Numerical Linear Algebra

Dr. D. N. Pandey
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 52
Householder matrices and their applications

Hello friends, welcome to the lecture. Here we will continue our discussion over Householder Matrices and their applications. So, in previous lecture we have defined what is known as a householder matrix; and we have discussed certain elementary properties of householder matrix and one of the application of the householder matrix that given two unit vectors say u and v. Then we can define a householder metric H given the form I minus 2 omega omega transpose; such that it map u to v.

So, and we have also discussed a corollary of this result says say that we in place of unit vector if we start with any two nonzero vector; then also we can define a householder matrix which map a given one vector to another vector y. So, and we have defined one algorithm and we have discussed one example also. So, let us continue our study and use the same algorithm to now to define householder matrix which map a matrix to a different matrix having certain properties.

So, let us continue, but before that let us recall the following algorithm: It says that, given a vector x in Rn, the following algorithm finds a householder matrix H, given as I minus 2 omega omega transpose, where omega is a unit vector; it means norm of omega is 1; such that H of x is equal to mu of ei, where mu is a constant given as minus sign of x 1 norm of x. Now here x 1 represent the first component of the vector x.

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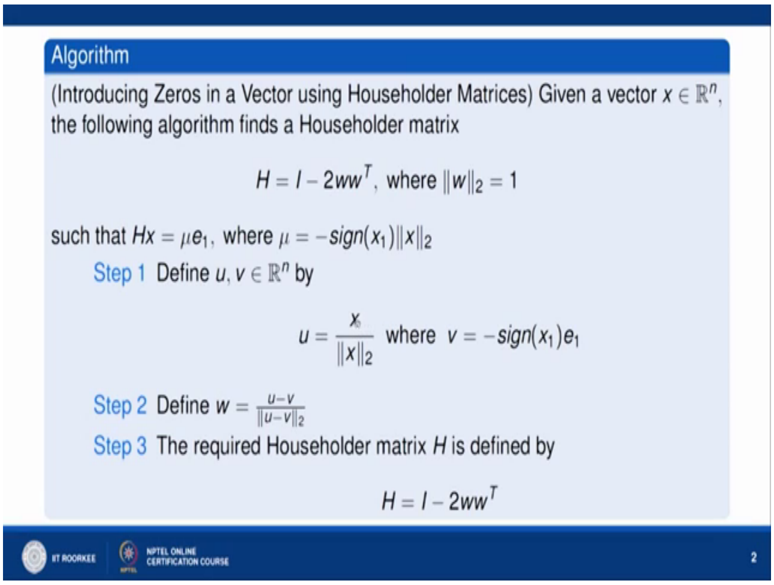

So, to find out this H we define u and v unit vector in Rn by u as x divided by norm of x, and v has minus sign of x 1 into e 1. Now, this will be a unit vector and u is by construction is it is a unit vector. So, with the help of u and v defined omega as u minus v divided by norm of u minus v. Now, once we have w then we can define H as I minus 2 omega omega transpose; and this will map this will map the x into mu of e 1.

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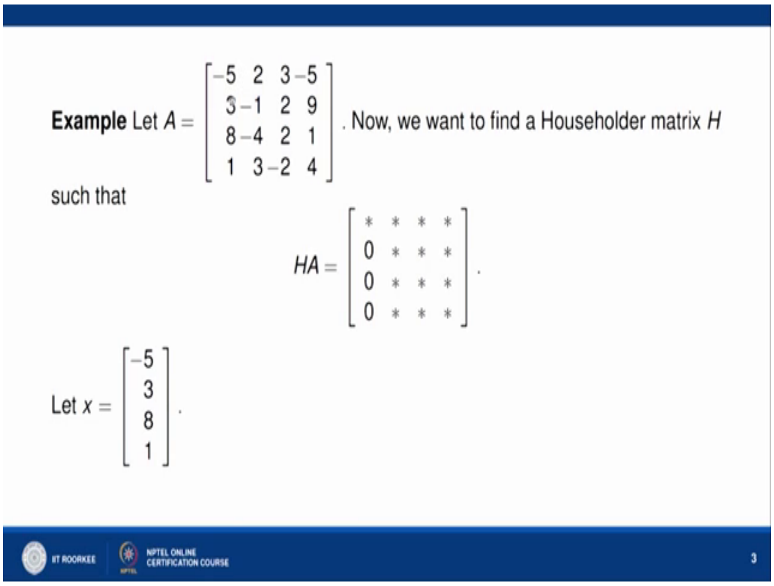

. So, we have seen one example. Now, let us consider one more application of the following the given algorithm. For that let a is this matric a is a 4 cross 4 matrix. And we want to find out a a householder matrix H; such that if you look at the first column. In first column these entries are 0 or we can say that only nonzero entries the first one and rest are all 0 entry.

So, we need to find out such a householder matrix. And the idea is that if we repeat this process then we can convert this matrix a in an upper triangular matrix. That is one very important application of householder matrix.

So, let us first do this step. So, for that you define x as the first column minus 5 3 8 1. So, we have defined let x equal to minus 5 3 8 1.

