Numerical Linear Algebra

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Lecture – 46
SVD and their applications

Hello friends, welcome to this lecture. We will in this lecture we will continue our study of a singular value decomposition and its application. So, if you recall in a previous lecture we have utilize the singular value decomposition of a given theorem a given matrix, a rectangular matrix A to find out say a structure of a algebraic structure of a algebraic and geometric structure of a given matrix A. And in previous class we have discussed how to find out orthogonal projection to range space of A, a normal a null space of A, range space of A transpose and null orthogonal projection to a null a null space of A transpose.

And in this lecture we will also continue our study basically what we try to do here is to find out the structure of a matrix using singular value decomposition. What do you mean by structure of a matrix? So, structure of a matrix may include say a rank of a matrix, a rank deficiency of a matrix or why we are discussing rank deficiency because a many a times when we solve this algebraic system these matrix is coming from a say the given data. Now, the given data may be polluted or we can say that the given data may contain some kind of error.

So, it means the actual matrix which you are handing may contains some kind of error. So, it means it may happen that this matrix is kind of a ill conditioned matrix. So, in that particular case when A is not well conditioned then handling situations are quite difficult, but how to find out say that a given matrix is a ill condition or well condition. So, we are going to discuss this with the help of singular value decomposition.

Other than this we also try to approximate a given matrix by a orthogonal matrix using SVD and then try to find out um a least square a solution of the system linear system A x equal to B.

So, now, let us start with finding say little bit about algebraic structure in terms of norms. So, given a matrix how we can relate the singular value decomposition with the norm of

A matrix. And with the help of this how we can say that the eigen values of A um matrix or we can say similar values of A matrix A is well conditions. So, let us consider that from. So, of we start with this theorem.

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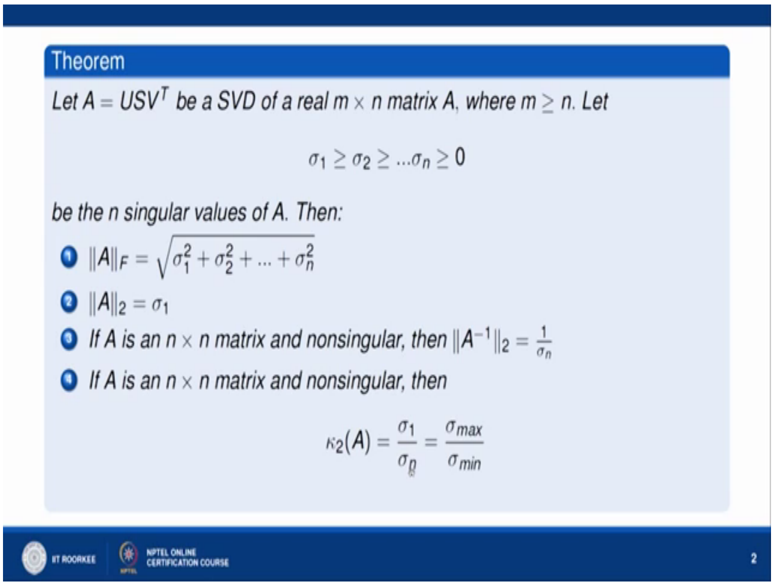

That let A is equal to USV transpose be a singular value decomposition of a real m cross n matrix A and here we are assuming m without loss of generality that m is greater than or equal to n. And let these singular values of A are arranged in this order that sigma 1 is greater than equal to sigma 2 greater than equal to sigma n and which is greater than equal to 0 here and here these are the singular values of A.

Then we say that the Frobenius norm of A matrix A can be given as under root sigma 1 is square sigma 2 square plus sigma n square and 2 norm of A is given by sigma 1 and if A is a n cross n matrix and a non singular matrix then in that case 2 norm of A inverse is going to be 1 upon sigma n. 2 norm or you can say the Euclidian norm of A inverse is going to be 1 upon sigma n.

And with the help of 2 norm of A inverse and 2 norm of A we can define condition number of a matrix A which is square matrix and non singular matrix given by then condition number of this matrix A is given by sigma 1 by sigma n. What is sigma 1 here? Sigma 1 a if you look at this order then sigma 1 is the sigma max and sigma n is the sigma minimum. So, it means that condition number of a given matrix A which is the square and non singular is defined as sigma max upon sigma min, where sigma sigma here sigma a is represent the singular values of this matrix A.

