Numerical Linear Algebra
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Lecture – 58
Applications of Power Method with Shift

Hello friends. So, I welcome you to this lecture on applications of power method with shift.

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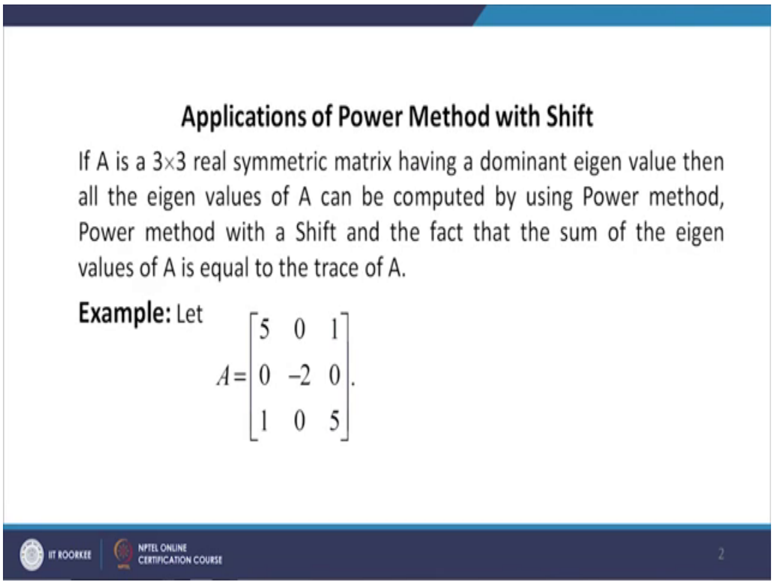

If A is 3 by 3 real symmetric matrix having a dominant Eigenvalue then all the Eigenvalues of A can be computed by using power method, power method be the shift and the fact that this sum of the Eigen values of a square matrix equals the trace of A. Trace of A means sum of the diagonal entries of A.

So, when the matrix A is given, we can easily find the sum of the diagonal entries of the matrix like here, we have in this example A equal to 5 0 1 0 minus 2 0 1 0 5. So, the diagonal entries are 5 minus 2 5 and therefore, their total is 8 ok.

So, we can say that the Eigenvalues of A will have sum equal to 8 ok. So, sum of the Eigenvalues of A is equal to 8. So, by if is A is 3 by 3 real symmetric matrix having a dominant Eigenvalue, then we will be able to compute all the Eigenvalues. see as such Eigen dominant Eigenvalue can be computed by using the power method. So, by using the power method, we will ah find out the dominant Eigenvalue of A, then by power method with a shift we shall be able to find the sub dominant Eigenvalue of A, and then the third Eigenvalue can be found by using the some defect that some of the Eigenvalues of A is equal to the trace of A.

So, let us take this example.

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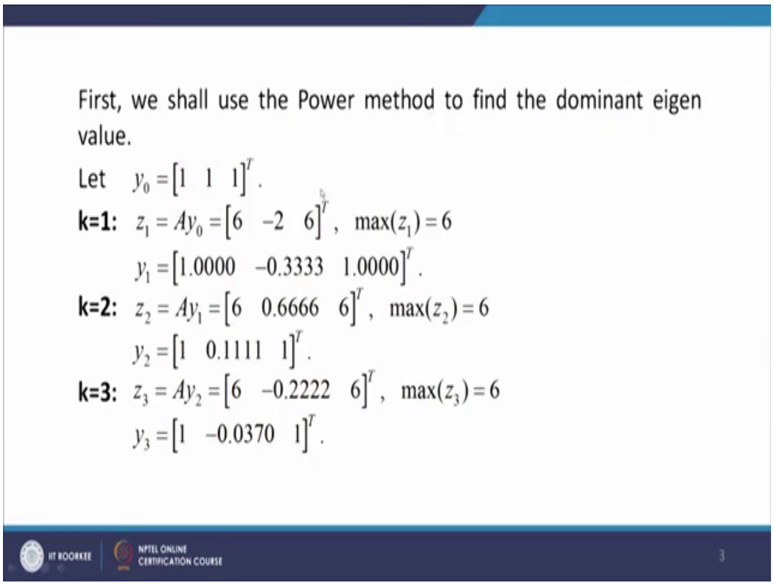

So, first we shall use the power method to find the dominant Eigenvalue. So, y naught equal to 1 1 1 let us take this y 1 1 1 vector, then in the first iteration we will get z 1 z 1 is equal to A y naught.

So, z 1 comes out to be the column vector 6 minus 2 6, and the maximum value of z z 1 is equal to 6 here as you can see. So, y 1 equal to z 1 divided by maximum of z 1. So, one we divide z 1 y maximum of z 1, we get y 1 equal to 1.0000 minus 0.3333 and 1.0000.

In the second iteration k equal to 2, z 2 is equal to A y 1 and so, Ay 1 is equal to 6 0.666 6 and 6 this column vector. So, maximum of z 2 is equal to 6 obvious; obviously. So, by 2 is z 2 divided by maximum of z 2 gives you 1; 0. 1111 transpose.

And in the third iteration what do we get? Z 3 equal to Ay 2. Z 3 equal to Ay 2 means when we multiply the matrix Ay by 2 we get 6 minus 0.2222 6 transpose and maximum of z 3 is equal to 6. then y 3 comes out to be 1 minus 0. 03701.

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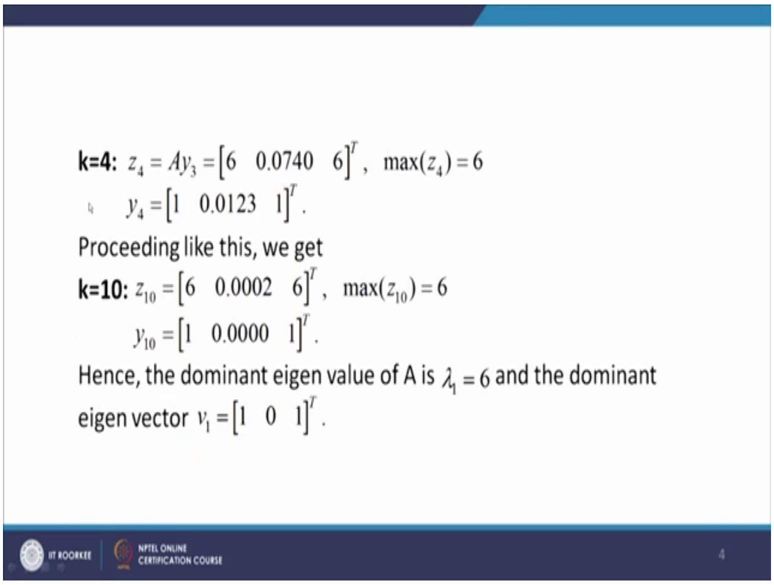