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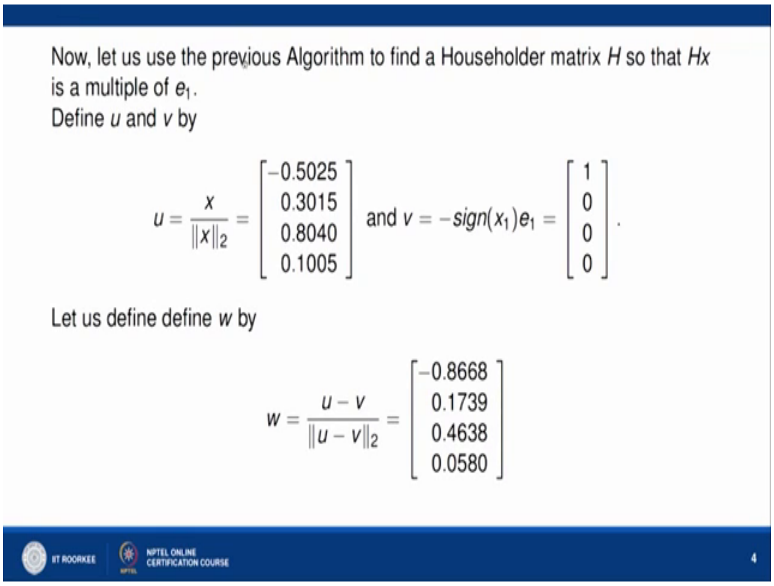

And you define. Now, we use previous algorithm to find a householder matrix H such that H of x is a multiple of e 1. So, define u and v by as we have defined in algorithm you as x divided by norm of x and it is given by minus 0.5025 and so on; and we as minus sign of x 1 into e 1.

Now, if you look at the first component of x is minus 5. So, sign of x 1 is minus 1. So, here v is nothing but 1 0 0 0. And now once we have u and v then we can define w as or omega as u minus v divided by norm of u minus v and we can calculate like this.

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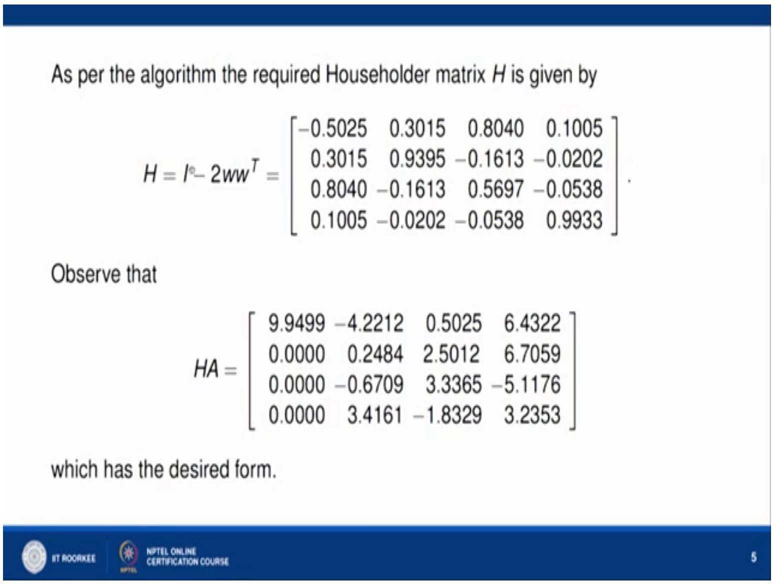

Once we have w then we can define H as I minus 2 omega omega transpose; and it can be calculated like this. And then once we have householder matrix; then we can calculate H of A and if you look at the H of A will have the desired property means in the first column; the entries below the diagonal term is basically 0. So, in the first column entry below the diagonal term is all zeros. So, only nonzero entry in the first column is the diagonal term that is this one. So, that is what we wanted to have this. So, we can show this thing in a MATLAB also.

So, now here if you look at we have we first we have find out what is w or omega and then we have calculated the entire H and then we form this. But here we say that to find out this H of A, we need not to define the entire of H. In fact, you can do this without calculating the entire householder matrix and you can do the same process only with the knowledge of omega.

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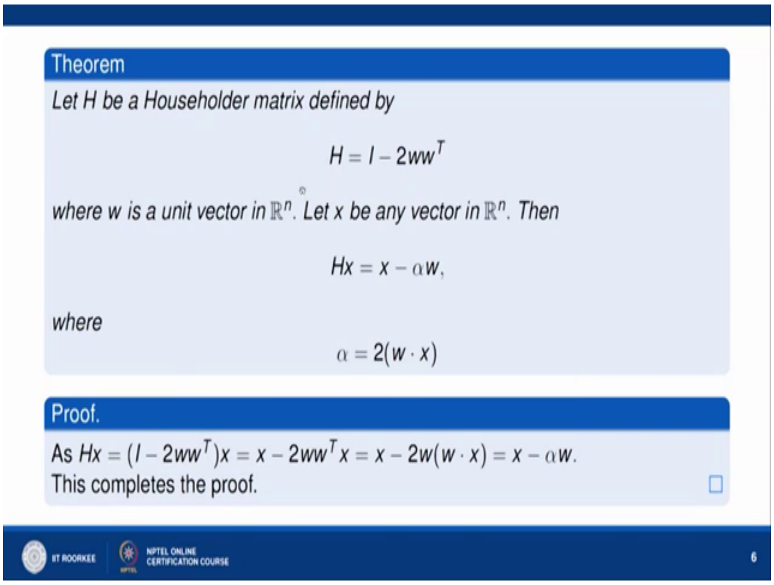

So, the following theorem, this theorem says that we really need not to calculate the entire householder matrix H. In fact, we can do only with the omega as a unit vector. So, with the help of omega only you can easily calculate the whole of H of A.