And this is a theorem has a small proof, so let us discuss this is a small proof.

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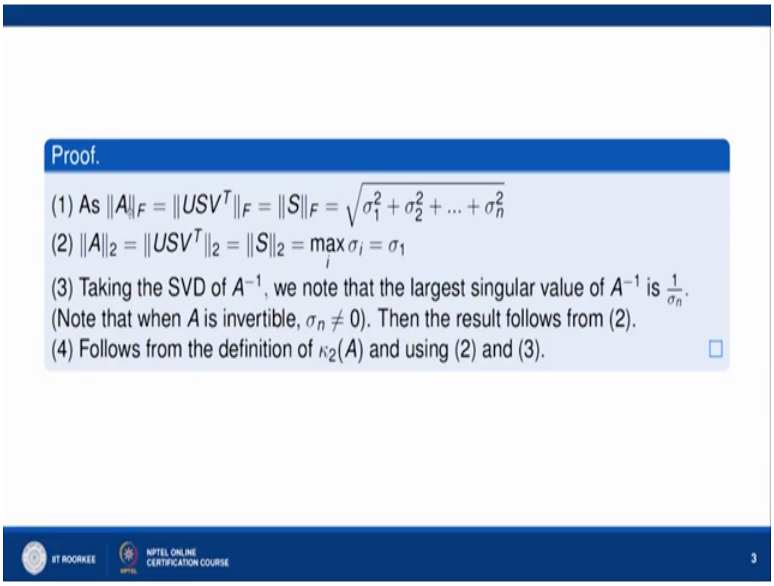

So, here if you look at Frobenius norm of A, A is given as a Frobenius norm of USV transpose, where U and V are a orthogonal matrices of a respective sizes U is a orthogonal matrix of size m cross m and V is a orthogonal matrix of n cross n.

Now, we have already seen in the lecture of a matrix norm that this Frobenius norm is independent are say invariant under the product of a orthogonal matrices. So, it means that the norm of USV transpose is nothing, but norm of S and a Frobenius norm of S here; so because U and V are orthogonal matrices.

So, now, what is the Frobenius norm of S? So, Frobenius norm of S is going to be trace of S transpose S, where S is a rectangular diagonal matrix. So, here you can find out S transpose S as a summation under root summation sigma 1 square plus sigma 2 square and sigma n square. So, it means that the Frobenius norm of A is given by this under root sigma 1 square plus sigma 2 square plus sigma n square.

Similarly, if you look at the at Euclidian norm of matrix A then it is again given as Euclidian norm of USV transpose and we have also seen that in this like Frobenius norm in this Euclidian norm is also invariant under the product of orthogonal matrices. So, the

2 norm of USV transpose is nothing, but 2 norm of S and 2 norm of S means it is 2 norm of S is basically lambda max of a H transpose of S and if you look at under root of a lambda max of H transpose of S and that is nothing, but maximum of a sigma i and this is nothing, but sigma 1, because we have ordered our singular a values of A in this order that sigma 1 is the maximum and sigma n is the minimum. So, 2 norm of A is going to be a sigma 1 here.

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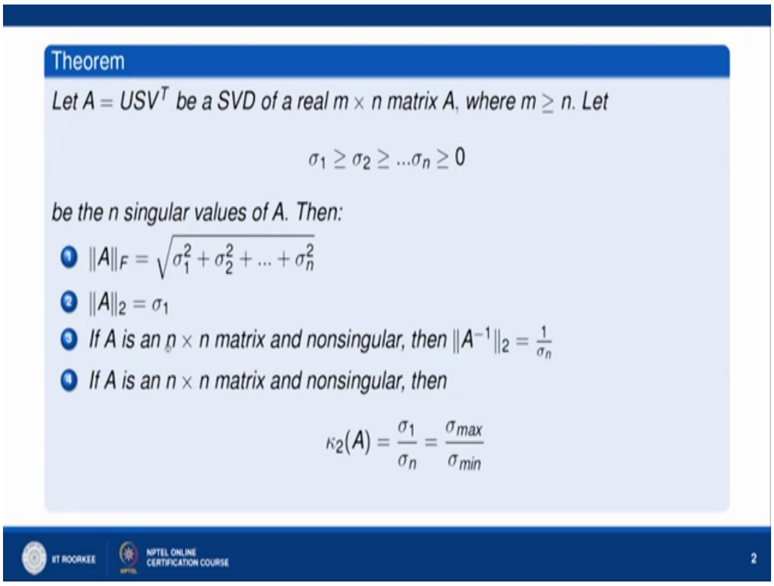

