Numerical Linear Algebra

Dr. P. N. Agrawal
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 05
Vector Space- II

Hello, friends. Welcome to my lecture on Vector Space 2. This is second lecture on vector spaces. We will begin with the linear span of a set in a vector space.

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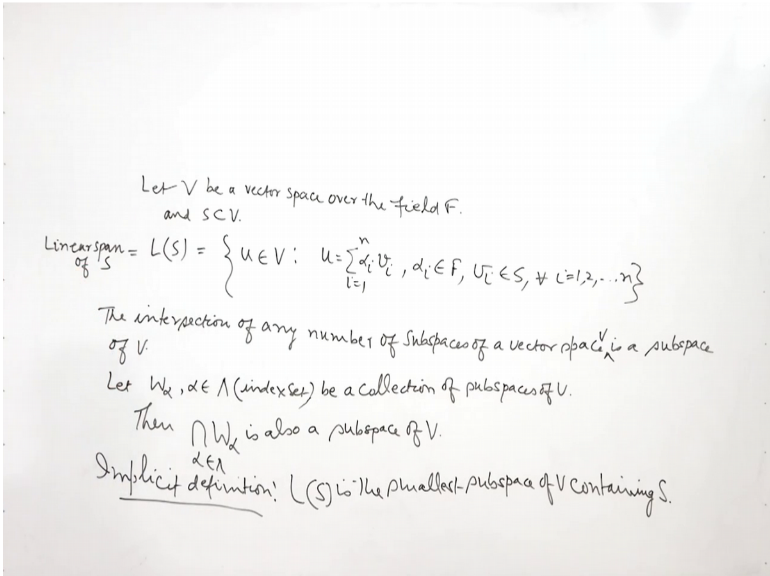

So, let V be a vector space and S be a subset of V, then we define a linear span last time. Linear span of S which we denote by L S. Linear span of S is the set of all linear combinations of vectors of S. So, this I can write as set of all u belonging to V such that u is equal to sigma alpha i V i, i is equal to 1 to n where alpha i is belong to the field F and v i is belong to S for all i equal to 1 to n. V is a vector space over the field F. F is equal to r when we take real vector spaces and F is equal to c when we deal with complex vector spaces.

So, there is another definition of linear span which we call as implicit definition. As I said in my last lecture the intersection of any number of subspaces of a vector space is again a vector space. The intersection of any number of subspaces of a vector space is a subspace if I take vector space as V then it is a subspace of V. So, it is very simple to prove you can say suppose let W alpha where alpha belongs to lambda; lambda being an index set be an arbitrary collection of subspaces of V, then intersection W alpha belongs to lambda is also a subspace of V. So, this can be easily proved.

So, making use of these result, let us consider all those subspaces of V which contain S. There is one subspace of V already there which is the vector space be itself which contains S. So, let us consider. So, what we are going to do is, we are going to show that the linear span of S whose implicit definition is the intersection of all subspaces of V which contain S and intersection one be considered intersection all subspaces of V which contain S then by this result it will be a subspace of V and moreover, this subspace of V will contain S and it will be contained in every subspace of V which contains S. So, it is called as the smallest subspace of V containing S.

So, we can prove that, L S implicit definition. The implicit definition says that the linear span of S, L S is the smallest subspace of V containing S. So, this is called as the implicit definition and the definition which we have written here, it is called the explicit definition both the definitions are equivalent. Let us see how we prove that.

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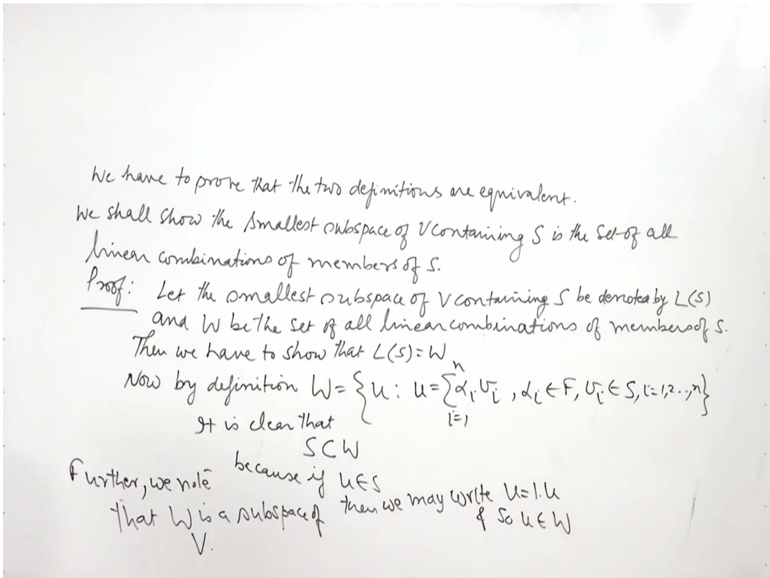