So, let us consider this theorem. Let H be a householder matrix defined by H as I minus 2 omega omega transpose, where omega is a unit vector in Rn. And let x be any vector in Rn. Then H operating on x is nothing, but x minus alpha omega; where alpha is given by 2 times omega dot x. This is the inner product of omega and x. And this is very easy to see that this result is quite easy to prove. So, to show that H of x and now H is I minus 2 omega omega transpose x; which is nothing but x minus 2 omega omega transpose x. Now, omega transpose x can be written as omega dot x. So, we can write this as x minus alpha omega; where alpha is 2 times omega dot x.

So, that completes the proof of this theorem. Now, this is this says that the action on of H on this x is nothing but x minus alpha omega; where alpha is easily calculated as 2 times omega dot x. So, here if you look at the operation of H transformation H operation of H on x is nothing but x minus alpha omegas; you can easily calculate alpha with the help with the knowledge of omega and x. And here we required only x and the omega we need not to find out the entire H here.

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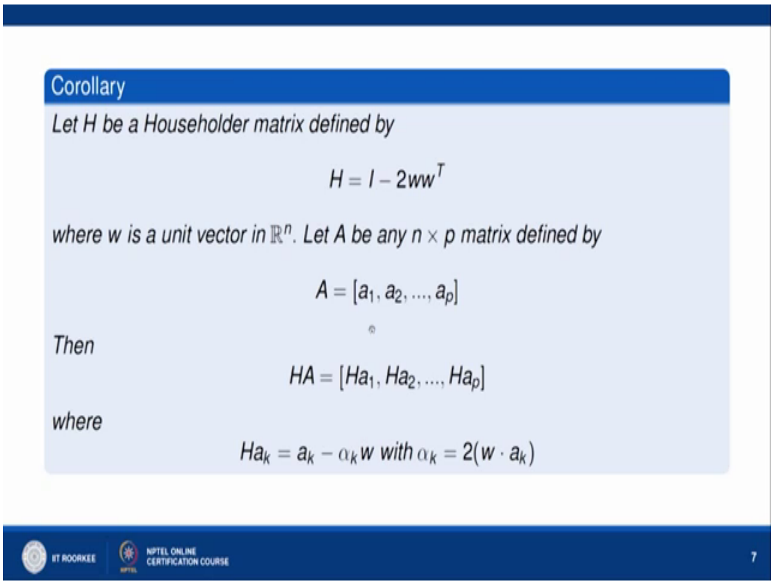

Now, with the help of this result; now, we have to corollary; and which if which enable us to find out the product H of A and B of H. So, let us say let H be a householder matrix defined by H as I minus 2 omega omega transpose; where omega is a unit vector in Rn. And let a be any n cross p matrix defined by a as a 1 to a p; where a 1 two a p define the p columns of a. Then H of A you can calculate as H of a 1, H of a 2 and H of a p. So, in in matrix H of A; your columns are given as H of a 1, a 2, H of a 2, and H of a p.

And you can use our the previous result, that H of a 1 can be given as a 1 minus alpha 1 w or in general you can say that H of a k can be given as a k minus alpha k w; where alpha k is nothing but 2 omega dot a k. So, here we need not to find out the entire H we can say that by knowledge of a is and omega. We can easily find out the matrix H of A.

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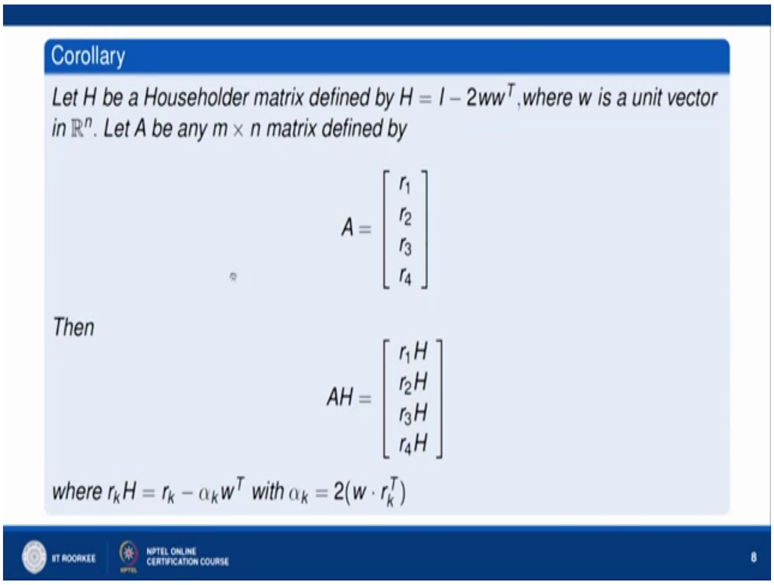

Similarly, if we have a matrix a given in terms of row form. So, here A as given as r 1 r 2 r 3 r 4 where r i is are basically say rows of your matrix A. Then you can calculate as r 1 H r 2 H r 3 H r 4 H; where this r 1 r k H you can calculate as rk minus alpha k omega transpose; where alpha k is given by 2 times omega dot r k transpose.

