Numerical Linear Algebra
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Lecture – 53
Householder QR factorization – I

Hello friends, welcome to this lecture, in this lecture we will discuss the concept of householder QR factorization and this can be considered as An important application of householder matrices, and this householder QR factorization has many applications in for example, least square theory in both the over determined case and underdetermined case and in Eigen values problem and in computing the singular value of a matrix. So, here in this lecture, we will discuss what is a householder QR factorization and how given a matrix we find out the QR decomposition of the matrix.

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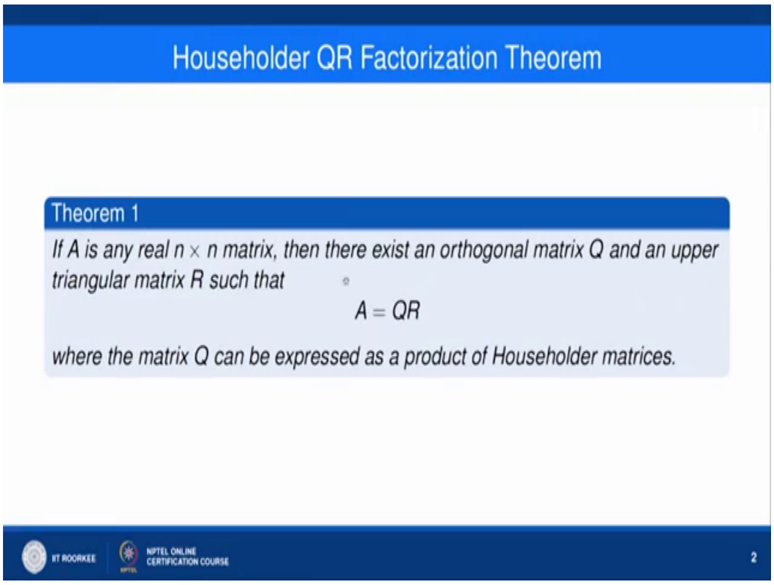

So, let us start with the lecture. So, the theorem says that if A is any n cross n matrix which is real then they exist An orthogonal matrix Q and An upper triangular matrix R such that A equal to Q into R where the matrix Q can be expressed as a product of householder matrices.

So, here we try to find out this QR factorization of A with the help of householder matrices. So, basically the proof of this theorem is basically a constructive proof.

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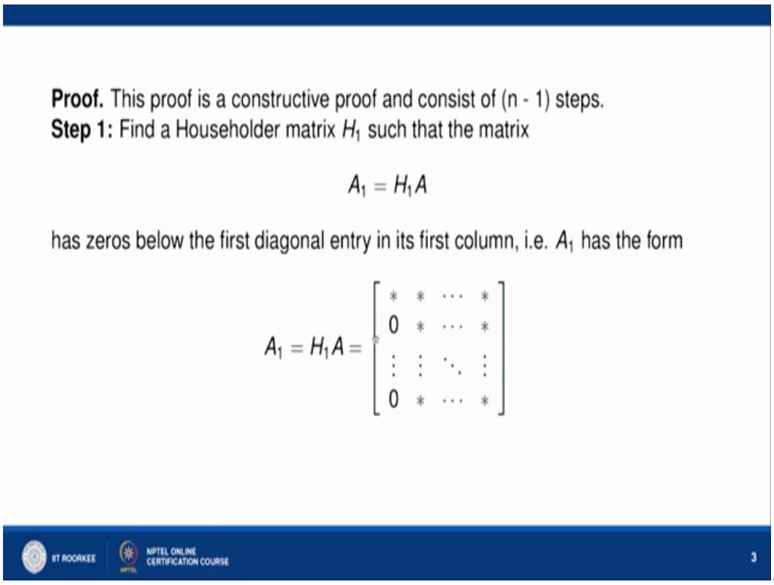

And it consists of n minus 1 step, where n represent the size of the matrix A. So, we have to do this n minus 1 steps. So, the idea is that given a matrix, we want to find out matrices such that if you multiply that then it will become An upper triangular matrix.

So, first it has n minus 1 step. So, a step one is find a householder matrix H1 such that the matrix A1 equal to H1A has the following form, it means that has 0s in the first column just before just below the first diagonal entry. So, H1 has a fact that only the first diagonal entry is nonzero, rest are all 0 in first column. Similarly, what we try to do this in first steps, second step what we try to do we repeat the same process, it means that in second column, all the entries below the diagonals are 0s and so on we can go up to n minus 1 terms.

So, how to find out these householder matrices which gives the required form that we are going to discuss in this theorem. So, first let us find this H1A. So, A1 1 equal to H1A where A1 has this form and H1 we need to find out.

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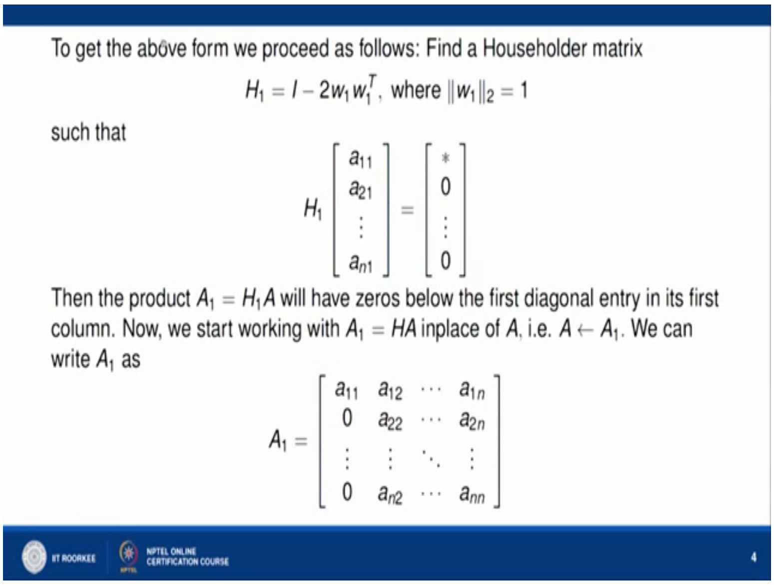

So, to get the above form, we proceed as follows. So, find a householder matrix H1 as I minus 2 omega 1 omega 1 transpose, where omega 1 is a unit matrix. So, the effect of H1 is what that if we apply it on the first column of A that is a11 to a1n then the image of this vector under H1 is going to be first non zero entries and rest all 0s entry. Then, the product A1 equal to H1A will have 0 below the first diagonal entry in its first column.

Once we have this H1 then we start working with A1 in place of A. So, now, without loss of generality, we can write A1 as in this follows though this when we apply this H1 on this A of course, this first entry will also be changed, but now we say that suppose we representing the first entry by a11 only and then we try to find out H2 such that if we apply H2 on this A1, then all the entries below this second column, all the entries below the diagonal term is going to be 0.