K equal to 4 when we carry out the fourth iteration, we get z 4 equal to A y 3 and we get 6 0.0740, 6 transpose maximum of z 4 comes out to be 6. By 4 is equal to 1 0.01 23 1 transpose. When we proceed in this manner what do we get in the tenth iteration? Tenth iteration gives you z 10 equal to 6, 0.0002 6 and so, maximum of z 10 equal to 6 while y 10 is equal to 1 0.0000 1. So, what do we get? We can say that the maximum of zk. Maximum of z k maximum of zk as k goes to infinity tends to lambda 1, the dominant Eigen value of A. So, maximum of z 10 here is equal to 6. So, when k is sufficiently large we see that lambda 1 is equal to 6.

The dominant Eigenvalue is equal to 6, and the dominant Eigenvector v 1 is equal to approximately y 10; y 10 is equal to. So, the dominant Eigenvector v 1 is given by this y 10 which is 1 and then this is 0 and 1. So, 101 transpose.

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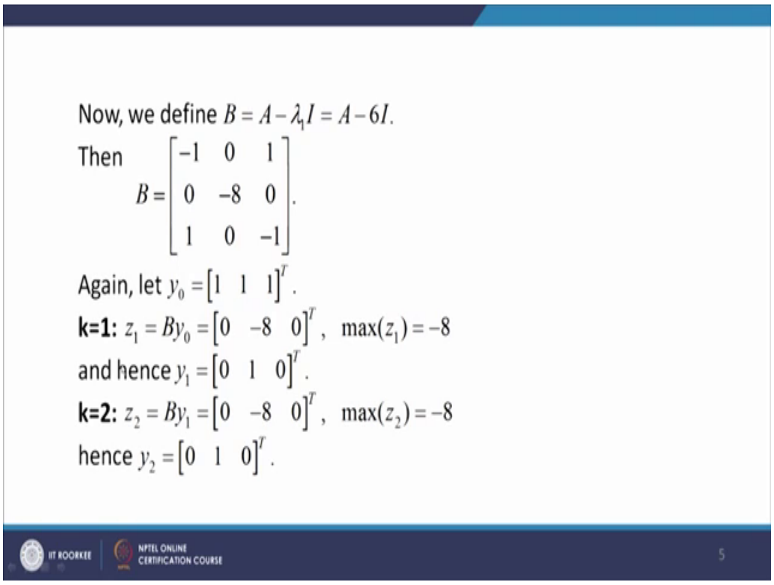

Now, let us define the. So, we have got the dominant Eigen value of the matrix A, it is equal to 6. Let us get another Eigenvalue of the matrix A by using power method with the shift. So, let us use B equal to, now here the rho we choose as equal to lambda 1. The lambda 1 value we have got that is 6.

So, we choose that as the value of rho. So, let us define B equal to a minus lambda 1 into I. So, we get A minus 6 I and A minus 6 I means from the diagonal matrix from the diagonal entries of the matrix A. We subtract 6, so we get the matrix Bs minus 1 01 0 minus 80 and 10 minus 1.

Again, we start with y naught equal to 111, we in the first iteration we get z 1 equal to be y naught equal to 0 minus 80 transpose. So, maximum of z naught z 1 maximum z 1 comes out to be minus 8 and hence by 1 which is equal to z 1 over maximum of z 1 comes out to be 010 transpose. In the second iteration z 2 is equal to By 1 z 2 equal to B y gives you 0 minus 80 transpose the same Eigenvector as z 1.

You can see same vector as z 1 z 2 is same vector as z 1. So, maximum z 2 2 is also minus 8 and. So, y 2 is also same as y 1 ok. So, y 2 is 010 transpose.

And since we are not getting ah any movement further, we are getting the same vector and in the second iteration. So, we stop here and we say that the dominant Eigenvalue of B which is dominant Eigenvalue of B is nu equal to minus 8.

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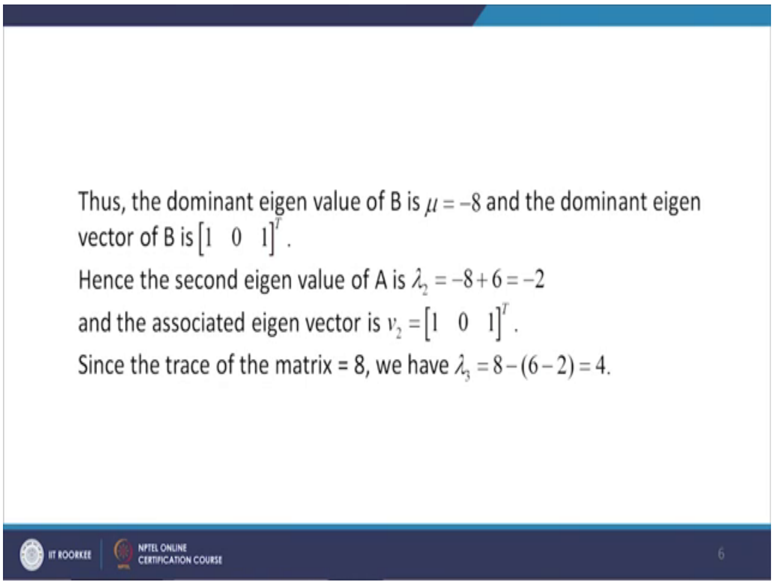

Since the, since when we write B as a minus 6 I, the Eigenvalue of comes out to be minus 8 ; that means, to get the Eigenvalue of A we add 6 to this Eigenvalue of B. So; that means, the Eigenvalue of A comes out to be minus 8 plus 6, which is equal to minus 2. So, we have got the dominant Eigenvalue which was lambda 1 equal to 6 and then we have got next Eigenvalue, second Eigenvalue, rather second Eigenvalue of A as minus 2 we have got two Eigenvalues. third Eigenvalue is obtained by using the fact that some of the Eigenvalues of the matrix is equal to its trace and trace was equal to 8 ok.