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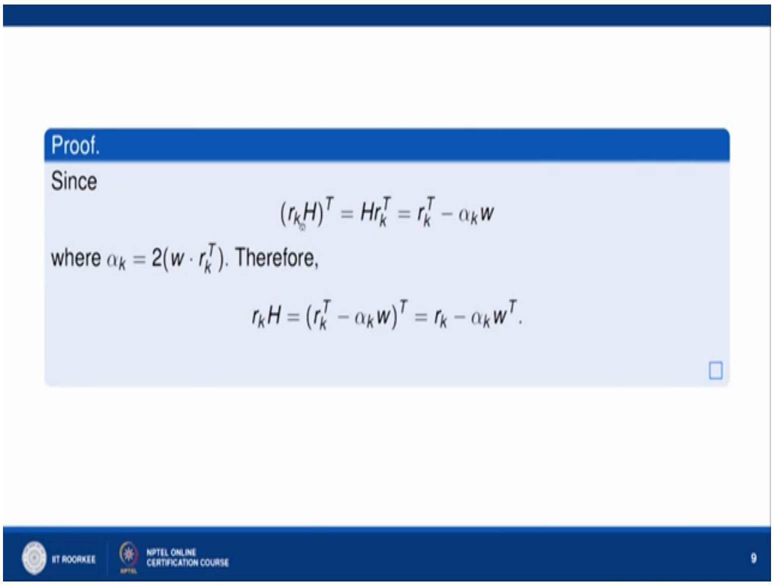

So, the proof of this is quite easy. In fact this you can say that r k H transpose is nothing but Hr k transpose. And for Hr k transpose we have this result r k transpose minus alpha k omega; where alpha k is nothing but 2 times omega dot r k transpose. So, here we can write r k H as transpose of this that is r k transpose minus alpha k omega transpose; and which is nothing but rk minus alpha k omega transpose.

So, it means that to calculate and H B we need not to find out the entire H; only information of omega is sufficient to calculate that.

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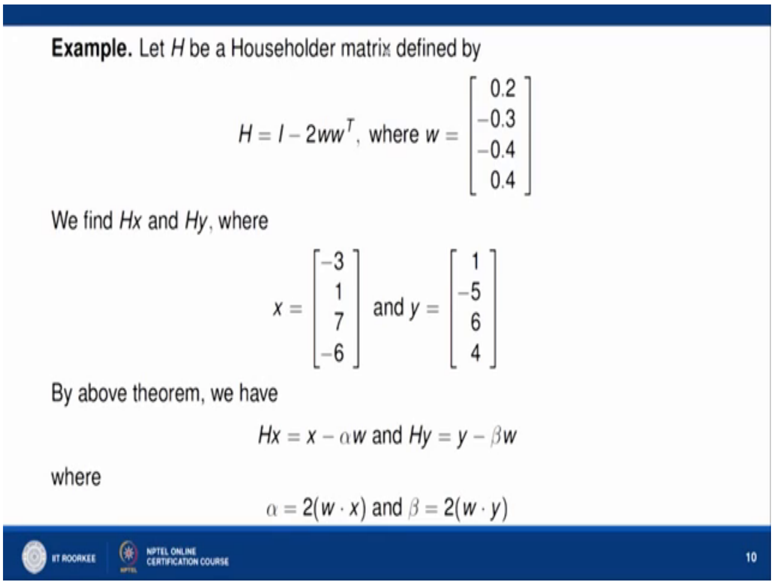

So, let us consider some example based on the previous results. So, here let H be a householder matrix defined by H as I minus 2 omega omega transpose; where omega is given by 0.2 minus 0.3 minus 0.4 and 0.4. And we want to find out H of x and H of y, where x is given as minus 3 1 7 6 and y as 1 minus 5 6 4.

And we this we want to find out without calculating the entire H. So, here if you recall the result then H of x is nothing but x minus alpha omega, where alpha is given as 2 times omega dot x and H of y as y minus beta omega where beta is given by 2 omega dot y. And that you can calculate very easily. And we can say that we can calculate alpha as minus 12.2 and beta as 1.8, for this particular example and H of x you can calculate as x minus alpha omega and H y as y minus beta omega and this is the result here.