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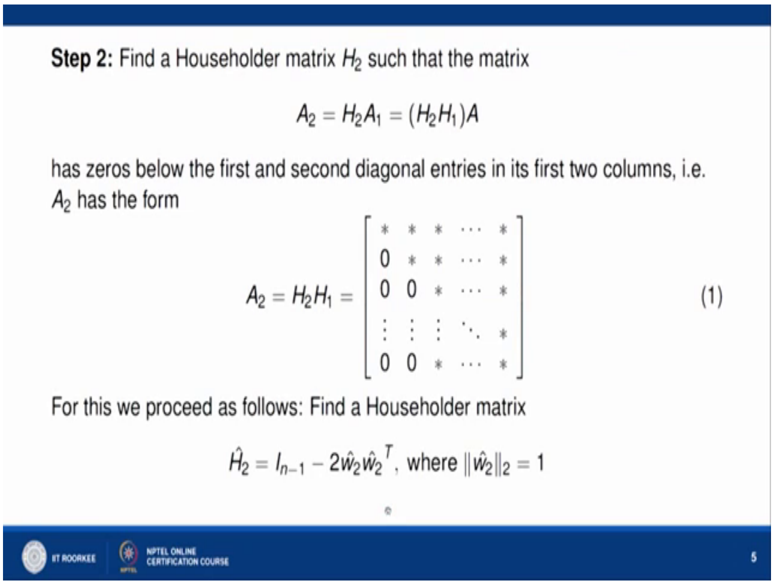

So, it means that in step two find a householder matrix H2 such the matrix A2 equal to H2 into A1, where A1 is nothing, but H1A, so, A2 is basically H2 into H1 into A. Has 0 below the first and second diagonal entries in it in its first two columns, that is in first column this is diagonal term and rest all these terms are 0. In second column, this is the diagonal entry and rest are all 0s, so that we need to find out. So, once we have H2 then we can write A2 as H2 into H1 of A. For this, we proceed as follows, find a householder matrix which is defined as H2 hat and it is given by In minus 1 minus 2 omega 2 hat into omega 2 hat transpose, where omega 2 is An unit vector in RN minus 1 having the 1.

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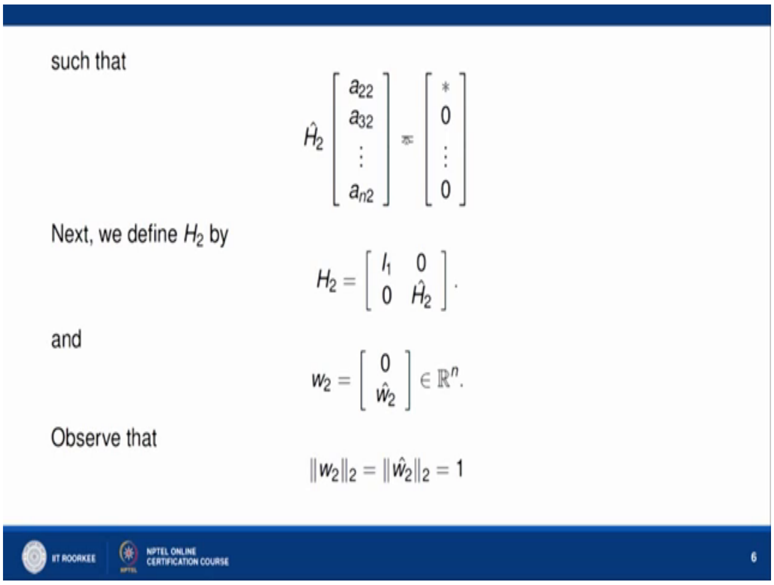

And such that the H2 hat if we apply on a22 to AN2 then it will give you in place of a22 it is some non zero quantity and all the other entries are simply 0. And once we have this H2 tilde H2 hat then we can define H2 by H2 as I1 0 0 H2 hat where this representation is a block diagonal representation. Here this is a 0 matrix and this is also a 0 matrix a of appropriate size. I1 is identity matrix of size 1 cross 1and H2 tilde H2 hat is a matrix of n minus 1 cross n minus 1 size and once we have H2 here we can define w2 as 0 w2 hat. Now by defining this w2 as this, we know the norm of w2 is nothing, but norm of w2 hat and it is given as 1.

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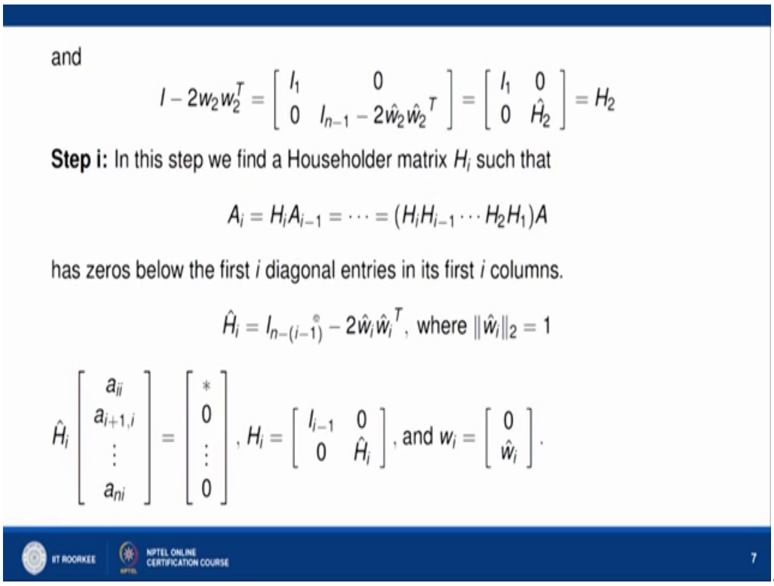

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So, you can define H2 as I minus 2 omega 2 omega 2 transpose and if you calculate it is given as I1 0 0 and here it is In minus 1 minus 2 omega 2 hat omega 2 hat transpose and this we are defining this is nothing, but H2 hat, so, your H2 is given as I1 0 0 H2 hat and this we are defining as H2.. So, that is what we are writing here.

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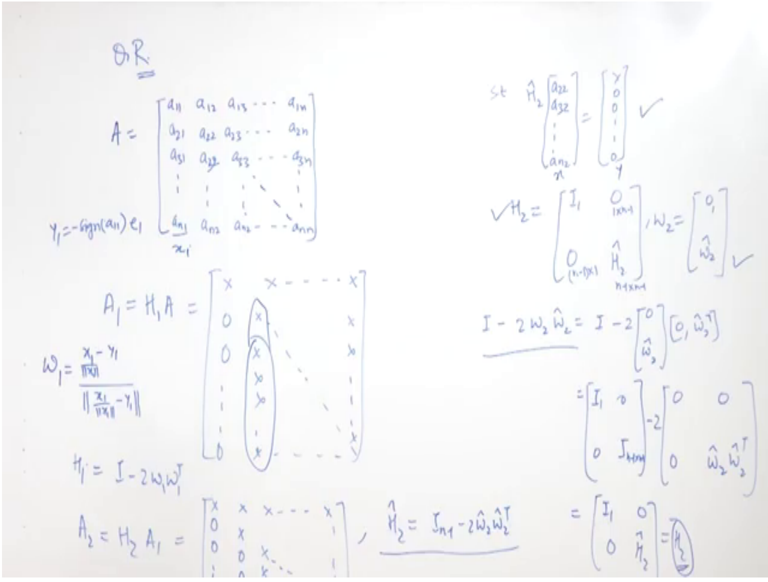

So, idea is this that we started with this matrix A which is a square matrix of size n cross n and what we try to do here we want to find out we want to write this A as Q into R, where R is the upper triangular matrix and Q is the orthogonal matrix which we are going to find out.

So, basically we are try to reduce this in upper triangular matrix. So, upper triangular matrix means we want to find out that Q such that all these entries below diagonal entries are simply vanish. So, that we are saying that this Q can be considered as product of householder matrices. So, this we are doing in the following steps.