So, lambda 3 is equal to, are 8 minus some of the Eigenvalues. one Eigenvalue is 6 the Eigenvalue is minus 2. So, 8 minus 6 minus 2 which is equal to 4. So, we have got all the Eigenvalues of the matrix A by using the power method.

We found the dominant Eigenvalue of A by using power method with A shift, where we use the value of rho as the value of the dominant Eigenvalue; that is lambda 1. We get second Eigenvalue of A and the third Eigenvalue of A is found by using the fact that some of Eigenvalues of A equals the trace of A.

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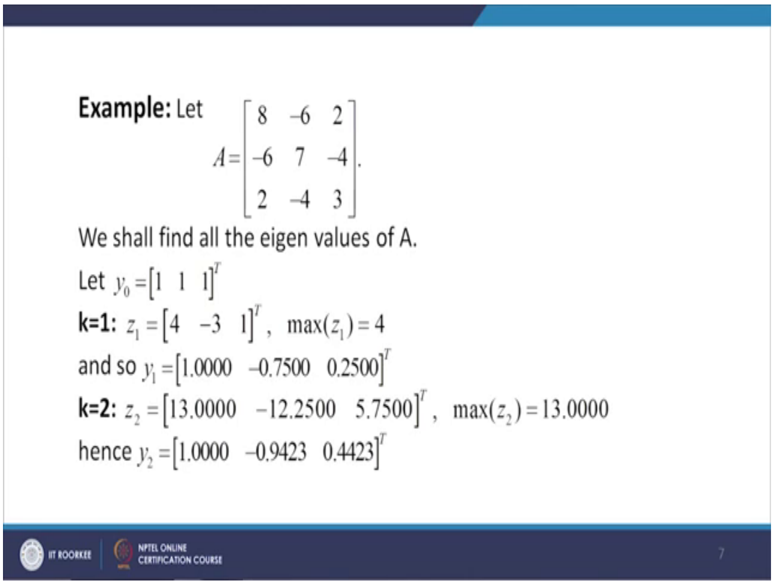

Now, let us take one more example on this applications. So, let A be equal to 8 minus 6 2 minus 6 7 minus 4 2 minus 4 3. So, you can see here, what is the trace 8 plus 7; that is 15 plus 3 that is 18. So, some of the Eigen values of A must be equal to 18.

So, we will find one Eigenvalue by using power method which is the dominant Eigenvalue of A, second Eigenvalue of A will be found by using the power method with shift and the third Eigen value will be found by using the fact that the, some of the Eigenvalues must be equal to trace that is 18.

So, let us first find the dominant Eigenvalue by using power method. So, y naught we start take as 111 transpose. In the first iteration we get z 1 equal to Ay naught Ay naught is equal to 4 minus 3 1 transpose and then maximum of z 1 you can see here is 4 and. So, y 1 is equal to z 1 divided by maximum of z 1.

And therefore, we get 1.0000 minus 0.7 500 0. 2500. In the second iteration z 2 is maximum, z 2 is equal to A into y 1 A when multiplied, when we multiply the matrix A by the vector by 1 we get z 2; that is 13. 0000 minus 12.2500, 5.7500 and maximum value of z 2.

You can see here is 13.0000 and y 2 then is equal to z 2 divided by maximum of z 2. So, we get 1.0000 minus 0.9423 0.4423 transpose k equal to 3. In the third iteration z 3 is equal to Ay 2 ok. So, we get 14.5385 minus 14.3654 7.0962 transpose and maximum of z 3 is 14.5385 y 3 is z 3 is divided by maximum of z 3.

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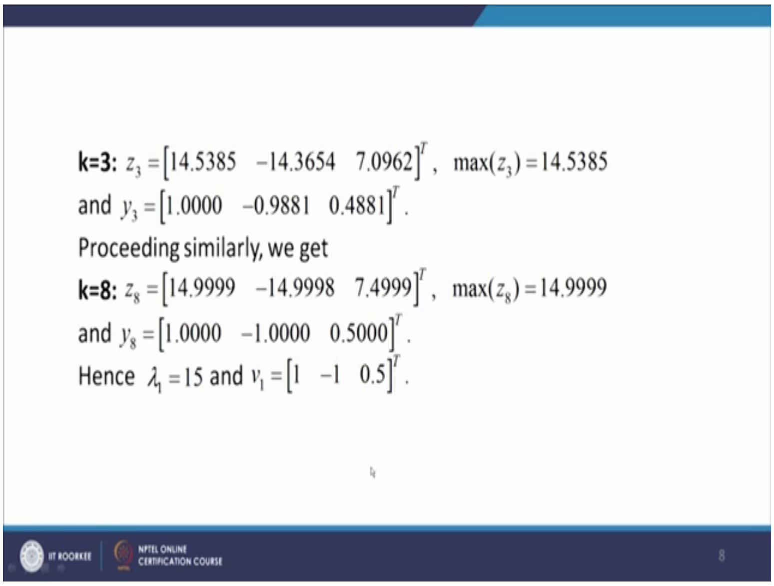

So, we get 1.0000 minus 0.9881 and 0.4881 transpose, proceeding like this we get in the kth in the eighth iteration k equal to 8 z 8 equal to 14.9999 minus 14.9998 7.4999 transpose and so, maximum of z 8 equal to 14.9999 and y 8. Y 8 which is equal to z 8 divided by maximum of z 8 is equal to 1.0000 minus 1.0000 and 0.5000. So, we can say that maximum of z 8, maximum of z k Ah goes to the dominant Eigenvalue as k goes to infinity.

So, we can say that 14.9999 is very close to 15. We can take lambda 1 as 15 and the corresponding Eigenvector is y 8; that is 1 minus 1 and 0.5. So, we denote that Eigenvector of the matrix Ay v 1.