The two definitions are equivalent. So let us say, we have to prove that the implicit two definitions are equivalent. So, in order to show this, let us take up one definition. Let us say let us take up the implicit definition that is let us assume that L S is the smallest subspace of V containing S and we show that this L S is the setup of all linear combinations of members of S. So, we shall show that the smallest subspace of V containing S is the set of all linear combinations of members of S. So, let us prove this.

So, let us say, the smallest subspace of V containing S be denoted by L S we take this notation of this implicit definition and W be the set of all linear combinations of members of S or elements of S. Then, we prove that we have to show that L S is equal to W. Now, W by definition it is the set of all elements u belonging W such that u is equal to sigma alpha i v i, i is equal to 1 to n, alpha i belong to F, V i is belong to S, i is equal to 1 to n and so on up to n.

So, let us first show that L S is contained in W and then we shall show that W is contained in L S. It is clear that S is contained in W. Because, you take any element u belonging to S then I can write u as 1 into u. So, u can be expressed as a linear combination of members of S and therefore, u belongs to W. So, W is a set which contains S. Now, further we can show that W is a subspace of V, further we note that W is a subspace of V.

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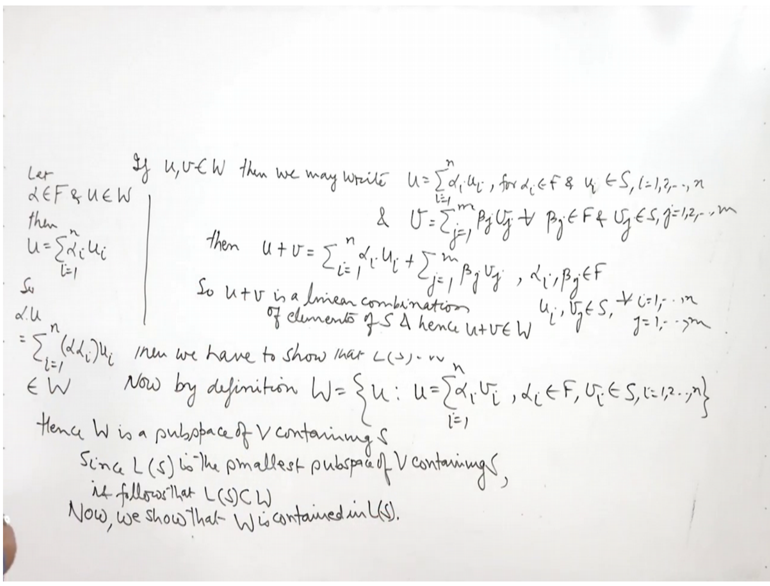

If we take, if u and v belong to W then we can write u as sigma alpha i say v i, i equal to 1 to n for alpha i belonging to F and v i belonging to S, for all i and similarly, I can write v as sigma j equal to 1 to m, beta j, let us take u I, here u i and here we take v j for all beta j belonging to F and v j belonging to S.

So, u plus v is a linear combination of elements of and hence u plus v belong to W. Now, let us show that, W is closed with respect to vector addition. Now, we have to show that W is closed with respect to a scalar multiplication. So, let us say let alpha belong to F and u belong to W, then u, I can write as sigma alpha i u i, i is equal to 1 to n and. So, alpha into u is equal to sigma i is equal to 1 to n alpha; alpha i into u i, F is a field. So, when alpha belongs to F and alpha i is belong to F alpha i is belong to F for all i. So, this is an element belonging to W. We have a linear combination of elements of S, u i belong to S.

Now, so, W is close with respect to it is scalar multiplication and therefore, W is subspace of V containing S. Now, since L S is the smallest subspace of V containing S, we have it follows that L S must be contained in W it is contained in every subspace of V that contains S. So, L S is contained in W. Now, let us prove that W is contained in L S.