Now, so this proves this second part, now, to look at the third part. So, here taking the SVD of A inverse, if SVD of A is given as USV transpose the only difference between this case and this case that here size is n cross n rather than the m cross n. Then here both U and V are n cross n matrices. So, if you look at the SVD of A inverse then if A has a U SVD USV transpose then a inverse will have just look at here this is very.

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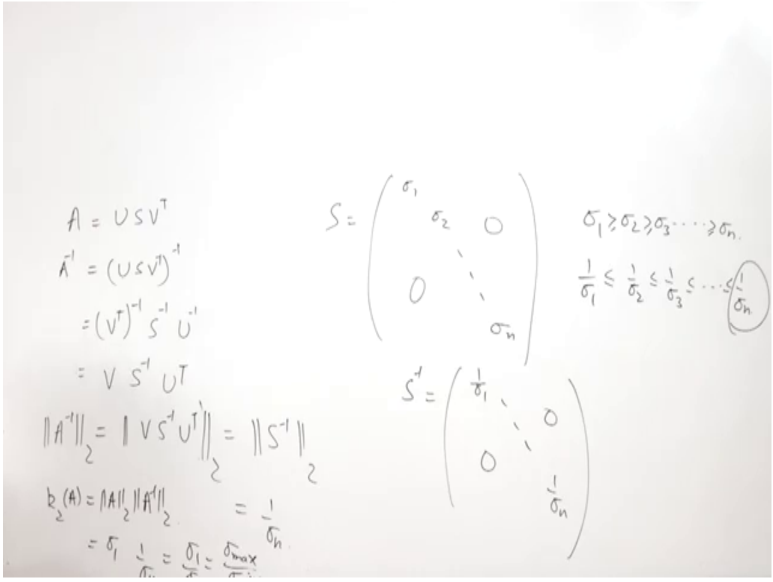

So, here we have S USV transpose then if A is invertible means inverse is defined. So, it is USV transpose inverse and this is nothing, but V transpose inverse as inverse and U inverse. And now, here since V transpose inverse is V inverse transpose and we can write this as a V here and this is your S inverse and this U inverse is nothing but U transpose. So, it is going to be VS inverse U transpose. Now, if you find out say 2 norm of A inverse or F norm of A inverse then it is going to be VS inverse U transpose and 2 norm of this and since 2 norm is invariant under a orthogonal matrices. So, this is going to be 2 norm of S inverse.

Now, we have already seen that 2 norm of S is going to be nothing, but the maximum of a similar values of S here, but now, here we have S inverse, it means that maximum value maximum singular value of this S inverse. Now, if you look at what is the structure of S, structure of S is going to be a sigma 1 to sigma 2 and so on sigma n. So, here and all these are 0 if you remember in case of square square matrix A non singular matrices these this S is going to be perfectly a diagonal matrix.

So, S inverse is defined as, so S inverse is going to be defined as 1 upon sigma 1 to 1 upon sigma n here. Now, if we look at the maximum singular value of this S inverse then it is going to be a 1 upon sigma n. Why because if you look at sigma 1 greater than equal to sigma 2 greater than equal to sigma 3 and so on greater than equal to sigma n then it will follow this pattern, that 1 upon sigma 1 is less than 1 upon sigma 2 less than or equal to 1 upon sigma 3 and so on less than or equal to 1 upon sigma n.

So, these are the singular values of S inverse. So, here if you look at the singular value of it the largest singular value of S inverse is going to be 1 upon sigma n. So, it is going to be 1 upon sigma n, that is what we wanted to proof here. So, it means that we know that the largest singular value of A inverse is 1 upon sigma n ok. And here this sigma n is nonzero because A is invertible. So, A is inverter means none of this singular value is going to be 0. So, this follow from this 2 ok.