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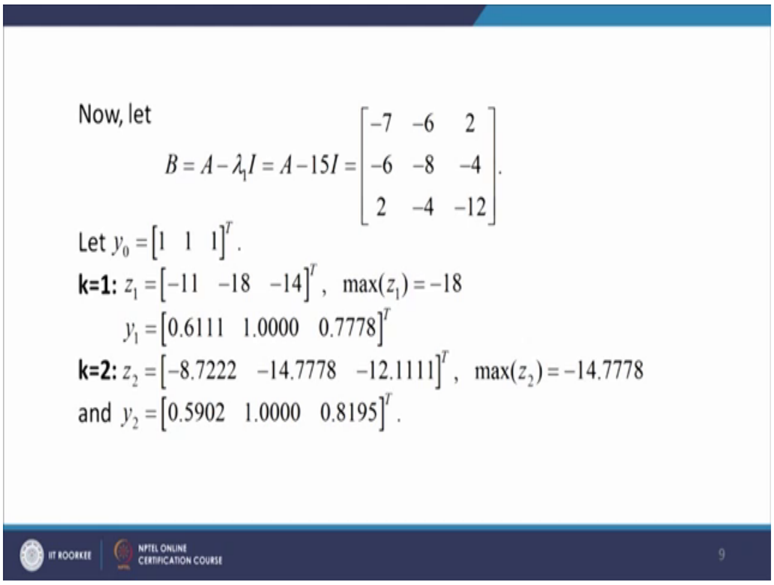

So, one Eigenvalue we have found it is 15. And now, find next second Eigen value of A by using power method with shift. So, we write the matrix B equal to A minus lambda 1 I rho, we have taking as lambda 1 and lambda 1 we have found to be 15.

So, A minus 15 I, so we subtract the diagonal entries of A by 15 and B arrive at the matrix B which is minus 7 minus 6 2 minus 6 minus 8 minus 4 2 minus 4 minus 10 and then we again start with initial vector y naught equal to 1111, 111.

In the first iteration k equal to 1 z 1 is equal to Ay naught, gives us minus 11 minus 18 minus 14 transpose maximum of z 1 therefore, is equal to minus 18. So, y 1 is equal to z 1 divided by maximum of z 1 and therefore, we get 0.6111 1.0000 0.77 78 transpose, sowe get y 1.

Now in the second iteration by k equal to 2 z 2 equal to Ay 1 gives us minus 8.7 22 2 minus 14.77 78 minus 12.1111 transpose and. So, maximum of z 2 is minus 14.778 and y 2 which is equal to z 2 over maximum of z 2 gives us 0.59 02 1.0000 0.8195 transpose.

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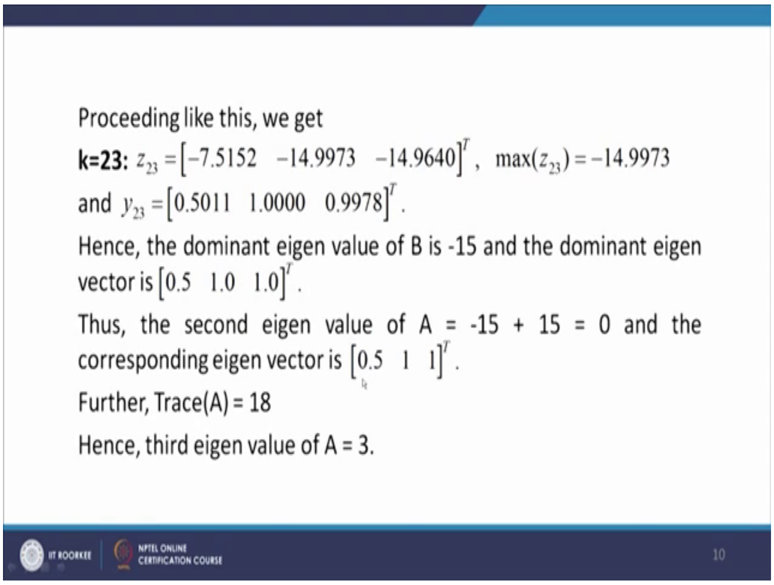

Proceeding like this in the 23 iteration k equal to 23 z 23 is equal to minus 7; 51 52 minus 14.99e 73 minus 14. 96, 40 transpose. So, maximum of z 20 3 is minus 14. 99 73.

And y 2 y 23 is 0; 50 11 1.00 00 0.99 78 transpose. Hence the dominant Eigenvalue of B is minus 15, because this z 23 is maximum of z 23 minus 14.9973 is very close to minus 15 and the dominant Eigenvector is 0.5. This is very near to 0.5 and this near this is 1 and this one near to 1. So, we can write 0.5 11 transpose.

So, second Eigen value of A will be found by adding lambda 1 to this Eigenvalue, because we took rho as lambda 1, so we. So, second Eigenvalue of A is this value minus 15 plus 15 and, so we get 0. So, second Eigenvalue is 0.

Now the corresponding Eigenvector we have seen it is 0.5 11.0. Now trace of A is equal to 18. So, one Eigenvalue of we found as 15 the other Eigenvalue we found as 0. So, the third Eigenvalue is; obviously, 3. So, third Eigen value of A is found by the, with the help of trace of A.

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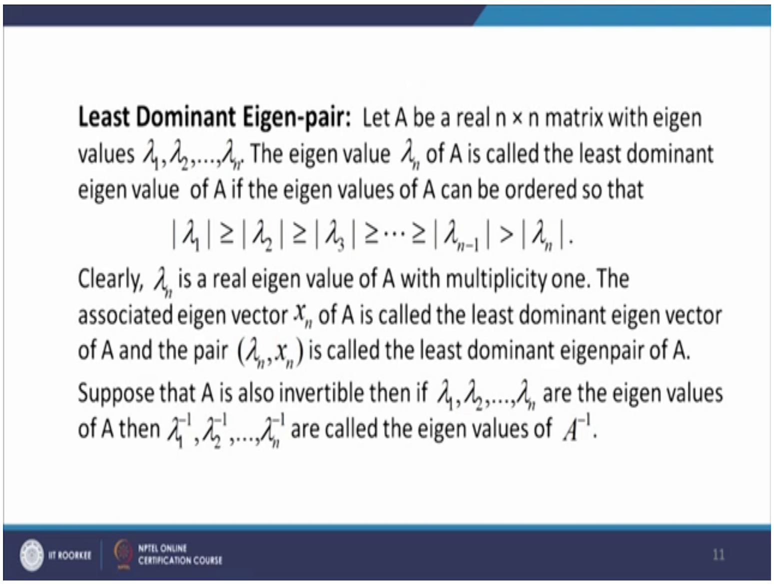

Now, we come to least dominant Eigen, but we seen how to find the dominant Eigen pair of A real 3 real n yn diagnosable matrix. Now let us see how we can find the least dominant Eigen pair.