So, in first step we find out a matrix H1 such that A1 is written as H1A and what is the form of A1, A1 has a form that in the first column below the first diagonal entries, all entries are 0 and how to find out this H1, this we already know that if you want to find out H1 having this property that it will convert this matrix into this matrix then we know that H1 is defined as I minus 2 omega 1 omega 1 transpose, where omega 1, is basically your this x you, if you call this vector say x.

Let us say call it x1 then we can find out x1 minus y1 and here x1 divided by norm of x1 minus y1 and y1 is what, y1 is the minus sign of x11 basically, the first entry that is a11. So, here I have to write a sign of a11 into e1. So, once we have y1 here then divided by norm of x1 upon norm of x1 minus y1. So, that we already know using algorithm you can find out w1 and then we can find out H1 and once we have H1 then A1 has the following representation.

Now, once we have A1 then we forget about this for the time being and start working with this. We want to find out now another a householder matrix which will remain say same structure in a column 1, but in column 2 it will make all these entries as 0 entry before the second column, it means that the effect of H2 on this A1, A1 which is written as H1A, the effect of H2 in over this A1 is the following that in first column below the diagonal entries, it is all 0s and here in second column also the below the diagonal term all the entries are 0.

So, basically what we want that H2 in second column reduce these entries as 0s and it will have no effect on this first column. For that we define H2 as this, as H2 as I1 0 0 H2 hat which is of size N minus 1 cross N minus 1 where H2 hat has this effect A22, A32, AN2 tending to cross 0 0 0 as on. So, it means what it is basically looking at these many vectors and keeping this as nonzero, rest all 0 entries. (Refer Time: 10:35)

So, here we define H2 hat as In minus 1 minus omega 2 hat omega 2 hat transpose. How to find out this omega 2 hat here the effect of H2 hat is on this. So, this is your x and this is your y. So, based on x and y we can use algorithm to find out this H2 hat. So, once we have H2 hat then we can define our H2 as I1 0 0 H2 hat and w2 we can define as 0 n2 w2 hat and we can see that by defining such H2 and w2 the effect of this I minus 2 omega 2 omega 2 hat is basically I minus 2, here omega 2 is this and omega 2 hat we can write it like that. And if you calculate, it is coming out to be that I1 0 0 H2 hat and which is nothing, but the proposed form of H2.

So, it means that the form of H2 is given by I minus 2 omega 2 omega 2 hat where omega 2 is defined by 0 1 omega 2 hat. So, in this way we can find out our matrix H2 such that if you apply this H2 on A1 then it has the following form and this we keep on doing for n minus 1 steps. So, for every step what we want that it will make the entries below the diagonal term as 0 and it will not touch any 0s which have already achieved.

So, in ith step, we can say that in step I, in this step we find a householder matrix Hi such that Ai can be written as Hi into Ai minus 1 where Ai minus 1 is is Hi minus 1 into Hi minus 2 up to H1A. So, here AI is written as the product of householder matrices into A and this has a property that it has 0s below the first I diagonal entries in its first I columns.

So, we can how to find out this Hi hat, Hi hat can be written as In minus I minus 1 minus 2 Wi hat into Wi hat transpose, where Wi hat basically is a unit vector. The effect of Hi hat on aii to ani is that at this place at the place of aii we have non zero entry and rest all these entries are your 0s entry and once we have HI hat defined then we can define capital HI as II minus 1 0 0 and HI hat and wi we can define as 0 wi hat.

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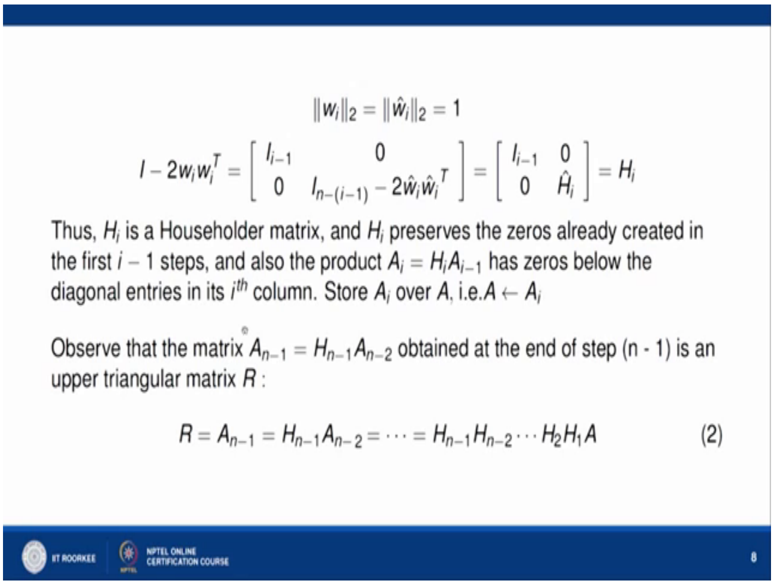

Once we have this Hi we can repeat this process and we can calculate that I minus 2 omega I omega I transpose and as we have done here the calculation we can show that it is coming out to be Hi. So, once we have w2 define as this, Wi define as 0 omega I hat then your form of Hi is I minus 2 omega I omega I transpose and we say that Hi is a householder matrix and Hi preserved the 0es already created in the first I minus 1 steps and also the product Ai equal to Hi minus I, Ai minus 1 has 0s below the diagonal entries in its ith column.

Then we start working with Ai rather than A and we can say that Ai is now stored as A and we start working with AI and this process we repeat up to N minus 1 step and at the end of n minus 1 is step we have a n minus 1 which is written as Hn minus 1 An minus 2. What is the form of An minus 1, here this An minus 1 has say the first n columns, we have 0 entries below the diagonal entries and a last column we have full column, we do not have any effect on the last column and we can say that that form is nothing, but the upper triangular matrix form.

So, here we denote this An minus 1 as R, so R can be written as An minus 1 where An minus 1 is nothing, but Hn minus 1 An minus 2 and this can be repeated as and we can say that this R can be written as Hn minus 1, Hn minus 2, H2H1 and A. So, R can be written as product of householder matrices into A. Now we say that if you look at the product of these householder matrices.

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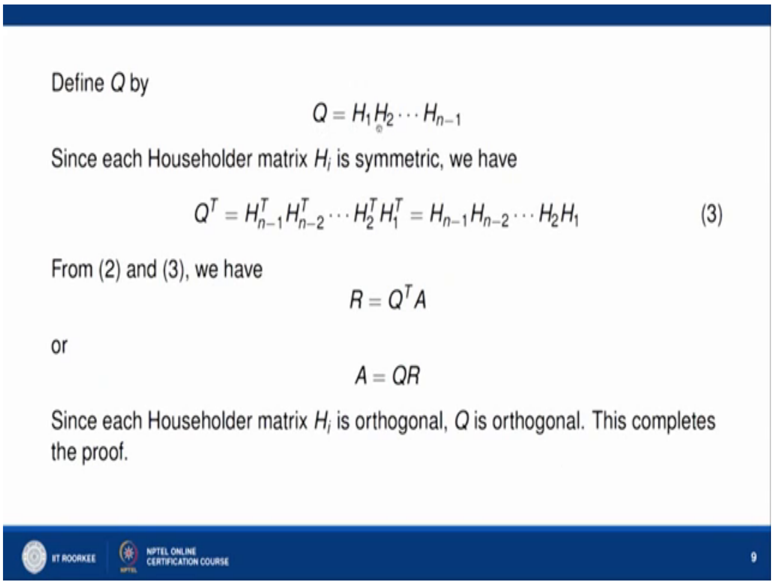

We define Q as H1H2 Hn minus 1 and since householder matrices are symmetric matrices, so it means that each H1 to Hn minus 1 all are in symmetric matrices and so Q is also symmetric and you can find out Q transpose as a a transpose of these matrices and we can write this as Hn minus 1 transpose Hn minus 2 transpose and so on H1 transpose. So, using this Q transpose we can write R as Q transpose A and since Q is also product of householder matrices and each householder matrices are orthogonal, so Q is also orthogonal matrices.