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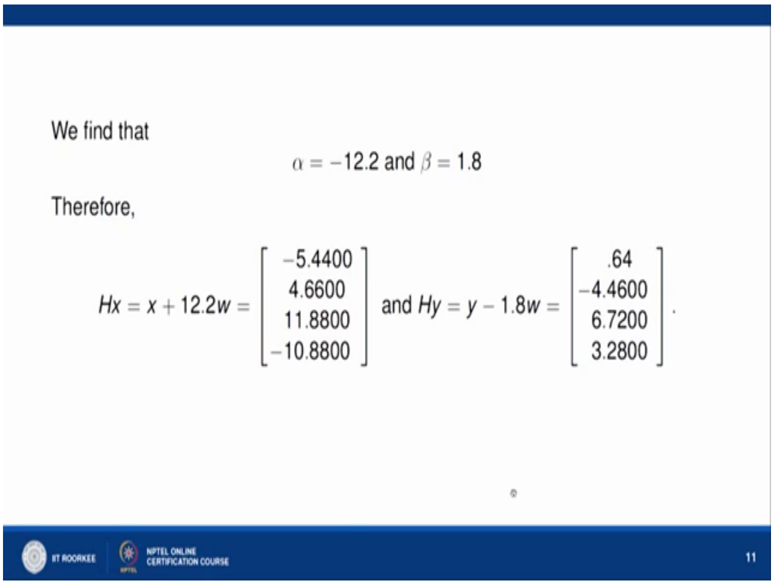

So, we can easily see this using MATLAB also. So, let me do this problem on MATLAB and see that how is it is.

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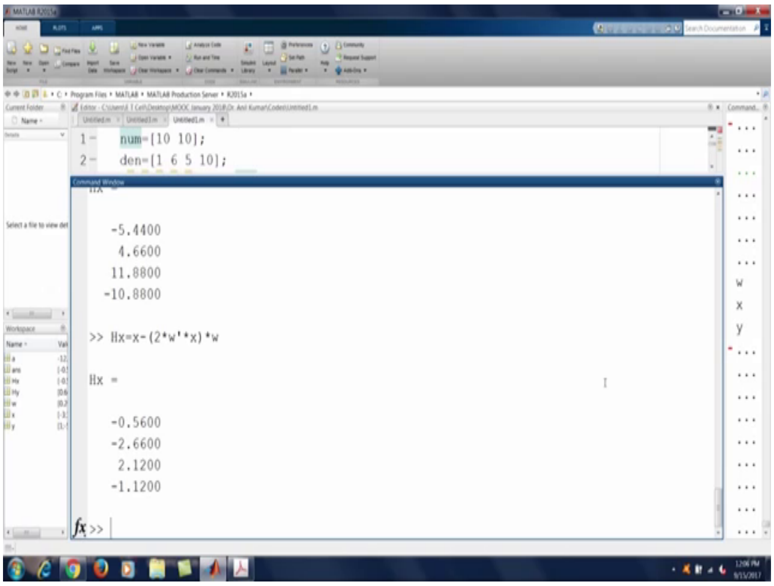

So, here we first define what is w. So, w is given by. So, here I have started with the 2 3 4 4 and ok. So, let me write it 0.2 and 0.3 with minus sign and then 0.4 again with minus sign and 0.4 and transpose of this. So, we if you take the transpose that is your w 0.2 3 4 4 [FL] and this is given by and x is minus 3 1 7 minus 6.

So, x you can write it as minus 3 1 6 and 7 I think last one is with minus sign; I think last one is minus 6. So, it is this. So, I think minus 3 1 7 minus 6 minus 3 1 7 minus 6 ok. So, it is like this. So, here x is given similarly you can write a y and y is given by this 1 1 minus 5 6 4. So, we have y also.

Now, to calculate H x; we use this formula x minus 2 times omega dash x into omega; where this represent your value alpha. And if you use this then the value of H of x is given by this minus 0.5600 and. So, you if you see this is actually it is coming out to be minus 0.560 minus 2.66 2.12 and minus 1.12. And regarding the value of alpha here you can easily calculate the value a here, where a is given by 2 omega dash cross x and it is coming out to be minus 12.2.

So, that just verify that whatever alpha we are have written here is the correct and similarly you can perform for beta also. So, for writing beta we write H of y and it is given as y minus 2 times omega dash into y into omega; and if you calculate it is coming out to be this quantity. So, and you see then it is matching with the quantity written here.

So, it means that here we have not calculated our H and we can calculate the fact of H on x by simple linear combination of x and omega. And with the constant alpha here where alpha can be evaluated as 2 omega dot x and similarly we can find out H of y; whereas y minus beta omega where beta is given as 2 omega dot y.

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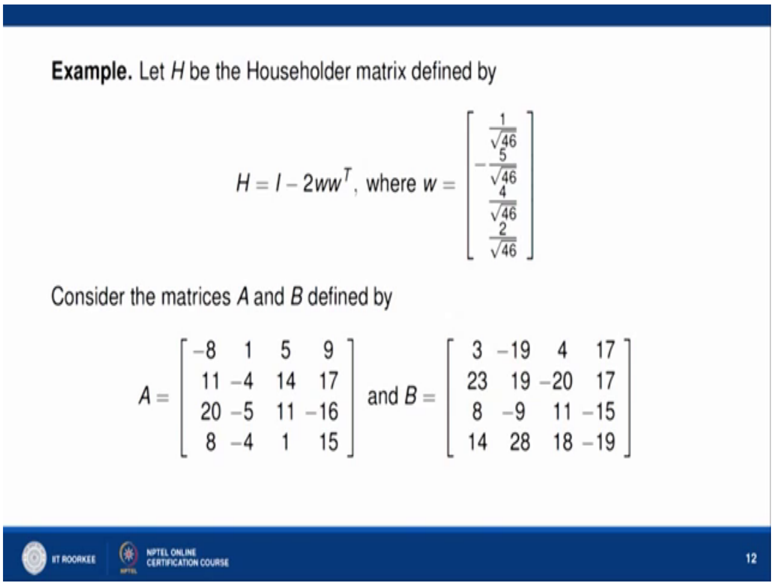

So, now moving on one more example, but this time we are considering the whole matrices A and B and we want to use the corollaries to find out H of A and H of B without finding the H. So, these we try to find out H of A without calculating H, but only the help of omega and the columns of A and rows of B here.