So, let A be a real n y n matrix with the Eigenvalues lambda 1, lambda 2, lambda n. The Eigenvalue lambda n of A is called the least dominant, least dominant Eigenvalue of A. if the Eigenvalues of A can be ordered, so that mod of lambda 1 is greater than or equal to mod of lambda 2, mod of lambda 2 is greater than or equal to mod of lambda 3 and so on greater than or equal to mod of lambda n minus 1 greater than mod of lambda n. So, mod of lambda n is the least dominant Eigenvalue and again it is 1 algebraic multiplicity is 1 and it must be real.

So, clearly lambda n is a real Eigen of A. this multiplicity 1, the reasoning is the same as in the case of dominant Eigen value of the matrix A. Now the associated Eigenvector xn of A is called the least dominant Eigenvector of A and the pair lambda n x n is called the least dominant Eigen pair of A.

Now, suppose that if the matrix A is also invertible. Let us assume that the matrix A is also invertible then if lambda n 1 lambda 2, lambda n of the Eigenvalues of A, then 1 by lambda 1, 1 by lambda 2, 1 by lambda n are the Eigen values of A inverse. So, you can see it is very easily.

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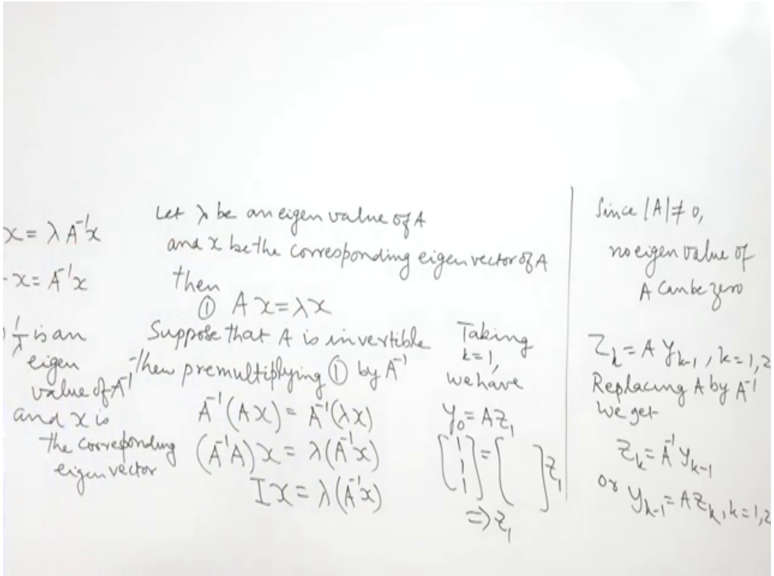

Let us say, let lambda be an Eigenvalue of A and x be the corresponding Eigenvector of A, then we have the matrix equation, then we have the matrix equation Ax equal to lambda x.

Now, suppose that A is invertible. Invertible means it is, its inverse edges or it is a non singular matrix, then we can pre multiply this equation 1, then pre multiplying equation 1 by A inverse what do we get?

Since matrix multiplication is associative, I can write it as lambda is a scalar, I can write it as lambda times A inverse x. So, A inverse A is the identity matrix. So, we have identity into x and from here, what do we get? Identity into x is equal to x. So, x equal to lambda time A inverse x ok.

Now, A is invertible matrix. Since A is invertible determinant of A is nonzero, since determinant of A is nonzero, no Eigenvalue of A can be A 0 Eigen value, no Eigenvalue of A can be 0. So, if lambda is an Eigenvalue lambda cannot be 0, I can divide by lambda ok. So, I get 1 by lambda into x equal to A inverse x.

So, from here we, if you compare this equation with this equation we see that 1 by lambda is an Eigenvalue of A inverse and x is the corresponding Eigenvector. So, if lambda is an Eigenvalue of A and x is the corresponding Eigenvector, then 1 by lambda is an Eigenvalue of A inverse and x is the corresponding Eigen vector.

So, this implies 1 by lambda is an Eigenvalue of A inverse. So, if the matrix is invertible and lambda 1, lambda 2, lambda n are its Eigenvalues. Then 1 by lambda 1 1 by lambda 2 1 by lambda n are be Eigenvalues of A inverse.

So, what we will do? We will determine the dominant Eigenvalue of A inverse. Dominant Eigenvalue of A inverse will give us the least dominant Eigenvalue of A, because if the least dominant Eigenvalue of A is lambda n, then 1 by lambda n will become the dominant Eigenvalue of A inverse.

So, we will determine the dominant Eigen value of A inverse; that is we will determine 1 by lambda n and then from that we will find the Eigenvalue of least dominant Eigenvalue of A; that is lambda n. So, let us, let. So, hence if lambda n is the least dominant Eigen value of A then by 1 by lambda n is the dominant Eigenvalue of A inverse with the associated Eigenvector x n.

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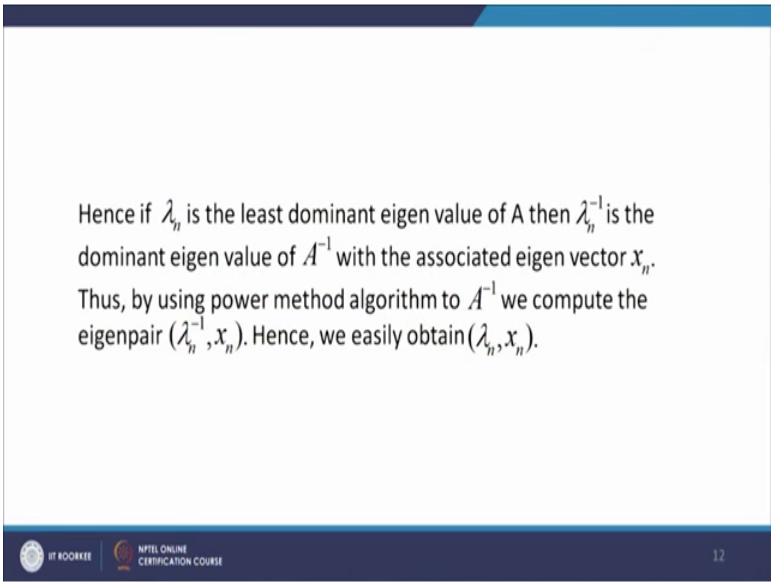

So, by using power method algorithm to A inverse we can compute the Eigen pair lambda 1 lambda n to the power minus 1 x n and then from that we can easily obtain lambda n x n ok.