So, Q is basically both symmetric and orthogonal matrices. So, we can apply Q on both side and we can write A as Q into R, where this Q is orthogonal and R is An upper triangular matrices, so this is the proof of this theorem and this is not only the proof, this is basically the method to find out the QR decomposition. Using this proof, we try to find out one example and we try to see that how we can find out this QR decomposition of a given matrix A.

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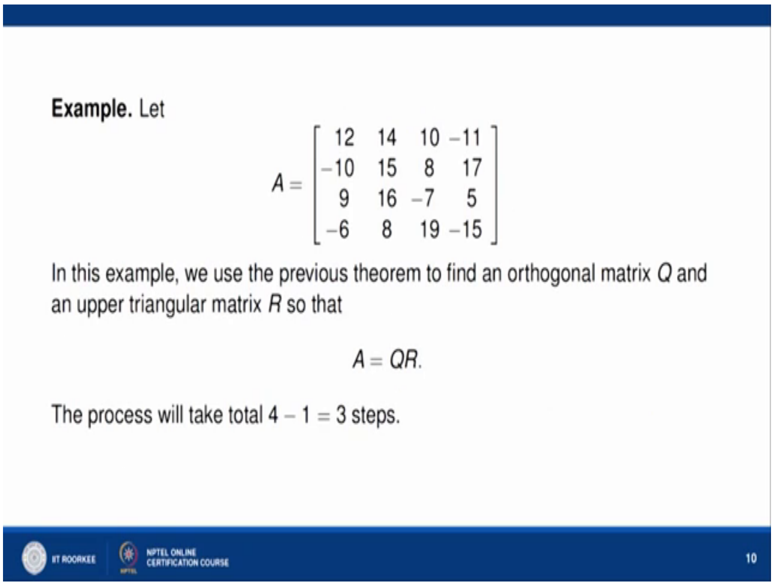

So, let us consider this example, where A is given as 12, 14, 10, minus 11, minus 10, 15, 8, 17 and if you look at this is a 4 cross 4 matrix and we know that in the beginning of the proof of the theorem 1, we have seen that we need a N minus 1 step where N is the size of the matrix N. So, here size is a 4 cross 4 so we need a 4 minus 1 that is 3 steps. So, in 3 steps we are able to convert the matrix A into QR form, where Q is An orthogonal matrix and R is upper triangular matrix.

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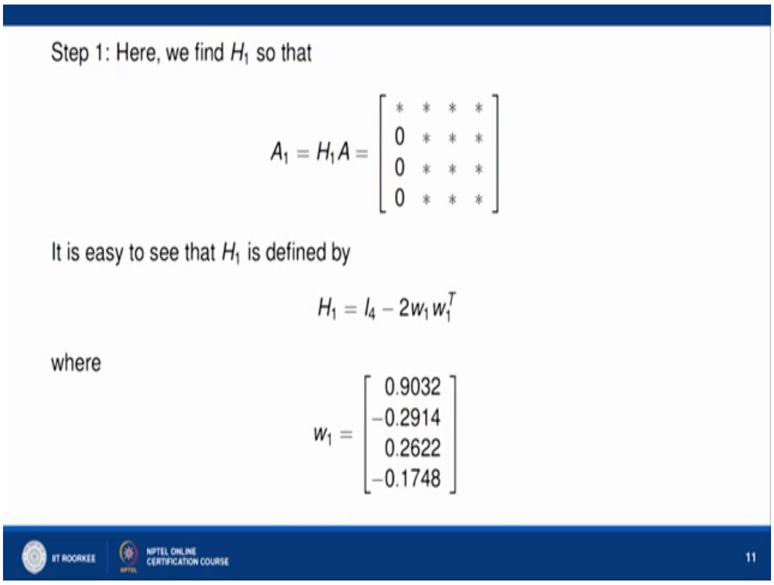

So, first let us start with the first step. So, step one is that we need to find out H1, so that A1 equal to H1 and what is the form of A1 that these entries are untouched, these entries may be anything, I really do not care, but in the first column below the diagonal entry that is this ,we all have 0s. So, to find out this H1 we take this H1 as I4 minus 2 omega 1 omega 1 transpose where omega 1 is given by this. So, to find out this omega 1 we are using the algorithm let me show this using a (Refer Time: 18:20) how we can get this omega 1.

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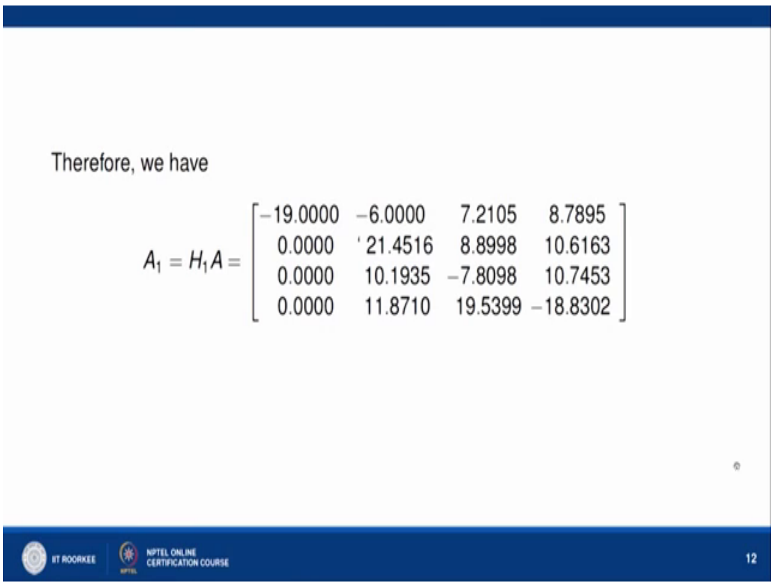

So, let us um just see how we can get this A1 form here. So, we look at given A how to find out this H1 and before finding this H1 how to find out this omega 1 and such that A1 which is nothing, but H1 into A has the following form. So, let us first write down the matrix a here, so let me write the matrix A here, A is this 12, 14, 10, minus 11 that we have written down.

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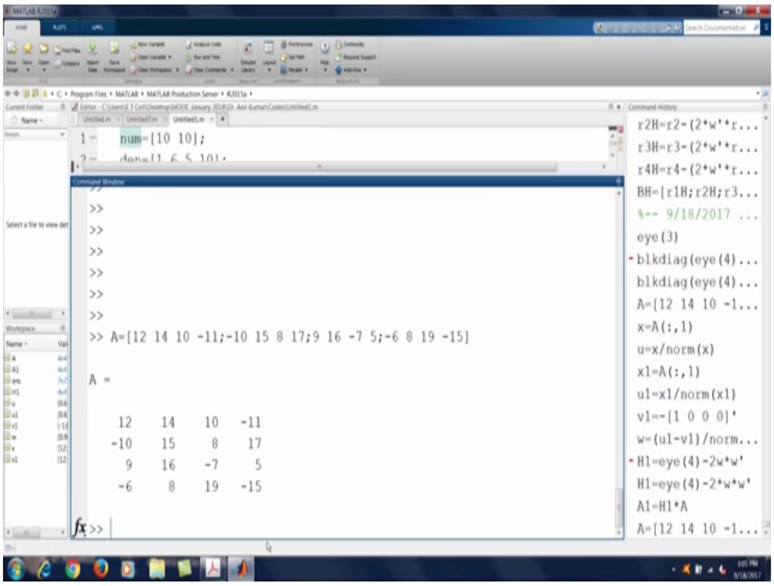

So, A is of this. So, it means that A I have written and then we try to find out the w1, so what is w1 here. So, to define w1, look at the first column here, 12, minus 10, 9, minus 6.