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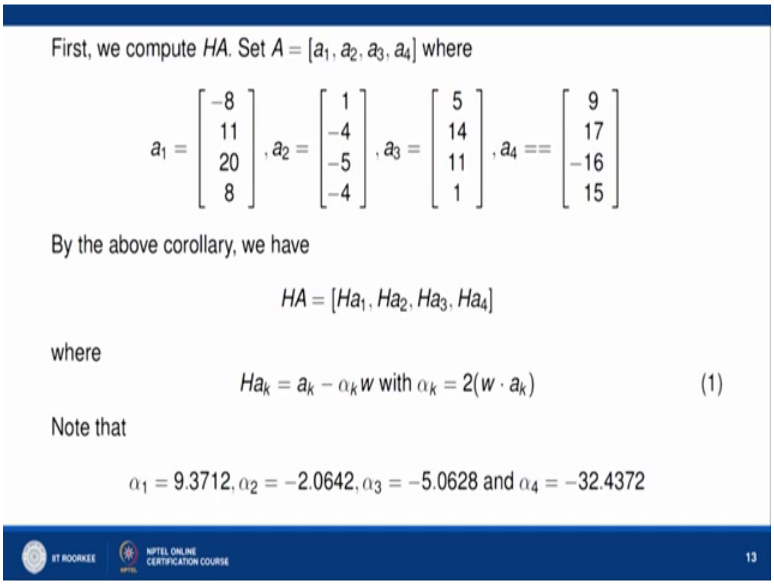

So, here first we try to find out a HA. So, here we write A as matrix of columns here. So, here we write A as a 1, a 2, a 3, a 4 where a i represent the i-th column of A. So, a 1 as minus 8 11 20 8 similarly we can write down all the columns here.

And by the corollary we know that H of A is nothing, but H of a 1 as first column, H of a 2 as second column, H of a 3 as third column and so on. And you can use the result for H of a 1 to calculate H of a 1. We have a 1 minus alpha 1 omega or we can say that for any k H of a k is equal to a k minus alpha k omega; where alpha k is nothing but 2 omega dot a k.

So, to calculate this we use this alpha k as 2 omega dot a k and we can write alpha 1 as this, alpha 2 is equal to this, and alpha 3 equal to minus 5.0628, and alpha 4 equal to this. So, once we have alpha 1, alpha 2, alpha 3, alpha 4 then we can write down H of a k a is equal to a k minus alpha k omega; and it is calculated as follows; so H of a 1 is given this, H of a 2 is this and so on.

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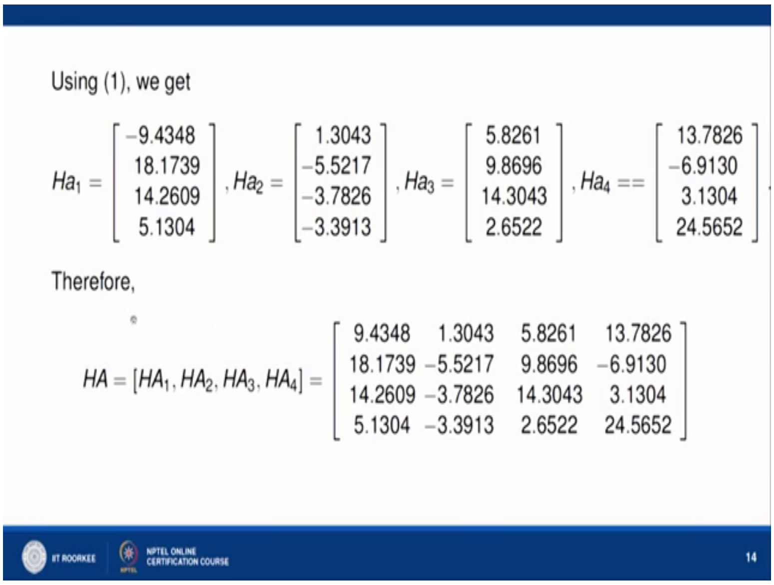

So, once we have H i H of a i's is given to us, then we can form our matrix H of A as H of A 1 as first, H of A 2, H of A 3 and H of A 4. So, that will give you the matrix H of A.