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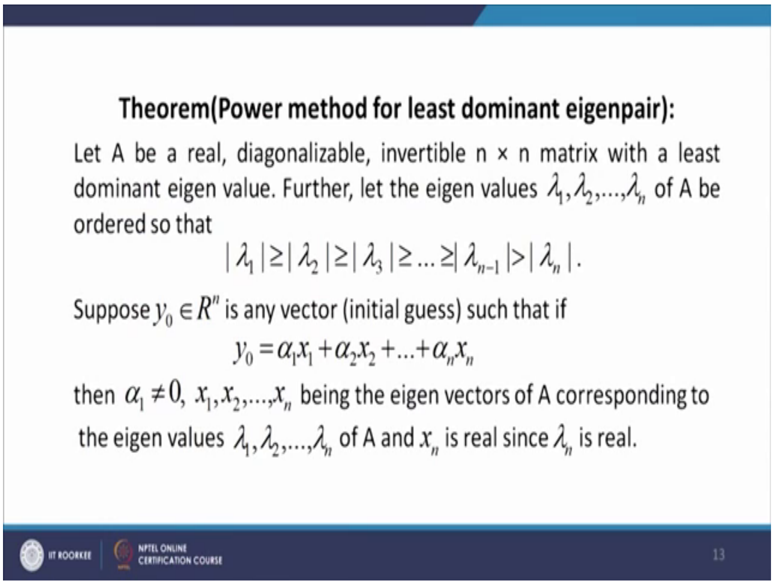

So, let us now see the power method for the least dominant Eigen pair. Let A be a real diagonalizable invertible n by n matrix with a least dominant Eigenvalue. Further let the Eigenvalue lambda 1, Eigenvalues lambda 1, lambda 2, lambda n of A be ordered.

So, that mod of lambda 1 is greater than or equal to mod of lambda 2, greater than or equal to mod of lambda 3 and so on. Greater than or equal to mod of lambda n minus 1 and greater than mod of lambda n and suppose we take the vector B naught in R to the power n as alpha 1 x 1 plus alpha 2 x 2 n.

So, an alpha n x n, then alpha 1 is not equal to 0. So, be write the linear combination y naught as a linear combination of x 1 x 2 x n in such a way that alpha 1 is not equal to 0 x 1 x 2 x n are the Eigen vectors of A corresponding to the Eigenvalue is lambda 1, lambda 2, lambda n of A and x n is real. Since lambda n is real you know that lambda n is the least dominant Eigenvalue of A. So, it its algebraic multiplicity is 1 and it is real.

So, as the same reasoning with we, which we gave in the case of dominant Eigenvalue that if lambda 1 is, lambda 1 is real and A is real then x 1 is real so, here also when lambda 1 lambda n is real and A is real matrix x n is real. So, x n is real Eigenvector, because lambda n is real and A matrix is real.

So, now, here the power method algorithm, we shall use for the matrix A inverse. So, what do we do? We write a z k equal to y k minus 1 in the power method algorithm will replace in the power method algorithm what we had? Z k was equal to A by k minus 1 k equal to 123 and so on.

For the power method algorithm for the matrix A we had z k, will given by this, but now A will be replaced by A inverse ok. So, replacing A by A inverse. We get z k equal to and which gives you or y k minus 1 equal to a z k k equal to 123 and so on.

Now we have y naught with us. So, take k equal to 1. Taking k equal to 1 we have y naught equal to A z 1. So, we solve this equation y naught is equal to 111. We choose the initial approximation as 111. The matrix A is known to us. The matrix A is known to us.

So, we write 111 equal to the matrix, which is given to us in to z 1 and solve this system of equations. Solve this system of equations we can use Gaussian elimination method with partial pivoting and determine the vector z 1.

So, we get the vector z 1. This we do, because we do not want to find the inverse of the matrix A and right z 1. So, because if the, if the A order of the matrix is very large ok. A inverse cannot be found easily.

So, we will write this equation in this manner and solve this equation matrix equation to determine the vector z 1 and once z 1 is known, we will determine by 1 by z 1 upon maximum of z 1.

Then s k will go to infinity, maximum of z k will go to 1 y lambda n, because we are replacing the matrix A by A inverse. So, maximum of z k will now go to 1 y lambda n and y k will go to c x n, where c is a real constant. So, the algorithm given is the here.

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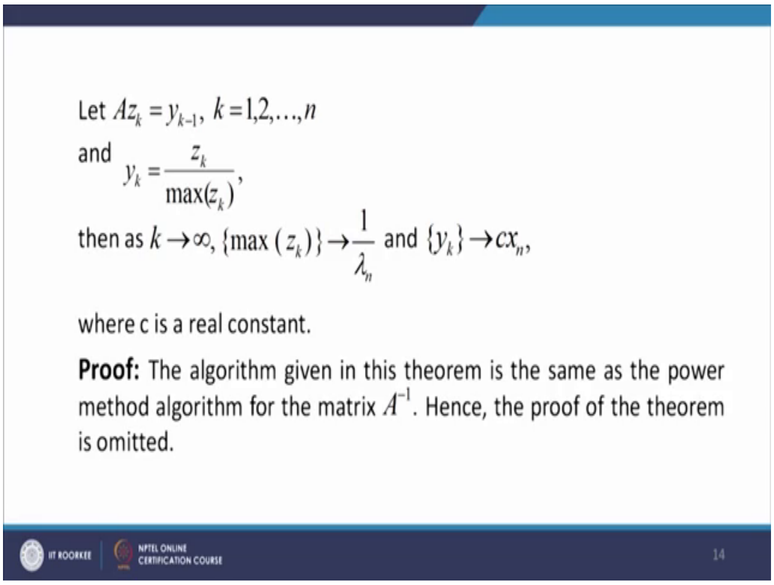

The algorithm is the same as the algorithm for the matrix A inverse. So, the, we do not write the proof. Let us illustrate this method by an example.