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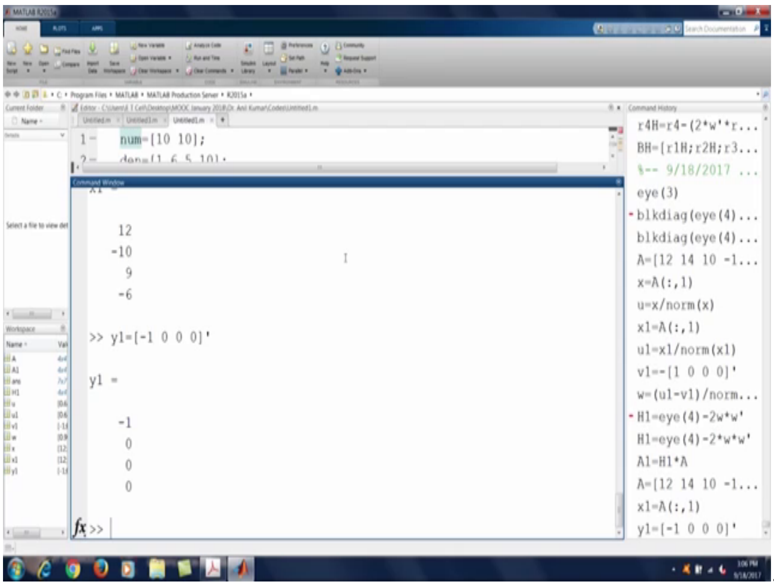

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So, let us call this as a x1 here and we write x1 as the first column of the matrix A here. So, x1 is this and similarly we can define y1. We are using the algorithm to find out H1, so y1 is basically this minus sign of the first component of x1 that is 12, so 12 is positive sign. So, here we can define this as minus 1 0 0 0 and dash.

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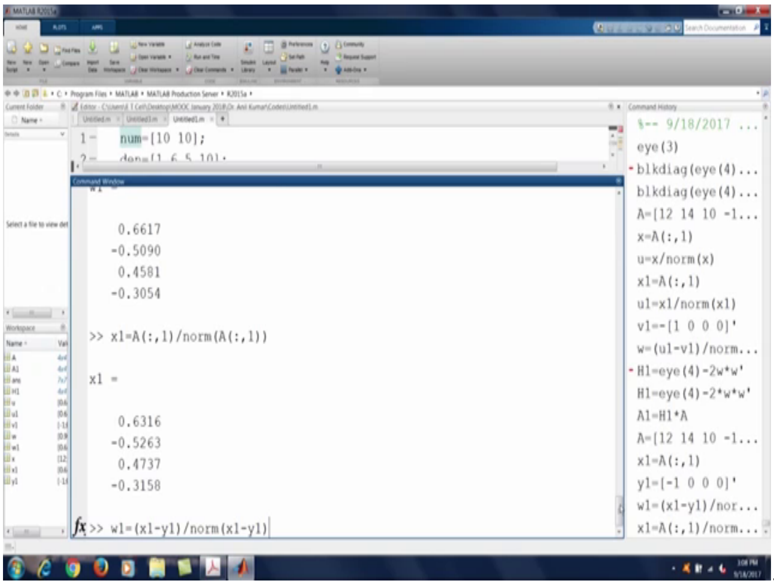

So, y1 is this here and then once we define this y1 then we can find out our w1, let us say that w1 is given by say x1 minus y1 divided by norm of this. So, here we can define by norm of x1 minus 1 norm of x1 minus y1 and if you calculate this is your w1. So, there is a small problem here that here our w1 will be this provided that x1 is the norm one, but here it is not, so, first we need to look at your x1 which is norm of this.

So, let me write it here x1 divided by, let me write it, x1 equal to this A comma 1 that will give you the vector x1 and divided by norm of this A , here we can find out 1 that will give you x1 and once we have x1 then we can define w1 like this. So, now you have w1 as 0.09032, minus 0.2914, 0.02622 and if you look at this is our w1. So, w1 is 0.9032, minus 0.2914, and so on. So, it means that once we have w1 defined to us then we can define our H1.

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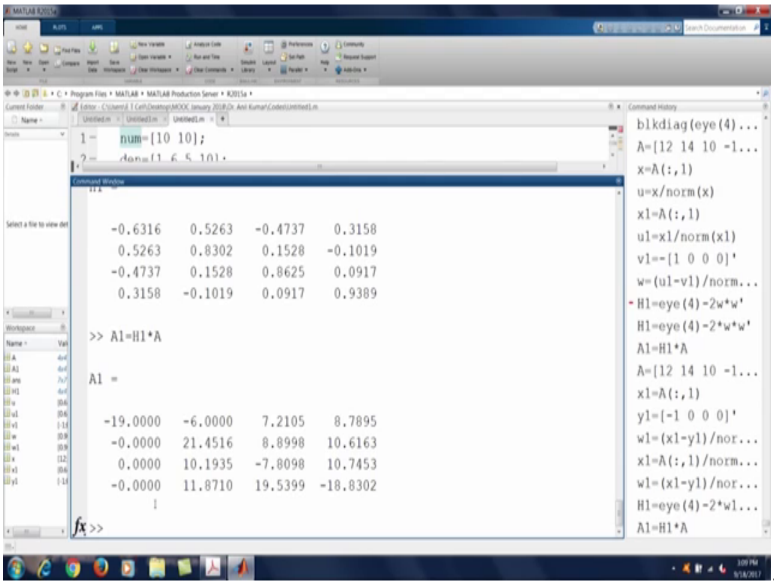

So, to define H1, let us say H1 is basically what H1 is equal to this identity matrix of size 4 cross 4 minus 2 omega 1 into omega 1 transpose. So, let me write it here omega 1 transpose and it is coming out to be this H1 of this form. So, once we have H1 then we can define our A1 as H1 multiplied by A and it is of this form. So, A1 will be minus 19 and if you look at in the first column entries below the first diagonal entries, this diagonal entries are all 0, rest are um it may be any arbitrary thing, it is basically the before one. So, A1 is of this form. So, A1 is that all these entries in the first column below the diagonal entry we have 0s, rest may be anything. So, for that we have first find out W1 with the help of algorithm and once we have w1 then we can define H1. So, now we have this A1.