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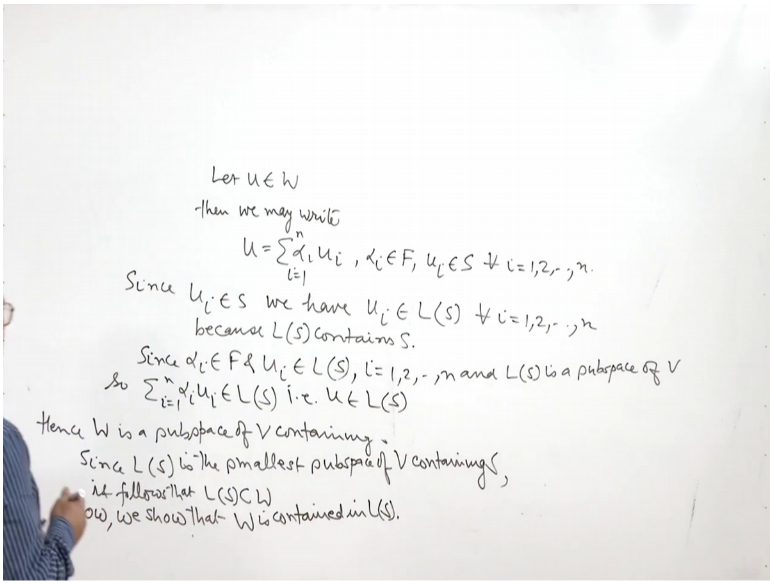

Now, we show, let us show that W is contained in. So, let us allot u belong to W. We want to show that u belongs to L S, then by definition of W we may write u as sigma sum alpha i into u i, i is equal to 1 to n where alpha i is belong to F, u i is belong to S for all i. Now, L S is a subspace. So, if you take any element if u 1 u 2, u 1 belong to S then alpha 1 u 1 plus alpha 2 u 2 since, u i is belong to S we have u i belong to L S for every I, because L S contains S. Now, u i is belong to L S for every i, i is equal to 1 to n and now if alpha is belong to F and u i is belong to L S, I think I should write since alpha is belong to F and u i belong to S, i is equal to 1 to n and L S is a subspace of V. So, it is close with respect to vector addition and scalar multiplication. So, i is equal to 1 to n alpha i u i belong to L S, that is, to say u belong to L S, so which means that W is contained in L S.

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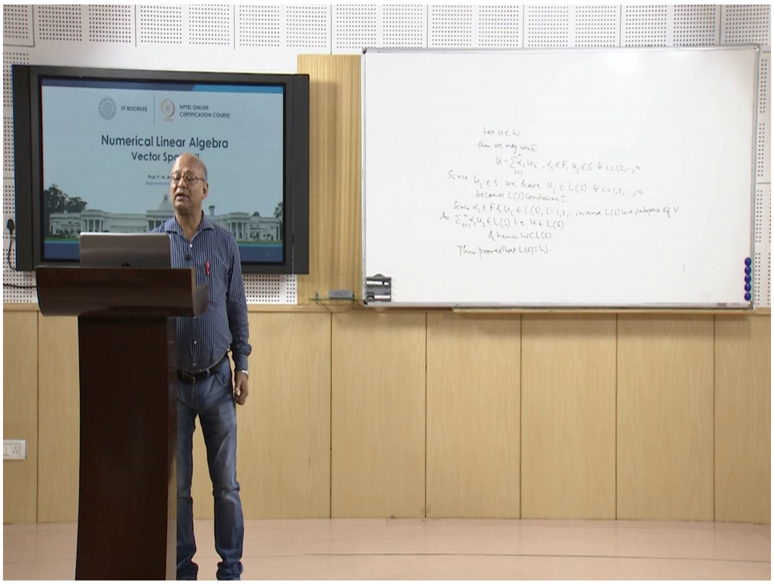

So, this proves that L S is equal to W. So, the linear span of a set S in a vector space is the set of all linear combinations of elements of S. So, this is what we have. We have characterized the linear span of a set S in a vector space, now let us look at some simple results.