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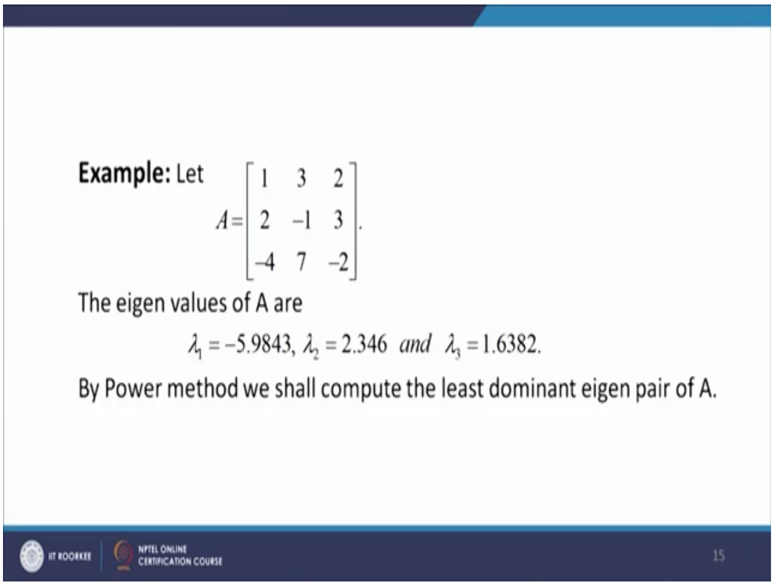

Say, let us take the matrix A to be 1 3 2 2 minus 1 3 minus 4 7 minus 2. Then if we find the Eigenvalues of A, they are minus 5.9 843 2.3 46 and 1. 63 82 and you can see from here that the least dominant Eigen value of A here is 1. 6382.

This dominant Eigenvalue here is 1.63 82. We are going to find the least dominant Eigen pair of A; that is the Eigenvalue lambda 3 1. 6382 and the Eigenvector x 3 corresponding to lambda 3 by power method.

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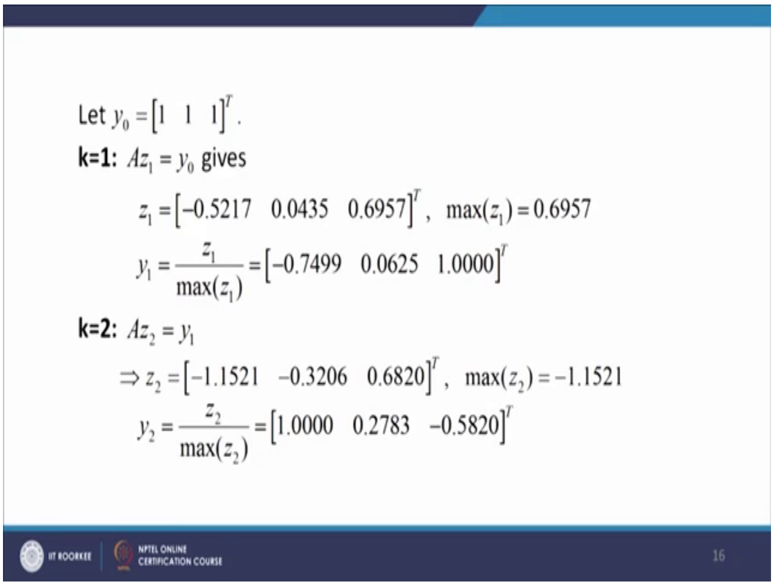

So, we begin with y naught equal to 111 and then solve the equation A z 1 equal to y naught A z 1 equal to y naught. When we solve we get the value of z 1 equal to minus 0.5 217 0.0435 0.6957 and you can see here maximum of z 1 is the 0.69 57. So, we divide z 1 by maximum of z 1 to arrive at the vector y 1 and it is 0 minus 0.7499 0.0625 1.00 00 transpose.

Then to find the next approximation vector by 2, we first solve A z 2 equal to y 1. So, we put here k equal to 2, so that A z 2 equal to y 1, we have, y 1 is given, y 1 is already. We have found A is known to us.

So, we can solve this equation A z 2 equal to y 1 to arrive at the value of z 2 z 2 is then minus 1. 15 21 minus 0.ah 3206 0; 68 20 and we can then find maximum of z 2 here which is 0, which is, sorry which is minus 1 1521 absolute value. We look at absolutely, if you take absolute values of the components of z 2 then this component is the highest, this is largest.

So, maximum of z 2 is minus 1. 1521. So, to arrive at vector y 2 we divide z 2 y maximum of z 2 and we get this vector 1. 0000 1 0. 2783 minus 0.58 20.

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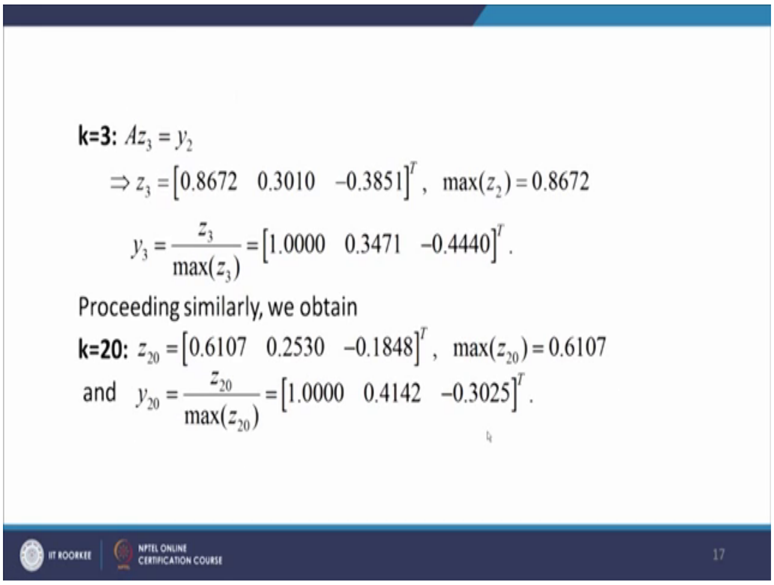

Then we performed third iteration and in the third iteration we will solve the question Az 3 equal to y 2. When we take k equal to 33 z 3 equal to A inverse by 2, so Az 3 equal to y 2.

So, A z 3 equal to y 2 we solve to get z 3 vector. Z 3 vector is this 0.8672 0. 30 10 minus 0.3851 transpose and if you find the maximum of vector maximum of z 3 here, then it is 0. 86 72.

So, we can find y 3, y 3 is equal to z 3 divided by maximum of z 3. So, what do we get? 1.0000 0.3471 minus 0. 44 40 transpose and proceeding similarly in the twentieth iteration z 20 is equal to 0.6 10 7 0.2530 and minus 0.18 48 transpose and you can find maximum of z 20 here maximum of z 20 is 0.61 07.