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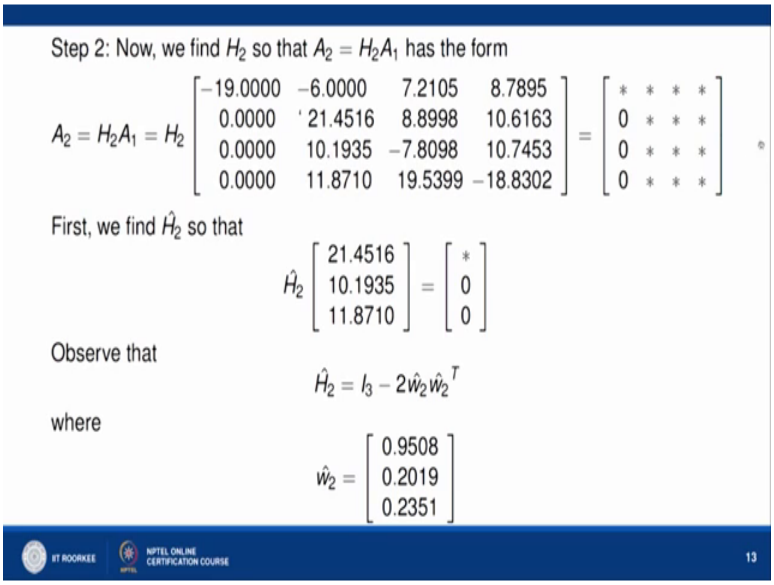

Now we start with this A1 and try to find out H2 such that this A2 will be the form that the entries in the second column below, the diagonal term is all 0s. So, now, we will find H2, so that A2 2 equal to H2A1 has the form A2 as H2A1 that is H2 into this A1 has the form that in the first column all the entries before the diagonal term is 0 and in the second column entries below the diagonal terms are 0 means these entries are 0 and all others may be untouched.

So, to find out this H2 hat we look at this second column and look at these entries 21.4516, 10.1935, 11.8710 and what we want to make here that it is these entries this second entries this is non zero and these entries are 0. So, these entries we want that should be 0. So, to define H2 hat we define H2 hat as I3 minus 2 omega 2 omega 2 hat transpose and we have this is x and y is minus of 1 0 0 and again using the algorithm we can find out w2 hat and we can say that w2 hat is nothing, but this that we can also look at here.

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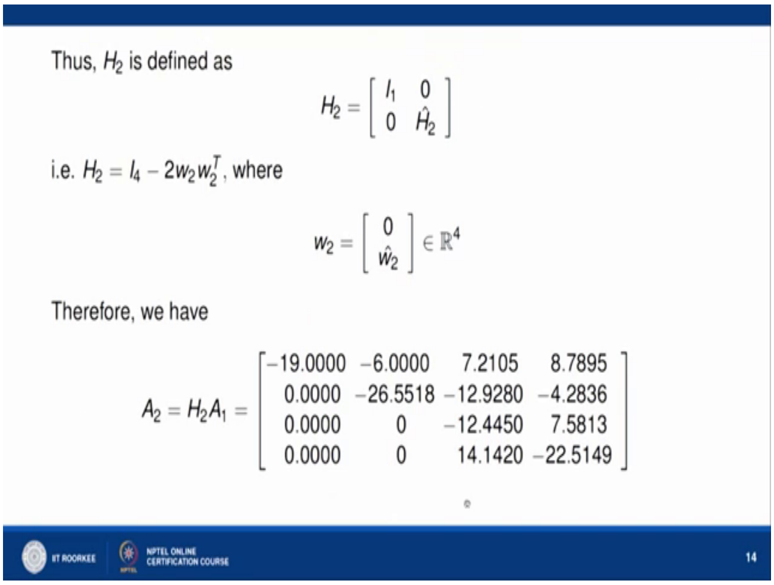

So, once we have w2 hat then we can define w2, w2 is nothing, but 0 W2 hat and once we define w2 then we can write H2 as I4 minus 2 omega 2 omega 2 transpose and we have already seen that this H2 will have this form, H2 as I1, 0, 0, H2 hat and if we operate H2 on this A1 then we have this form that in first column entries below the diagonal term it is all 0 and the second column also entry below the diagonal term that is this term all entries are 0.

So, that is the effect of H2. So, here you can find out this w2 hat with the help of algorithm and then we can define w2 with this setting that it is 0 w2 hat once we have w2 then we can define H2 as I4 minus 2 omega 2 omega 2 hat and we can have H2, so once we have H2 we can write A2 as H2 into A1. So, here we have this form. Now what we want we need to find out H3, so that A3 which is given as H3A2 has the following form.

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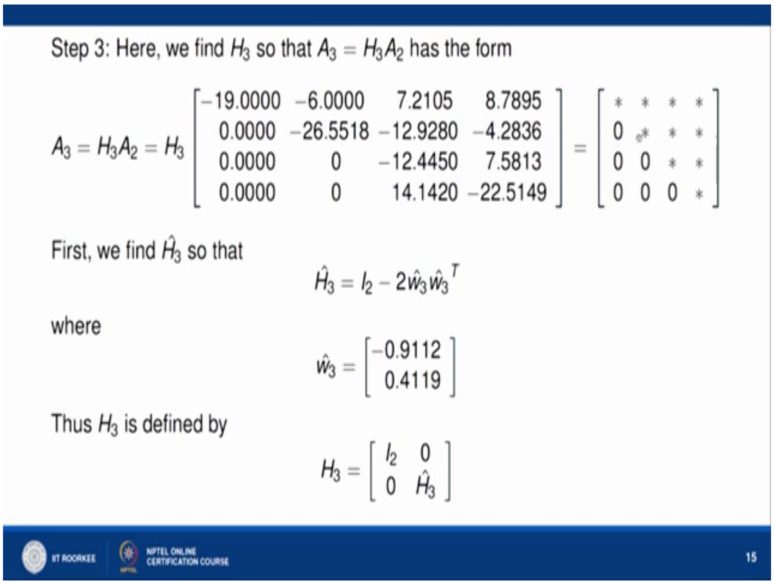

So, here A3 is equal to H3A2 H3 as this, A2 we have just calculated as this, we write it here and what we want that this entry has to be 0, it means that we have this form that in first column these three are 0, in second column these two are 0, in third column this entry has to be 0, means this we want to um replace by 0. So, for that we define H3 hat, H3 hat is I2 minus 2 omega 3 omega 3 hat, how to find on this omega 3 hat, this omega 3 is this is your x3 and y3 is 1 0 and H3 hat will map this minus 12.4450, 14.1420 to 1 0.

So, for that you can find out W3 hat as um this x3 divided by norm of x3 minus Y that is 1 0 divided by norm of x3 minus y3 that is we are using algorithm that is all we want to say. So, once we have w3 hat that is minus 0.9112, 0.4119 then we can define H3 as I2, 0, 0, H3 hat, where H3 hat is defined by I2 minus 2 omega 3 omega 3 hat. Let us just verify that W3 hat is given by this and H3 we can calculate in what form. So, to find out this w3 hat we need to find out that your H3 hat will send this to 1 0, so, here it is minus 12.4450, so let me write it x3 as um, (Refer Time: 27:49) so, we need to find out this w3 hat.

So, here we let us try to find out this H3 hat which have this effect that it make these 2 entries as a some non zero into entries and 0 entries. We need to find out such H3 hat where H3 hat is I2 minus 2 omega 3 hat omega 3 hat transpose omega 3 hat, we can find out like this and once we have omega 3 hat.