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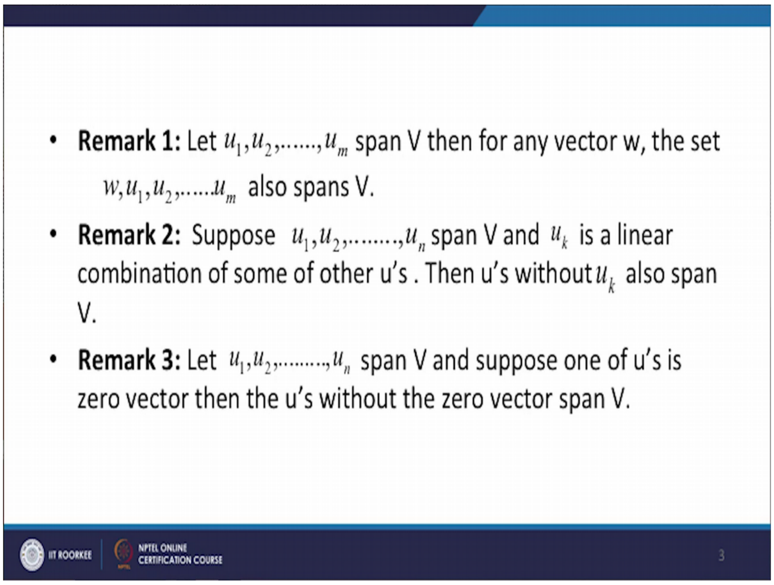

Let u 1, u 2, u m be a set of m vectors which span S, then you take any vector W in the vector space W u 1, u 2, u m will also span V. So, this let us see we have easily we can prove this.

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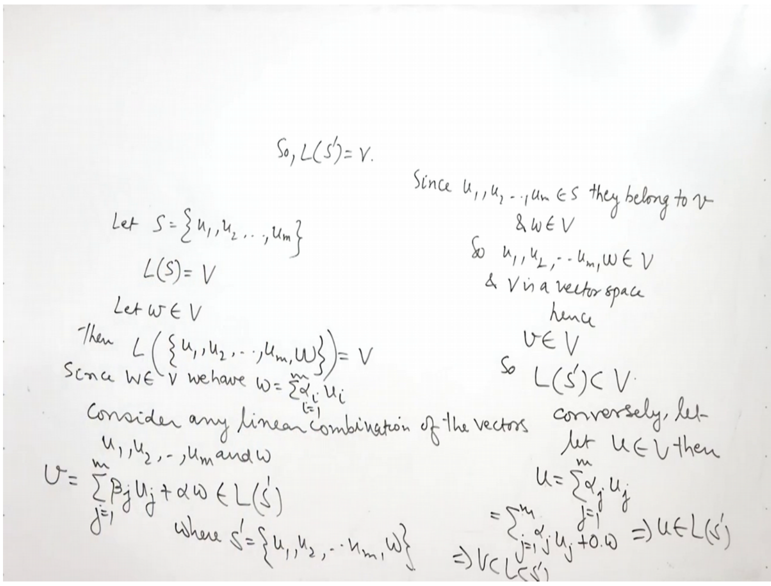

So, let us say let S be equal to set of all vectors u 1, u 2, u m and that span V. So, L S been span of S is equal to V. So, the set of vectors span V. So, L S is equal to V now you take any vector W, let W belongs to V; then we have to show that V the set L of u 1, u 2 and so on u m, W is also equal to V.

So let us take any vector any say u belonging to V then we have to show that u can be written as a linear combination of u 1, u 2, u m and W. From here W belongs to V we can say that W can be written as a linear combination of u 1, u 2, u m. So, since W belongs to V we have W equal to sigma alpha i u i, i is equal to 1 to m. Consider any linear combination of the vectors u 1, u 2, u m and W. We can write it as we have sigma say beta j u j; j equal to 1 to m plus some scalar alpha times W. Linear span we have already seen, linear span of a set of vectors is nothing, but the set of all linear combinations of members of S. So, it we are taking a linear combination of u 1, u 2, u m and W, so, this is an element of this set L S. I can this is let me call it L S dash. So, this is an element of L S dash where S dash is u 1, u 2, u m and W.

So, when you take any linear combination of u 1, u 2, u m and W it is an element of L S dash. Since u 1, u 2, u m belong to S they belong to V and W is also an element of V. So, u 1, u 2, u m and W belong to V and V is a vector space. Hence, this linear combination let me call this linear combination as the element v. So, then hence v belongs to V. So, this L S is contained in V.