Now, to find out same condition number of A; so condition number of A is going to be defined as a norm of A, 2 norm of A and 2 norm of A inverse. Now, we have seen that 2 norm of A is going to be sigma 1 and 2 norm of A inverse is going to be a 1 upon sigma n. So, this is nothing but sigma 1 upon sigma n. Now, what is sigma 1? Sigma 1 is the largest singular value of A, so we can say that sigma max divided by sigma min.

So, it means that condition number of A is going to be this ratio, ratio of sigma maximum divided by sigma minimum. So, in case of A is n cross n square matrix and non singular matrix then condition number of A you can calculate by this ratio sigma max upon sigma min. So, that completes the proof of the previous theorem.

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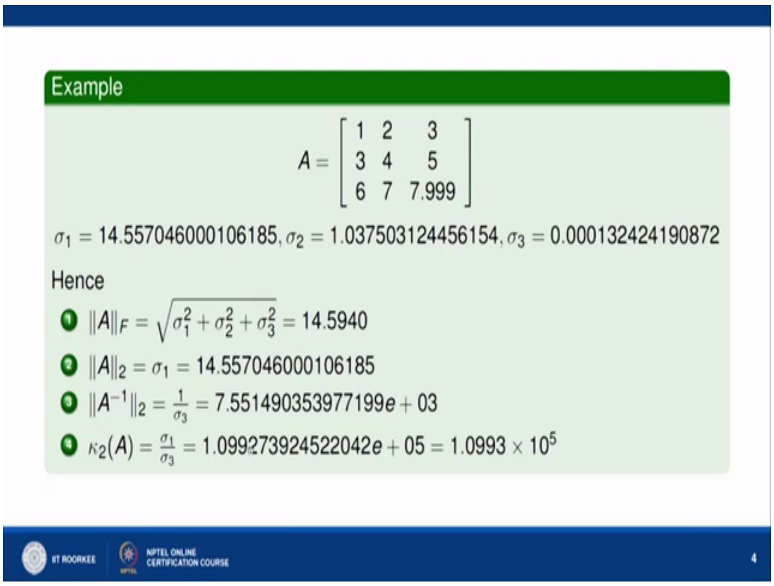

Now, let us consider some example here. So, here we can calculate a singular values of A as this. Now, to find out say Frobenius norm of A which is given as a under root of sigma 1 square plus sigma 2 square plus sigma 3 square and you can you have all these sigma is, so you can calculate this quantity and it is coming out to be 14.5940.

Similarly, you can find out a Euclidian norm of A or 2 norm of A which is given as a sigma 1 a sigma max. So, sigma max is a sigma 1 here. So, it is given by 14 point this quantity. So, and to find out say 2 norm of A inverse which is given by 1 upon sigma 3. So, sigma 3 is this quantity. So, you can calculate and it is come given by 7.55 into 10 to power 3 here. So, and we can calculate the condition number of A as a sigma 1 upon sigma 3 which is given by 1.09927 into 10 to power 5 or you can simplify this as 1.0993 into 10 to power 5 here.

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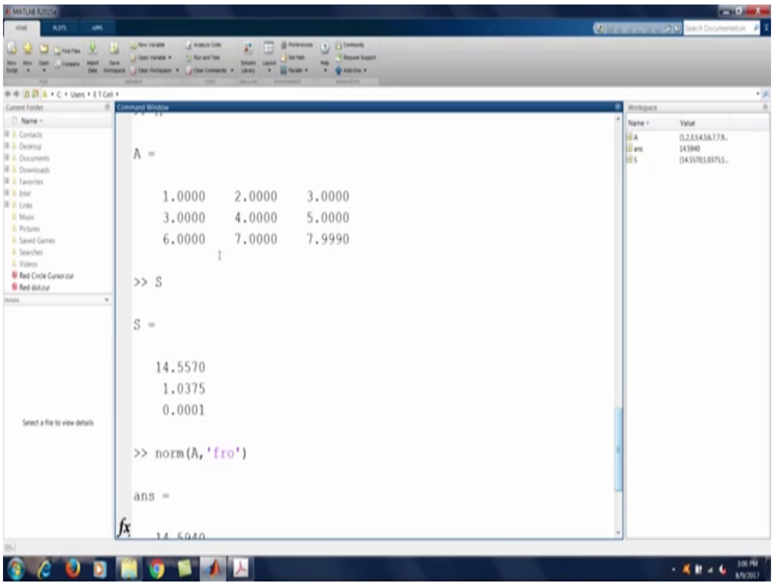

So, let us look at the same thing with the help of a MATLAB here. So, here we have calculated here we have entered about matrix A and S which is nothing, but SVD of this matrix A which is given by 14.5570, 1.0375, 0.0001. You can take a long format, but let us find out say Frobenius norm of A using simply command. So, norm Frobenius norm of A matrix A is coming out to be 14.5940.