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i.e. $H_3 = I_4 - 2w_3w_3^T$, where

$$w_3 = \begin{bmatrix} 0 \\ \hat{w}_3 \end{bmatrix} \in \mathbb{R}^4$$

Therefore, we have

$$R = A_3 = H_3A_2 = \begin{bmatrix} -19.0000 & -6.0000 & 7.2105 & 8.7895 \\ 0.0000 & -26.5518 & -12.9280 & -4.2836 \\ 0.0000 & 0 & 18.8381 & -21.9106 \\ 0.0000 & 0 & 0.0000 & -9.1826 \end{bmatrix}$$

and

$$Q = H_1H_2H_3 = \begin{bmatrix} -0.6316 & -0.3846 & 0.5087 & -0.4410 \\ 0.5263 & -0.6839 & -0.2461 & -0.4413 \\ -0.4737 & -0.4956 & -0.5304 & 0.4988 \\ 0.3158 & -0.3727 & 0.6320 & 0.6017 \end{bmatrix}$$

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Then we can define H3 as I blocked diagonal matrix I2, 0, 0, H3 hat and once we have H3 then we can we can find out A3 as H3A2 and which have this form. If you look at this form, this is An upper triangular matrix. So, here we stop and we say that this is our upper triangular matrix, we call this as R, so R can be written as A3 and A3 is nothing but H3A2 and we can write down the expression for A2 that is a H2 basically it is H2A1 and A1 we can write it H1AA.

So, we can write this R as H3, H2, H1A and we can write Q as H1, H2, H3 and this is given by this. So, it means that we can find out our Q and R. So, let us ah stop little bit here and we try to show that how we can find out this W3 hat which makes this entry to 0 0 entries here.

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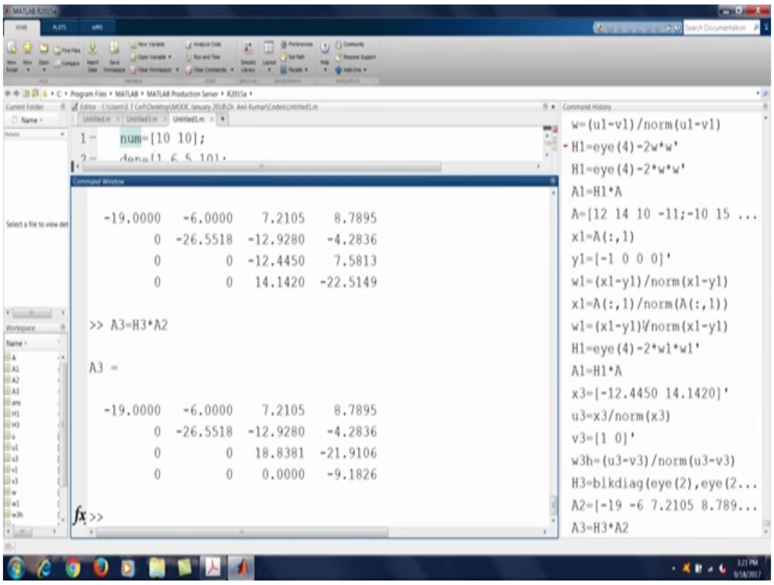

So, let us do it in An algorithm. So, here we first defined what is x3 here , So let us say the x3 is this minus 12.4450, 14.1420 that is here, minus 12.4450, 14.1420, but if you look at this x3 is not a unit vector, so for that we need to find out a unit vector.

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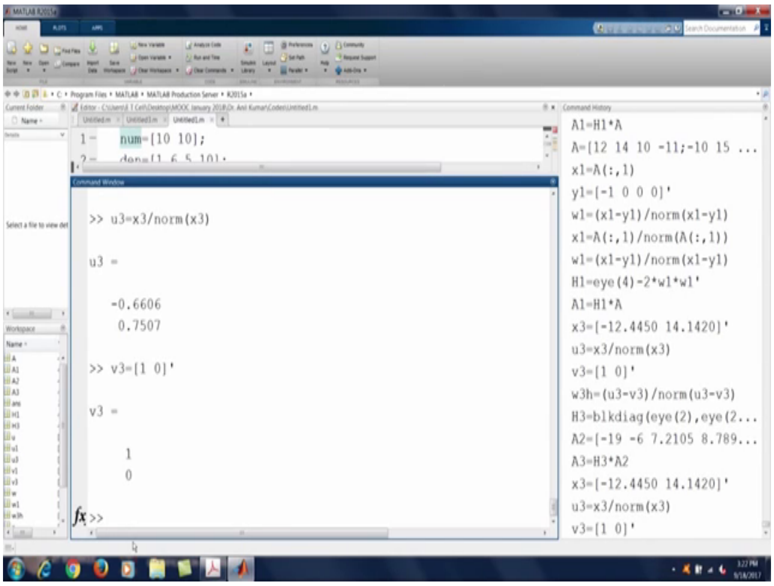

So, let us call this as u3, u3 is nothing but x3 divided by norm of x3 and that makes this as a unit vector. So, it is minus 0.6606 and 0.7507. Now we want to define v3 unit vector and that is given by minus sign of this, now sign of this is basically this is negative. So, v3 is going to be nothing, but a 1 0. So, v3 is this. Now once we have u3, v3 then we can define our w3, since here we are using w3.

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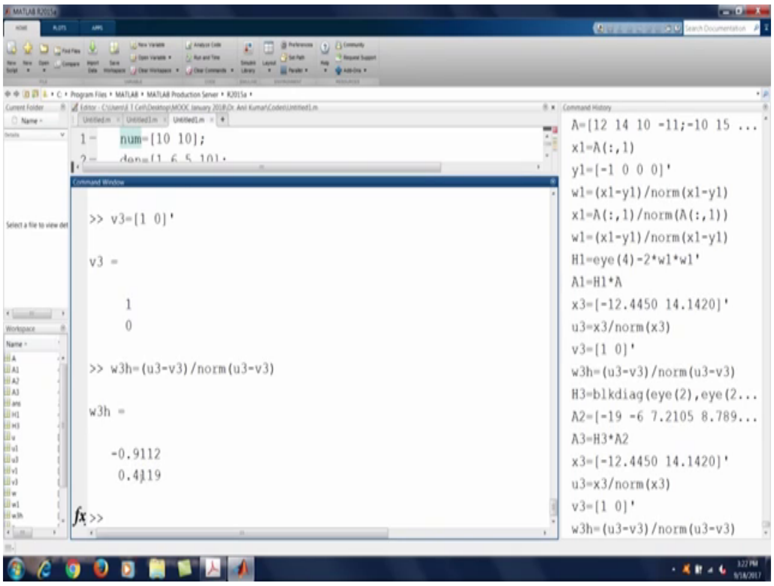

So w3 hat is given by u3 minus v3 divided by norm of u3 minus v3 and it is coming out to be this w3 hat is equal to minus 0.9112, 0.4119 and that is what it is written here, w3 hat as minus 0.9112, 0.4119. So, once we have w3 hat with us

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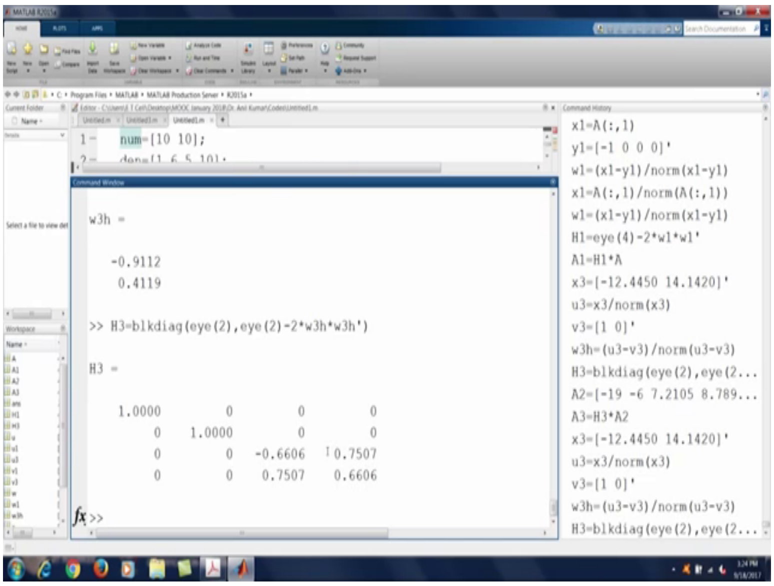

Then we can define our H3, so, H3 is given by , we need to define first of all H3 hat and H3 is what, H3 hat is basically I2, this is your H3 hat, that is I2 minus 2 omega 3 omega

3 hat omega 3 hat transpose and once we have H3 hat then we can write H3 as a block diagonal matrix like this. So, let us write H3 as the following thing and this is the representation of a writing a block diagonal matrix.