Now, if I take any element in V, I should show that I can write it as a linear combination of u 1, u 2, u m and W or if I take conversely let us take any element. Conversely I take any element u belonging to V then I can then since u 1, u 2, u m is span V then I then u is equal to sigma alpha j u j, j equal to 1 to m and this is also equal to sigma j equal to 1 to m alpha j u j plus 0 into W. I can write it like this where 0 is the 0 scalar and W is the W vector. So, u is a linear combination of u 1, u 2, u m and W. So, this means that, u belongs to L S dash and therefore, V is subset of L S dash. So, L S dash is equal to V.

So, if u 1, u 2, u m span V than W is any vector in V then W u 1, u 2, u m also spans V. Now, in the remark 2 we say that if u 1, u 2, u m span V and u k is a combination one of the elements u k here in u 1, u 2, u n is a linear combination of some other use, then the use without u k will also span it is also very simple we can easily show this and let u 1, u 2, u n span V and suppose one of the vectors one of the u is a 0 vector, then the use without the 0 vector will span V. So, these are simple observations now let us look at some examples of subspaces.

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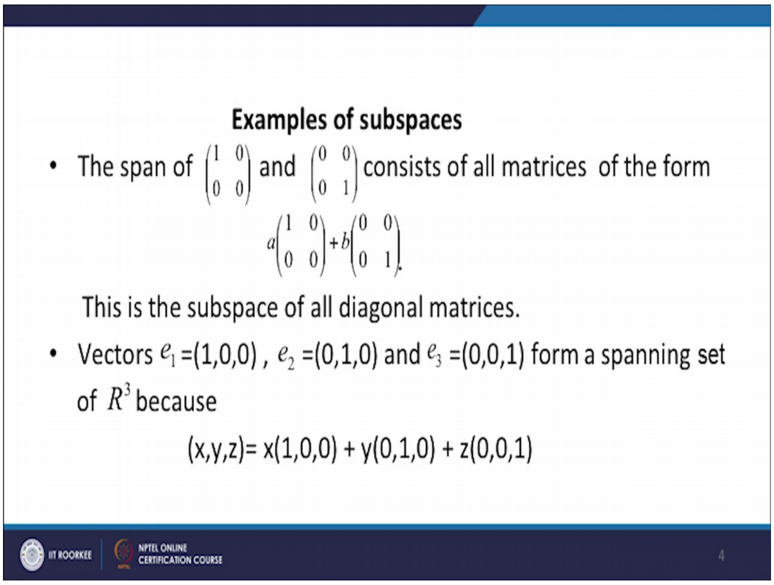

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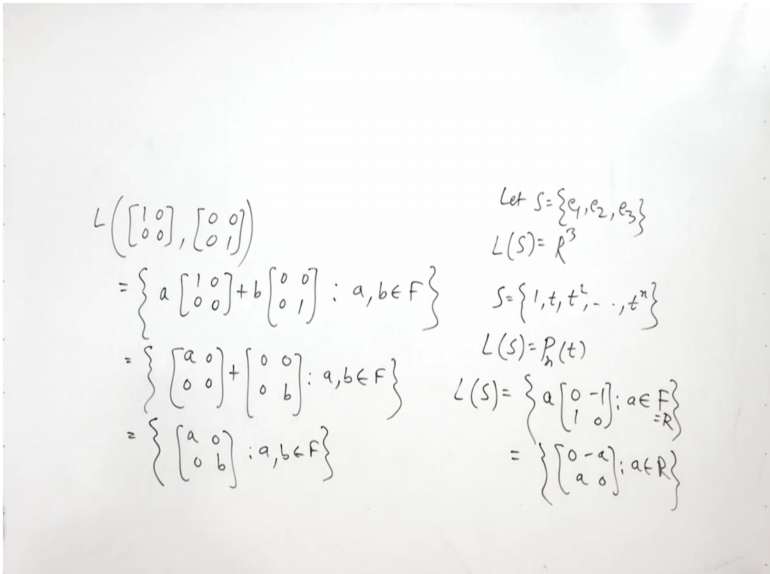

So, the first example is let us consider the linear span of 1 0 0 0 and 0 0 0 1, this our set S. So, S is the set of two matrices; one is 1 0 0 0 the other one is 0 0 0 1. Now, L S will be the set of all linear combination of. So, these two, a times 1 0 0 1 plus b times 0 0 0 1, where a, b belongs to the field F and this is what a 0 here 0 we have. So, here all non diagonal elements are zeroes. So, L S where S is these 2 matrices L S is the set of all 2 by 2 diagonal matrices. So, linear span of these 2 matrices is the subspace of all diagonal matrices.