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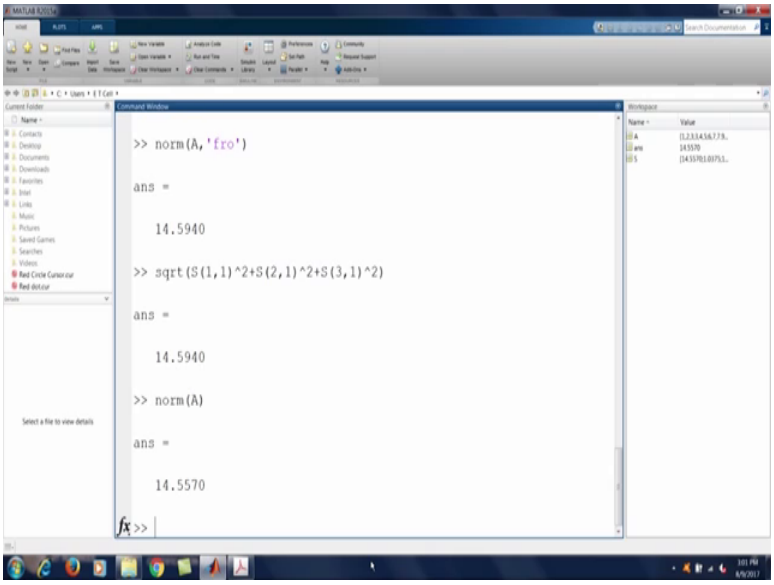

And if you calculate using the SVD then it is nothing, but a square root of S 11, S 11 represent the first element of this matrix and a square plus S 21, S 2 represent the 21 entry of this matrix. So, S 21 square plus S 31 square if you calculate it is coming out to be 14.5940.

So, it means that it is matching here that your Frobenius norm of A is given by sigma under root of sigma 1 square plus sigma 2 square plus sigma 3 square equal to 14.5940. Now, look at to find out 2 norm of A. So, norm of A is you write norm of A it is coming out to be 14.5570 or you can write it a norm of A, 2 so that we will also give the same result.

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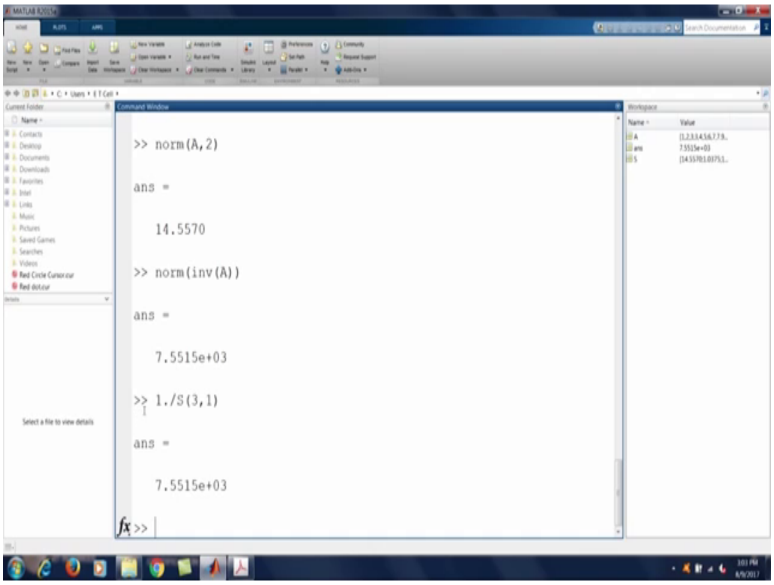

So, by default norm of A is 2 norm of A. So, it means that I am your you are getting 14.5570 and which is nothing, but the a largest singular value of this A. So, it is the 14.5570 is the largest singular value of matrix A. So, here also 2 norm of A is matching with a sigma 1, similarly we need to find out a sigma 2 norm of A inverse. So, for that we simply write 2 norm of, so we can write norm of inverse of A matrix. So, here since we have taken a non singular matrix matrices. So, we can write it this and it is coming out to be 7.551 or 5 into 10 to power 3. So, here it is a 7.551 into 10 to power 3 here, so that is given by 1 upon sigma 3.