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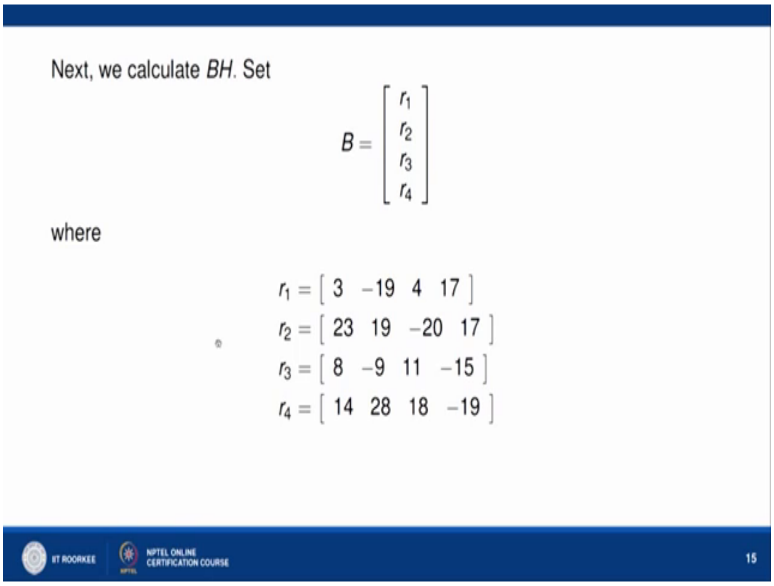

Similarly if you want to find out matrix B of H, then to find out say operation H on the right hand side. So, we write in this to calculate B of H we write B as say column vectors or we can say that r 1 r 2 r 3 r 4 are rows of your matrix B. So, we can easily write r 1 as 3 minus 19 4 17; that we can write it from the matrix B listed here. So, B r 1 is this, r 2 is this, r 3 and r 4 we can calculate.

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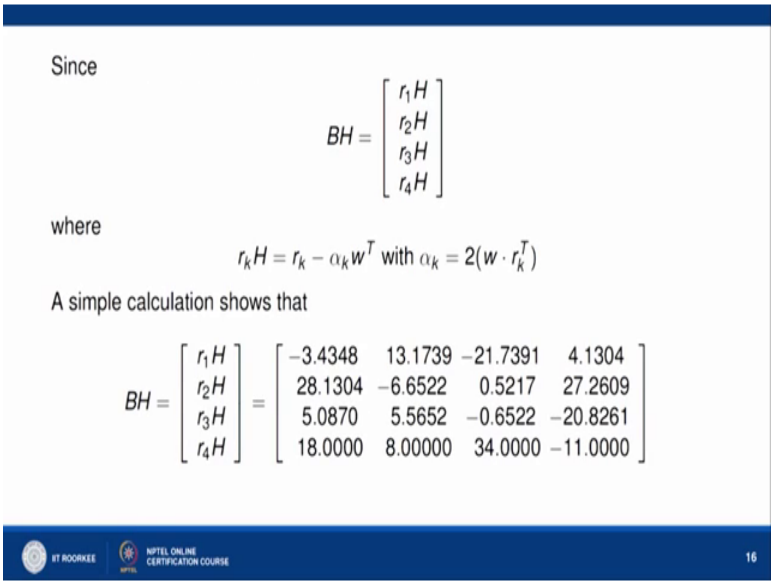

And then using corollary, we can write BH as r 1 H r 2 H r 3 H and r 4 H; where r i H or r k H is given as r k minus alpha k omega transpose, where alpha k is nothing but 2 omega dot r k transpose. So, we can calculate r k H and we can write it BH as r 1 H r 2 H r 3 H and r 4 H so that we will give us the product HA and BH without actually calculating our H.

So, let us just verify it just one verify it with the help of MATLAB here. So, let me write it this whole thing in your MATLAB. So, we are going to verify the calculation which we have shown here in this example with the help of MATLAB. So, for that the first we define what is our matrix k A here. So, let me write A as minus 8 1 5 9; 11 minus 4 14 17; and that is given here minus 8 1 5 9 11 minus 4 14 17 that I am writing here and if you enter you will get your matrix A.

Similarly, we define what is your B. So, B as capital B here, and we have this 3 minus 19 4 17 and this I have written here. So, we have written our A and B like this. And then we write down this w. Now, what is w here, w is if you look at it is the vector minu[s]- 1 minus 5 4 2 divided by the norm of that vector; so we to define your vector w.

Let me define first your x. So, x we are taking as 1 minus 4 minus 5 4 2; and then we calculate when we can calculate the norm of x here. So, we can calculate norm of x and it is coming out to be this and you can define w as x divided by norm of x. So, that will give you the w here. So, w is given by this.

Now, to find out H of A, we need to calculate H of a 1, H of a 2, H of a 3, and H of a 4. So, first we need to look at what are these a i's. So, to find out a i's let us say that a 1 is basically our matrix a the first column. So, first column I can write it here as this as here. So, this will give you the first column that is minus 8 11 20 8. So, are you get huh. So, a 1 is minus 8 11 20 8; similarly you can calculate a 2 as A here we have this 2 that will give you a 2 1 minus 4 5 minus 5 4 that we are getting, similarly we can find out a 3. So, a 3 is let me write it here a 3 as the third column let me write it here a 3 is this. Similarly we can calculate a 4. So, a 4 is coming out to be this thing. So, a 4 is coming out to be 9 minus 17 minus 16 15. So, that is given here.

Similarly, we can calculate our rows of B, but that i will discuss little bit later. So, once we have a i's, then we can calculate our alpha i's. So, to calculate our alpha i's. Let us say that we have say we can write it a 1 is equal to 2 times omega dash a i a 1 and this is this ok. So, we here we have not put this thing. So, now, it is 9.7312; and it is coming out to be 9.3712; so 9.3712 ok. So, here we have a 1 as 9.7312.