So, this is the command block blk diag block diagonal matrix where the first diagonal is I2 that will give you the identity matrix of size 2 cross 2. The remaining diagonal entries are I2 minus 2 omega 3 hat into omega 3 hat transpose, that is nothing, but your H2 hat. So, you should calculate your 3 is coming by this. So, this will represent your H3 hat, so once we have H3 hat we can define H3 as the I2, this H3 hat 0 0.

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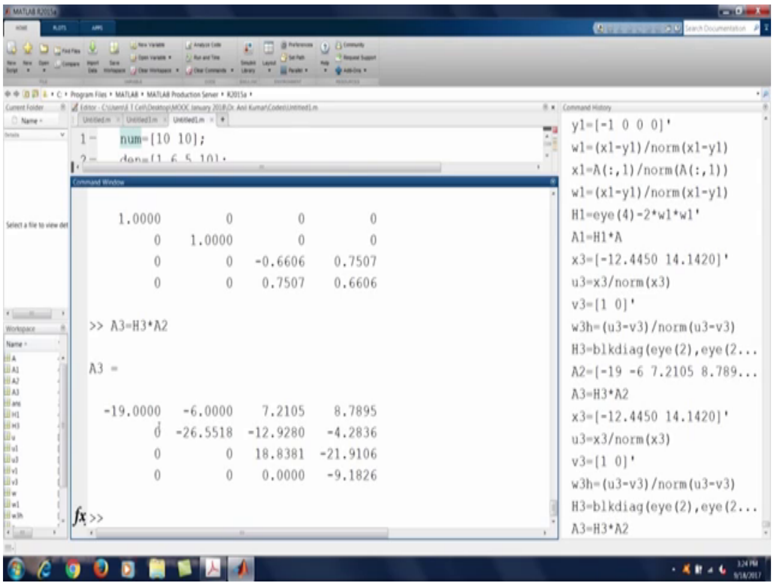

So, once we have H3 then we can calculate our A3, our A3 is basically what, A3 is basically your H3 into A2, where A2 is what, A2 we already know, using this A2 is given by this. So, we can write A3 as H3 into A2 and that is this thing A3 is minus 19.0000, minus 6.0000 and minus 26.5518 and here 0. And if you look at the third column here, 7.2105, minus 12.9280 and if you look at the last term is 0 and this is maybe non0.

So, here we call this A3 as R here and that is what we have done here, we have defined R as A3, because now it has this upper triangular form and Q as the product of H1, H2, H3. Now, since each one is symmetric and each one is orthogonal, so Q is your orthogonal matrix and R is also in upper triangular form, so we can write um A as this Q into R. So, that will complete the verification part of this theorem and here one thing you may note it down.

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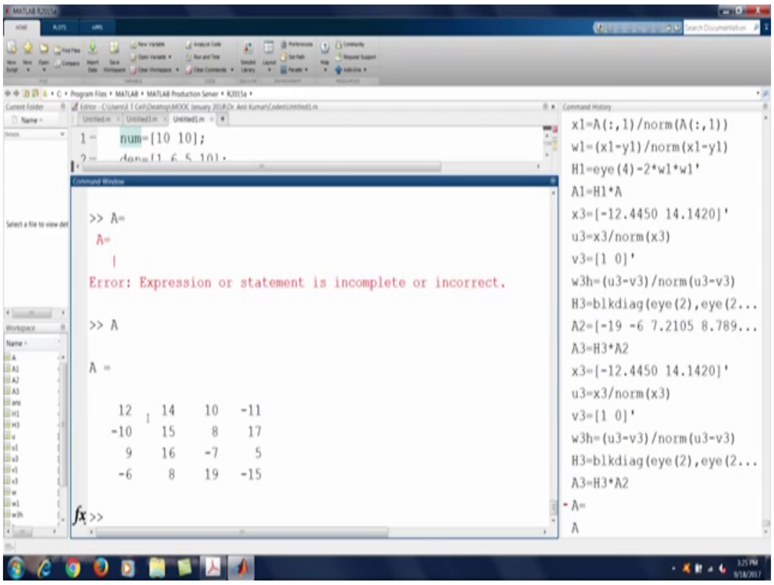

That your matrix A is here this and we try to find out this Q and R then this is command here this Q and R , you can write it here as QR of A.

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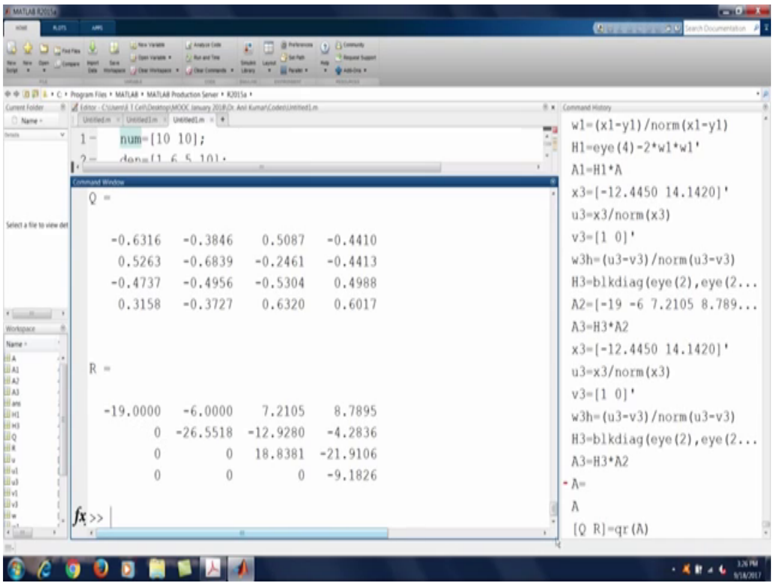

So, this is the command of (Refer Time: 33:52) command, this will give you the upper triangular matrix that is R and if you look at this R and A3R seen here, we have seen your R that we have obtained through the proof of the theorem. And if you look at here, we have just used a command QR of A and that will give you the value RM and Q here is this and the matrix that is your H1 into H2 into H3 that is your Q. So, this command QR equal to QR of A that will give you the QR decomposition of a given matrix A. So, here I will stop and in next lecture we will discuss the uniqueness of this, QR factorization and how to find out the QR decomposition of a rectangular matrices and once we are done with this QR decomposition method then we will see some application of this QR decomposition as in the problem of Eigenvalue problem. So, here I will stop. Thank you for listening us.

Thank you.