So, we have we just want to see that if you calculate the norm of inverse a using the command then it is coming out to be this, but if you want to calculate this by say singular value decomposition or from singular value then whether it is coming out to be same or not. So, if you look at if you calculate 1 divided by a singular value of the minimum singular value of this a then it is coming out to be the same one.

So, so it means that this is also true whether you calculate using the command or using the a singular value decomposition of a matrix A here.

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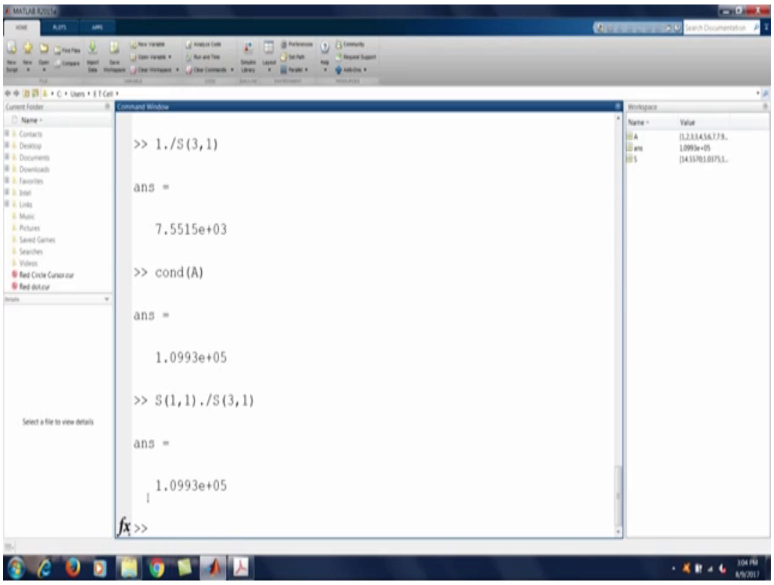

So, similarly you can find out say condition number of A. So, for condition number of A, you write this command condition number condition of A with 2 norm you can write it or if you do not write 2 norm then also it is by default the 2 condition number A, condition A with 2 norm. So, it is coming out to be 1.0993 into 10 to power 5 which is given here 1.0993 into 10 to power 5 when you use directly the definition here that kappa 2, A.

So now, let us find using the a singular value of a then it is given by a sigma 1 divided by sigma 3. So, you can say that sigma 1 is your S 1 comma 1 divided by S 3 1. So, S comma 3 comma 1 and it is coming out to be same. So, it means that condition number of A you can calculate by sigma 1 upon sigma 3 here. So, you can say that sigma max divided by sigma min. So, condition number of A you can easily calculate with the help of your SVD. So, once you have your SVD you can calculate the norm of this matrix whether it is 2 norm of matrix, a web norm of a matrix. You can find out say a condition number of a matrix here.

And also with the help of a singular value decomposition you can find out say orthonormal basis is for it is different a subspaces of this matrix A.

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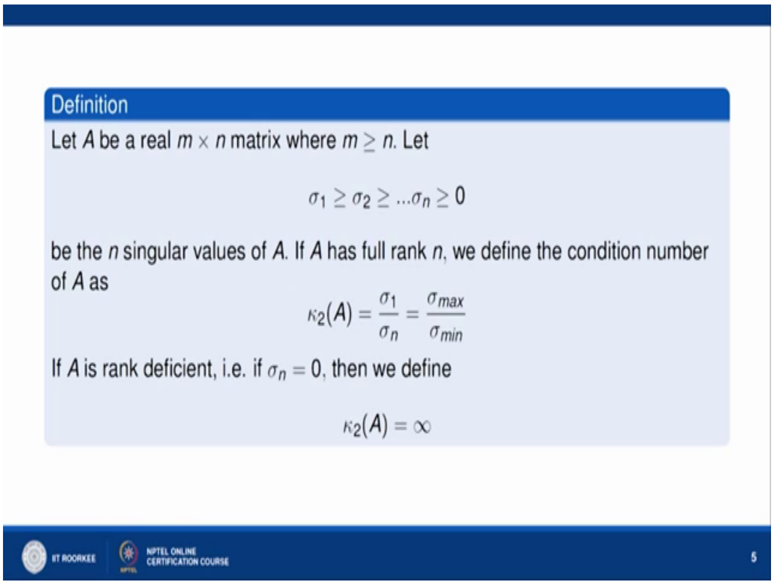