Similarly, we can calculate alpha 2. So, here our a 2 is given as 2 times you can write it; here you can write a 2 as 2 times omega dot a 2 and it is coming out to be minus 2.0642 and which is matching here. Similarly we can write a 3 a 3 as alpha 3 alpha 3 I am writing and a is a underscore 3. And it is 2 cross omega dot a 3. So, that is minus 5.6028 minus 5.0628. And then we can write a 4. So, a 4 we can write it like this; a 4 as a 4 here and it is minus 32.4372. So, for your alpha 4 is minus 32.4372. So, it is alpha 4 is equal to minus 32.4372.

So, once we have alpha is with us then we can calculate H of a 1 as a 1 minus this thing a 1 minus alpha 1 w. So, that we write it as H of H of a 1 as a 1 minus a 1 into w. So, w let me write it here and then it is coming out to be minus 9.4348 and here we can see it is minus 9.4348 and 18.1739, 18.1739 and 14.2609 it is coming out to be.

Similarly, we can calculate H of a 2. So, H of a 2 a is given by a 2 minus a underscore 2 into w and it is coming out to be 1.3043; so 1.3043 and minus 5.0. So, you can calculate similarly H of a 3. So, H of a 3 also you can calculate in a H of a 3 as a underscore a 3 minus a underscore 3 w. So, a 3 you can also calculate, similarly we can calculate H of a 4 as a 4 minu a 4 minus a underscore 4 w that is given by this so that you can verify right.

So, here it is not very difficult to see and then you can write H of A as H of a 1, H a 2, H a 4, H a 3 and H a 4 right and it is coming out to be the H A which we have calculated here.

So, here without calculating our H just by writing your just by knowing the information about say a i's and omega; we can write down H of a i as a a i minus alpha i omega where alpha i is given as 2 times omega into omega dot a i; where this is an inner product between omega dot a i. So, omega dot a i is omega transpose a i. So, we have seen that we can calculate like this.

Similarly, to find out this B of H we can find out r 1 r 2 r 3 r 4 as follows. So, here to find out b 1 let me write b 1 as, now here in place of column I need say rows here. So, to show that it is root row, so it is first and row so that we will show you the first show let me 3 minus 19 4 17 3 minus 19. Let me use some other notation. So, let us call this as r 1; rather than let us use the same notation. So, r 1 is this right 3 minus 19 4 17 ok.

Similarly, we can calculate r 2, r 2 as the second row of matrix similar then r 3 and r 3 is the third row and similarly r 4. So, here we have calculate r 1 r 2 r 3 r 4 as listed here. Then we can calculate B H by calculating r 1 r 1 H r 2 H r 3 H r 4 H; I am not going to show the entire r 1 r 2 r 3 r 4. We simply write down say let us calculate r 1 H. Now, r 1 H is nothing but r 1 minus alpha 1 omega transpose.

So, let me calculate first. So, your this r 1 H let me write it r 1 H as follows. So, r 1 H is nothing but your r 1 minus; now here we write alpha 1 w transpose. So, here r 1 minus here alpha 1 is basically 2 times omega dash into r 1 transpose and that is your alpha 1 into here we have r 1 transpose r 1 transpose. I think that is sorry it is omega transpose omega transpose and here we can say that it is r 1 H is minus 3.4348; so minus 3.33 minus 3.4348 13.1739 13.0 and minus 21 7391 and 4.13 4 1.

Similarly, we can calculate r 4 and all this thing. So, that we can write it here r 2 H as r 2 minus 2 times r omega you can calculate like this, 28.1304. And similarly you can calculate r 3 H. So, r 3 H is r 3 minus here we have r 3; so 5.0870 that is listed here. Similarly we can calculate r 4 H. So, r 4 H is r 4 minus alpha 4 alpha 4 is given by this and we can calculate r 4.

So, here we have 18.18 8 34 and minus 11. Then we can write BH as this. So, to write B of H; we write r 1 H. So, r 1 H then r 2 H; then r 3 H; then r 4 H; we can write it like that and we have this matrix BH. So, BH is coming out to be this ok.

So, here what we have shown here that how we can use the result and we can implement the calculation and BH and BH without exactly calculating your H, but only using this result that H of A can be written as H of a 1, H of a 2, H of a 3, H of a 4; where H of a k is given by a k minus alpha k omega. Similarly we can calculate B of H as r 1 H r 2 H r 3 H r 4 H; where r k H is given by r k minus alpha k omega transpose. So, here we have seen how to find out this product without calculating your H.

So, So, we have seen that how to calculate the product of matrices and householder matrix without actually calculating the house holder matrix H; just and we can do this without with the help of omega only.

So, in this lecture we will do only this much. And in next lecture we will utilize this information which we have gathered today and the previous lecture to find out the QR decomposition of a increment matrix and with the help of QR decomposition or QR factorization of a given matrix. A how we can use this in finding the eigenvalue and solving the eigenvalue problem it means finding the eigenvalue of a given matrix.

So, that we will discuss in coming lectures; so here we will stop our lecture and we will meet in a next lecture. Thank you very much for listening us.

Thank you.