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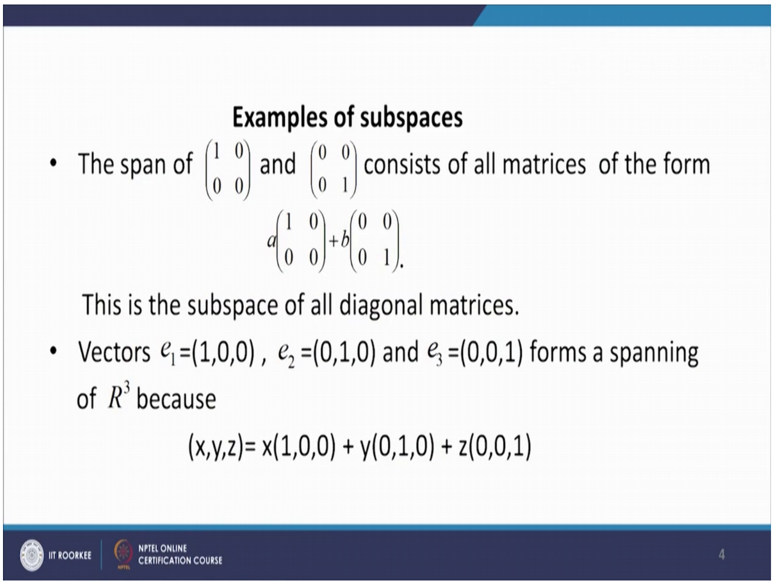

Now, let us take another example of suppose we take the vectors e 1 equal to 1, 0, 0, e 2 equal to 0, 1, 0 and e 3 equal to 0, 0, 1 then the linear span of these 3 vectors L e 1, e 2, e 3 is nothing, but the whole of R cube. So, let S be equal to e 1, e 2, e 3 then linear span of S is equal to R cube, because any element in R cube that is x, y, z can be written as a linear combination of 1, 0, 0, 0, 1, 0 and 0, 0, 1. So, the span subspace of R cube.

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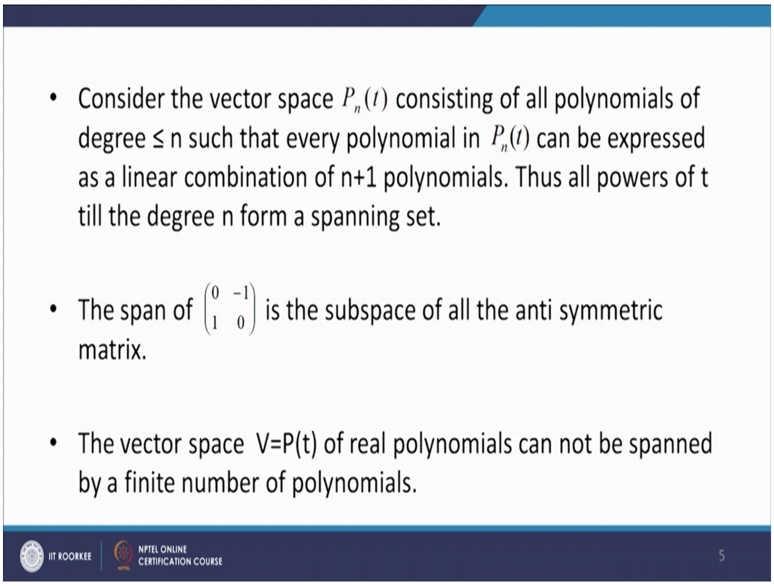

Let us consider the vector space P n t consisting of all polynomials of degree less than or equal to n, such that every polynomial in P n t can be expressed as a linear combination of n plus 1 polynomials, then all powers of t till the degree n say, one t t square t to the power n their linear span will be the vector space P n t.