Now, moving on a next since we here we have defined condition number of a condition number of a matrix which is of a square which is a square matrix and non singular matrix, but now, with the help of this since the singular values may be defined for a rectangular matrixes as well. So, here using this a last result that kappa 2 A is given by sigma max of a sigma min.

So, using this observation we can define condition number of a rectangular matrix and it is given by this. So, here we say that let A be a real m cross n matrix where m is greater than equal to n and let sigma 1 is greater than sigma 2 greater than equal to sigma n greater than equal to 0 be the n singular values of A. Now, if A has full rank means none of these a singular values are 0 then we define the condition number of A as kappa 2 A equal to sigma 1 divided by sigma n and it is given by sigma max upon sigma min. But if A is a rank deficient means the rank of this A is going to be less than n. So, it means that some of the eigen values are singular values of A is going to be 0 or in particular you can say that sigma n is going to be 0.

Then your kappa 2 A is going to be infinity because if you try to calculate then it is sigma max upon 0. So, it is going to be an infinity. So, in case of when A is rank deficient your condition number of A is going to be infinity, if A is having full rank then condition number of A is given by sigma max upon sigma min.

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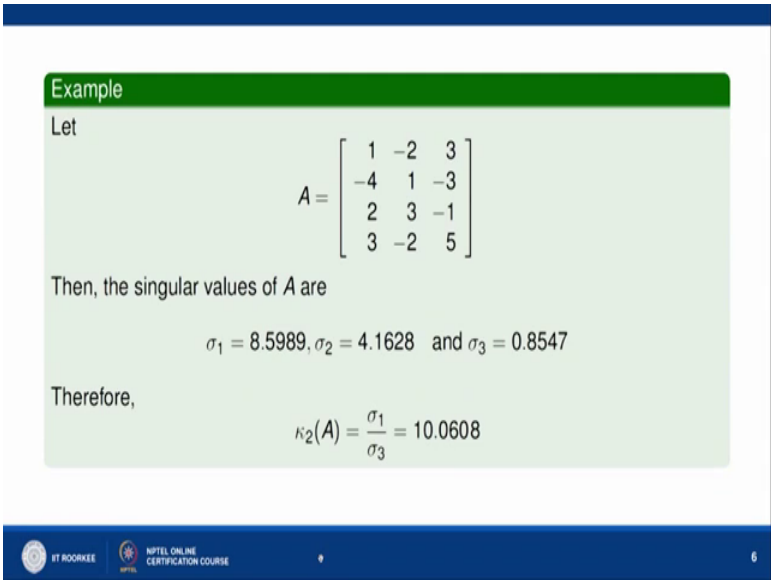

So, let us consider one example. So, here the example is this. Now, this time we are taking a rectangular matrix it is 4 cross 3 matrix and we can find out singular values of A it is given by sigma 1 sigma 2 and sigma 3 and you can see that a none of these sigma is are 0. So, it means that it is this matrix A is of full rank. So, rank of matrix A is going to be 3.

So, here we can define condition number of a matrix A as sigma 1 divided by sigma 3 and it is coming out to be 10.0607, and if you want to calculate you can calculate it using MATLAB also. So, here let us verify this using MATLAB. So, here let us called this as B matrix, B as a 1, minus 2, 3. So, first row is 1, minus 2, 3 and second row is minus 4, 1, minus 3 and third row is 2, 3, minus 1 and 4th row is 3, minus 2, 5. So, when you form this matrix then B is this matrix and then we calculate the singular value of this matrix B.

So, S is a SVD be here. So, S is going to be this 8.5989, 4.1628 and 0.8547. So, let us find out say condition of a condition of this matrix. So, condition number of this matrix B is going to be 10.0608. So, that is using the formula and now, if you look at the singular value of this matrix B. So, sigma 1 sigma max is 8.5989 divided by 0.8547.