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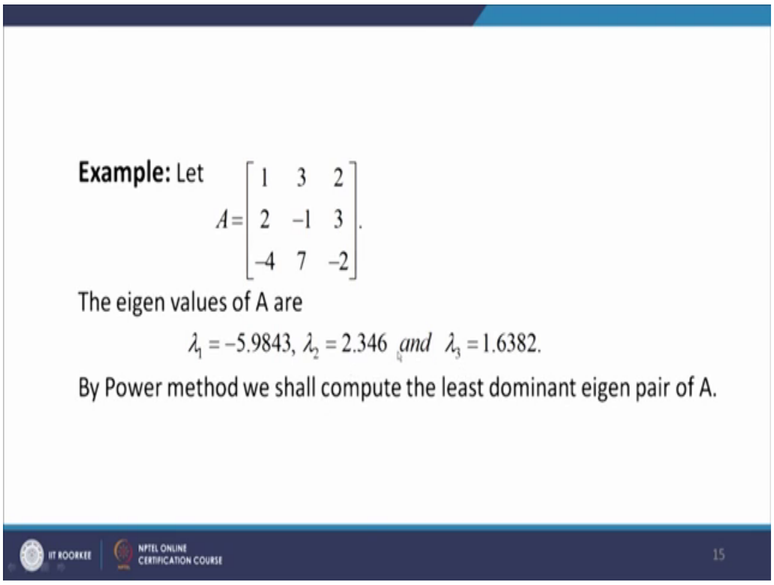

Now, you see, let us go to this the least dominant is 1. 6382 if you find 1 over 1. 63 82, it comes out to be, it comes out to be ah this one 0.610 4.

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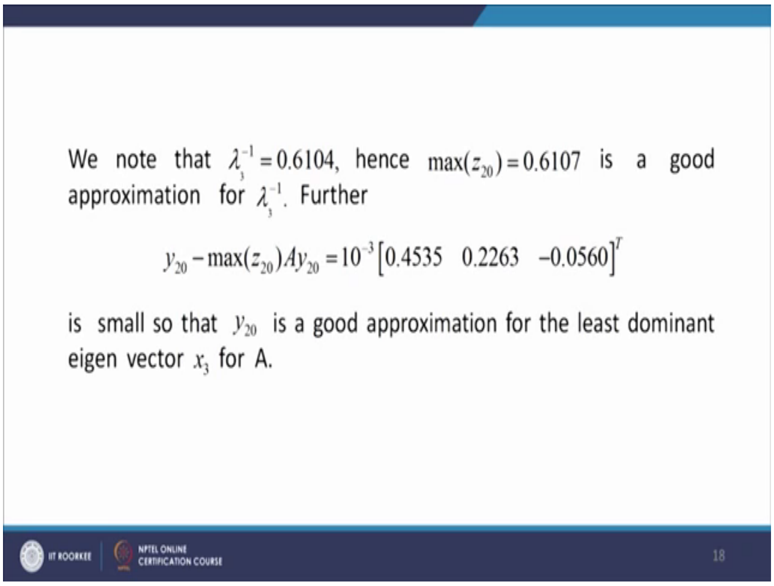

So, you can see here maximum of z 20 which is 0.6 107 and when we divide z 20 by maximum of z 20, we arrive at y 20. Y 20 is 1.00 00 0.41 42 minus 0. 3025 and transpose of this vector.

So, we note that 1 over lambda 3. 1 over lambda 3 is 0. 6104 and maximum of z 20, we have found to be equal to 0.610 7. So, you can see this maximum of z 20 which is 0. 6107 is a good approximation for 1 by lambda 3 ok.

Further y 20, this y 20, let us look at this y 20. y 20 is this one, this y 20 minus maximum of A z 20 into Ay 20. Maximum of A z maximum of A z z 20 is 1 by lambda 3 ok, This 1 by lambda 3.

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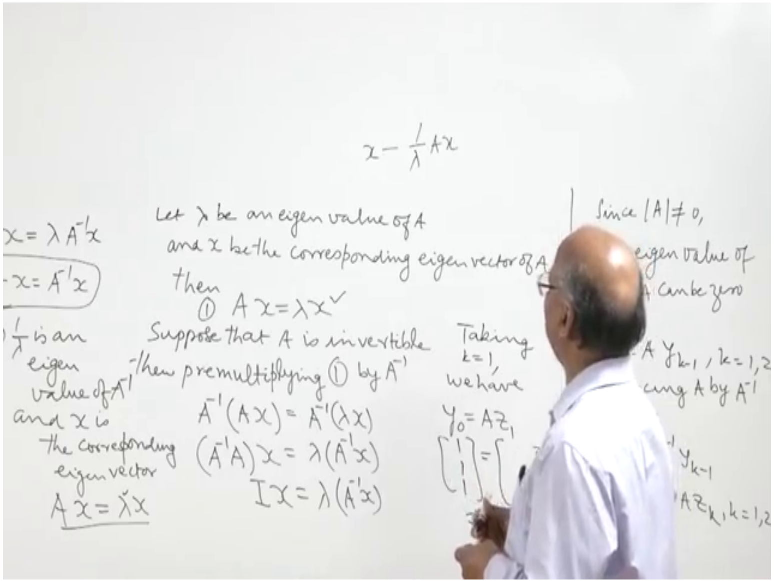

So, see what we have, we have the question 1 by lambda 1 by lambda A x x minus 1 by lambda A x. This is what we are finding. this difference x minus x is the equal to 1 by lambda A x. So, x minus 1 by lambda A x must be nearly 0. So, what do we get here; x is y 20.

That is x 3 y 20 minus 1 by lambda 3 maximum of z 20, then A x by 20 which is x 3. So, this difference is 10 to the power minus 3 0.45 350.2263 minus 0. 0560 transpose and you can see that, these components of the column vector are multiplied by 10 to the power minus 3.

So, this difference of y 20 with the maximum of z 20 into Ay 20 is very small and therefore, we can say that y 20 is a good approximation for the least dominant Eigenvector x 3 of A. So, we can find the least dominant Eigenvalue of the matrix A and the least dominant Eigenvector of the matrix A by using the power method, when we assume that the matrix A is real diagnosable matrix and also invertible. With this, I would like to conclude my lecture.

Thank you very much for your attention.