Now, let us go to the span of the linear span of 0, minus 1, 1, 0 will be the subspace of all anti symmetric matrices. 0 see linear span of. So, where F we are taking as equal to R the set of real numbers the field is of real numbers. So, here we have 0, minus a, a, 0. So, it is the subspace of all S symmetric matrices. Now, it is clear that the vector space of all real polynomials cannot be spanned by a finite number of polynomial.

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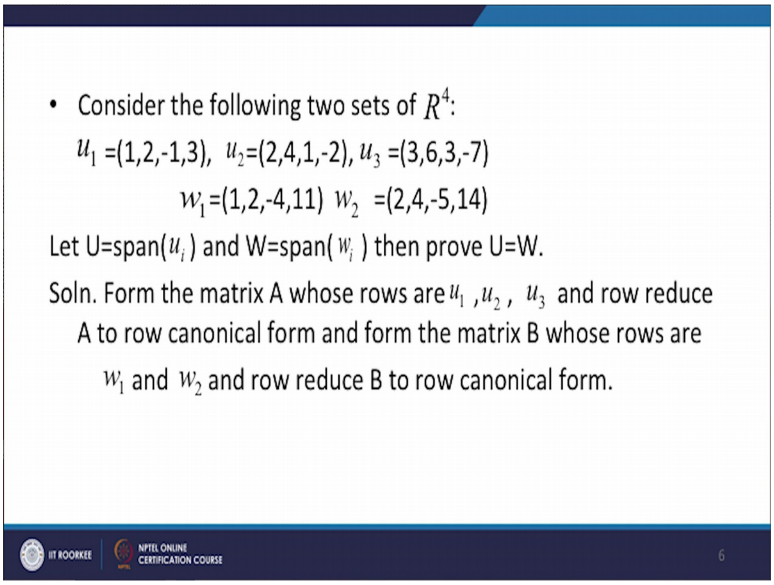

Now, let us go to the following set 2 sets of R 4; one set of R 4 is consisting of 3 vectors u 1, u 2, u 3. u 1 is 1, 2, minus 1, 3; u 2 is 2, 4, 1, minus 2; u 3 is 3, 6, 3, minus 7. The other set is consisting of two vectors w 1 and w 2. w 1 is 1, 2, minus 4, 1; w 2 is 2, 4, minus 5 and 14 and then we can find the span of u 1, u 2, u 3 and also the span of w 1, w 2 and then show that the two are equal to generate the same space. So, let us see how we prove this.

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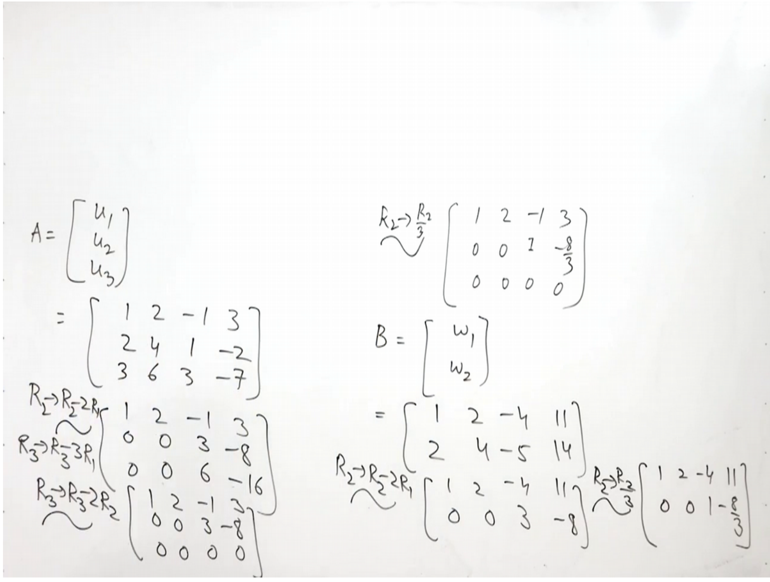

So, let us form the matrixes whose rows are u 1, u 2 and u 3. So, first we take A whose rows are u 1, u 2, u 3; u 1 is 1 2 minus 1 3, u 2 is 2 4 1 minus 2 and u 3 is 3 6 3 minus 7. Let us; then row reduce this matrix to canonical form. Canonical form means V will reduce it to echelon form and then row reduce it to echelon form, that is, will make the leading coefficients of all non zero entries in each row to be equal to 1, such that it is the only non zero entry in it is corresponding column.

