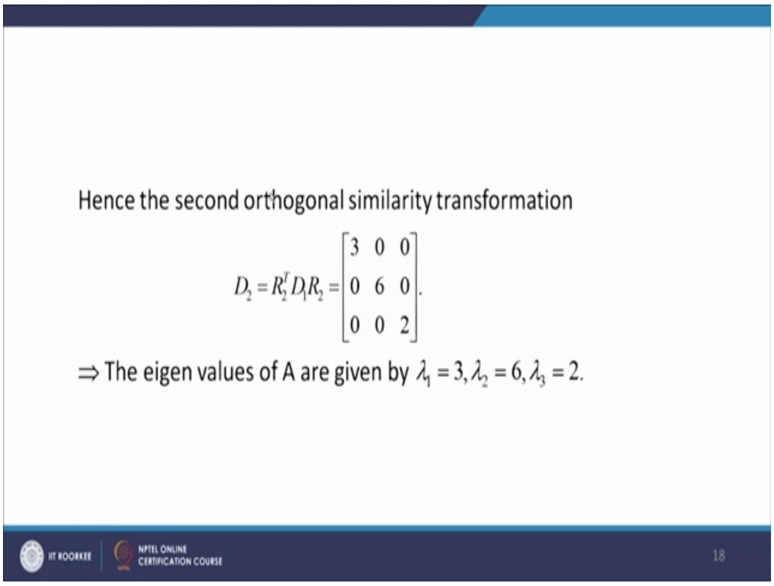
(Refer Slide Time: 26:54)




So, mu is done, now w is positive, right? W is equal to 0.353, since it is positive. So, sign w equal to 1, and we have mode w as 0.3536 then under root 3 5 3 6 0.3536 is squared plus 1, and when we calculate this value it is 0.7071.

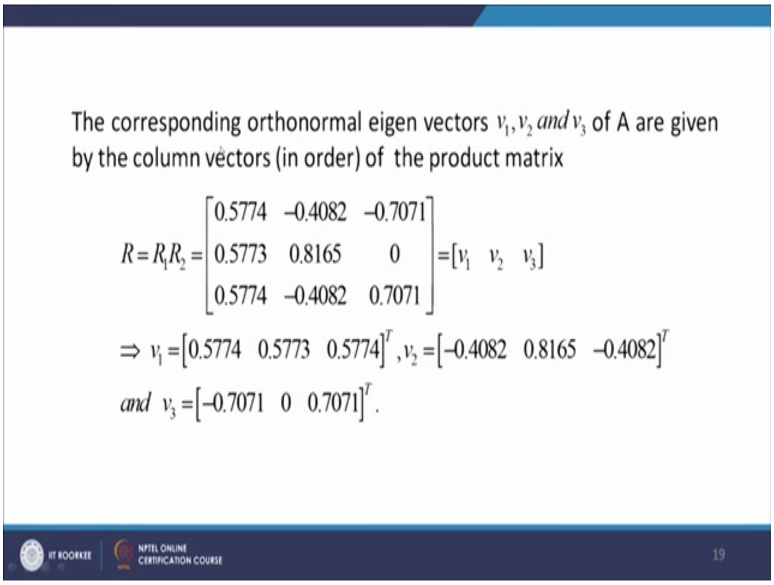
So, c is equal to 1 over square root 1 plus mu square. We put this value of mu and get the value of c. It is 0.8165, and s is equal to c in to mu. C we have found, mu is also known to us. So, c into mu we can find, and c into mu comes out to be 0.5773. And therefore, we put the values of c and s, in the R 2 matrix we get 0.8165, which is the value of which is the value of c ok; c and then we have s, s is equal to 0.5773. So, we put this value and arrive at this matrix R 2.

(Refer Slide Time: 27:58)



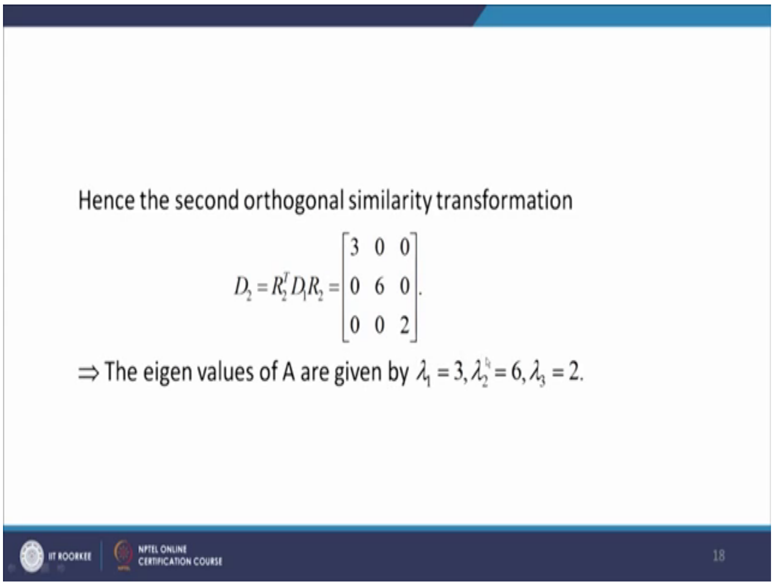

So, the second orthogonal similarity transformation D 2 is equal to R 2 transpose D 1 R 2 gives you, 3 0 0 0 6 0 0 0 2. The diagonal entries of the D 2 matrix give the eigenvalues of the matrix A. So, eigenvalues of the matrix A are given by lambda 1 equal to 3, lambda 2 equal to 6, lambda 3 equal to 2. And then the eigenvectors of the matrix A are then found from R equal to R 1 into R 2.

(Refer Slide Time: 28:33)




So, the corresponding orthonormal eigenvectors v 1 v 2 and v 3 of A are given by the column vectors in order of the product matrix R equal to R 1 into R 2, ok; so, we get this R 1 into R 2, ok. You have the matrix R 1 with you have R 2 known to you so, you multiply R 1 and R 2, we arrive at this matrix. So, first column gives the orthonormal eigenvectors v 1 corresponding to eigenvalue lambda 1; that is, 3 and the second column minus 0.4082, 0.8165, minus 0.4082. This gives the orthonormal eigenvectors v 2 corresponding to the eigenvalue lambda 2 equal to 6.

(Refer Slide Time: 29:20)



And lastly the third orthonormal eigenvectors minus 0.70710, 0.7071 the eigenvector this eigenvector is give is the v 3 corresponding to the eigenvalue lambda 3 which is lambda 3 equal to 2.

So, this is how we find the eigenvalues and eigenvectors; that is the eigen pairs all the eigen pairs of the given real symmetric matrix by applying the Jacobi method. With this I would like to conclude my lecture.

Thank you very much for your attention.