So, what we do? We will do elementary row operations on this. So, in order to reduce it to echelon form we will make this entry 0, this entry because number of zeros preceding the first non zero entry should increase row by row. So, we will put an equivalent sign here, now to make this 0 with the help of this we multiply this by 2 and then subtract from here. So, let me call this as R 1 row this R 2 row, this R 3 row. So, will do R 2 goes to R 2 minus 2R 1 and then from this 3 times this we subtract from here. So, R 3 goes to R 3 minus 3R 1. Let us make this 2 operations. So, 1 2 minus 1 3, here we will get 0 we will get 0 here again. So, we multiply 2 and subtract here, this should become one plus 2 that is 3 and this will be 6, we will get minus 8 here and then this will be 0, this will be 0 and then 3 plus 3 will become 6 and here we will get 9, 9 will give you minus 16.

Now, number of zeroes preceding these first non zero entry here is 2 and here also it is 2. So, let us make this entry 0 with the help of this one. So, this we say R 3 goes to R 3 minus 2R 2 and we write 1 2 minus 1 3 and then 0 0 3 minus 8 and then 0 0 0 0. Now, this matrix is in echelon form, but we have to reduce it to canonical form. So, here the first leading coefficient is 1, this is not 1. So, let us make this also equal to 1. So, we divide second row by 3. So, we write R 2 goes to R 2 by 3 and then we get 1 2 minus 1 3 0 0 1 and then here minus 8 by 3 and then 0 0 0 0.

So, the matrix A is reduce to row canonical form. The similarly we form the matrix B with the vectors w 1, w 2. So, let us consider the matrix B equal to B we will form with the vectors w 1 and w 2. So, this equal to 1, 2, minus 4, 11 and then 2, 4, minus 5, 14. So, reduce it to equivalent form. So, we get we do R 2 goes to R 2 minus 2R 1 then this is 1, 2, minus 4, 11 this will become 0, this will become 0, here we will get 3, here we are subtracting 22 from 14. So, we get minus 8. Now, here the first coefficient first entry is 1, but here it is 3. So, make it 1 by dividing second row by 3. So, this is change to R 2 goes to R 2 by 3 and we get 1, 2, minus 4, 11, 0, 0, 1, minus 8 by 3.

Now, we can see that A is row reduced to this matrix, third row is a 0 row. When we take linear combination of these elements third row does not contribute anything. So, this matrix we can regard will generate the same space as this matrix because the 0 vector does not contribute anything to the linear combination. So, we can say that when we reduce A and V to canonical form, row canonical form their span the same space. So, this they span U and W, then U and W must be equal.

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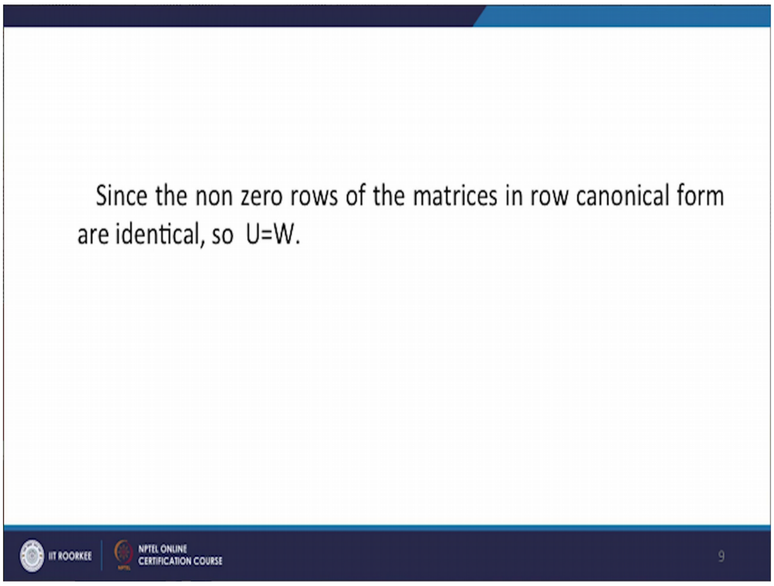

This is what we are saying in this example, the non zero rows of the matrices in row canonical form are identical, so U is equal to W. So, with that we would like to conclude our lecture. Thank you very much for your attention.