So, that we are writing as S 11, S 11 is this entry and S 31 is this entry. So, S 11 divided by S 31 and it is coming out to be 10.0608. So, here we say that whether we use this command or we simply use the ratio of a sigma max upon sigma min then your value is going to be same. So, here we say that you can find out the condition number using the singular value decomposition theorem.

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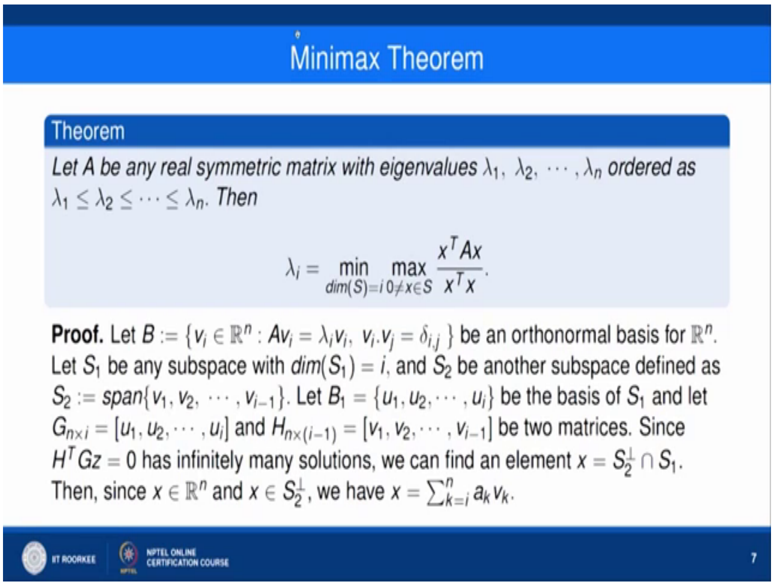

Now, moving on a next result let us first theorem first thing we want to discuss is minimax theorem.

So, what is this theorem. So, let a be any real symmetric matrix with eigenvalues lambda 1, lambda 2, lambda n which are ordered as a lambda 1 less than or equal to lambda 2 less than or equal to lambda n. Then you can find out the ith eigenvalue of this matrix A as a minimum over all those subspaces whose dimension is i and maximum over all those x which are nonzero and element of this subspace S of this quantity x transpose A x divided by x transpose x.

So, it means that if you find out say a a maximum of x transpose A x divided by x transpose x over all nonzero x and belonging to this set S and this S is having dimension i and we are considering the minimum of this quantity and whatever quantity you are getting that quantity is nothing, but the ith eigenvalue of this matrix A. So, let us first consider a brief proof of this minimax theorem and with the help of minimax theorem how we are going to utilize this minimax theorem in our case that we are going to discuss here.

So, first let us prove this theorem. So, let since we have already know that A is a real symmetric matrix, it means that it is a a diagnosable matrix or we can say that we have n linearly independent orthonormal basis orthonormal eigen vectors are available. So, it means that considering those eigen vectors we form a basis for R n. So, let B is the set of all v i such that Av i equal to lambda i v i. So, it means that your lambda i comma v i is basically eigen pair for this matrix A and not only that we are we are simply orthonormalizing the set it means that v i dot v j is basically delta ij. So, it means that these this is orthogonal set as well as all these vectors are having norm 1. So, B and so let us assume that this is an orthonormal basis for R n.

So, to prove this theorem we have considered that B 1 be the basis of S 1 and this S 2 is spanned by this or so, so we can call this B 2 as a basis of S 2 and consisting these v 1 to v i minus 1 element. Now, with the help of a B 1 and basis of S 2 we can find out two matrices G and H, where G consists the columns as a basis element of a basis of S 1 and H consists H has columns basis element of S 2 here.

So, we have a two matrices and now, we can say that this if you form this product H transpose a Gz equal to 0 then we say that we claim that it has finally, many infinitely many solutions. And if it has infinitely many solutions we get one solution and we claim that such a solution will belongs to S 2 th of intersection S 1. So, how it is let us consider this. So, here we have S 1 where whose dimension is i and we say that B 1 is a basis here u 1 to u i and H 2 which is span of a v 1 to v i minus 1.

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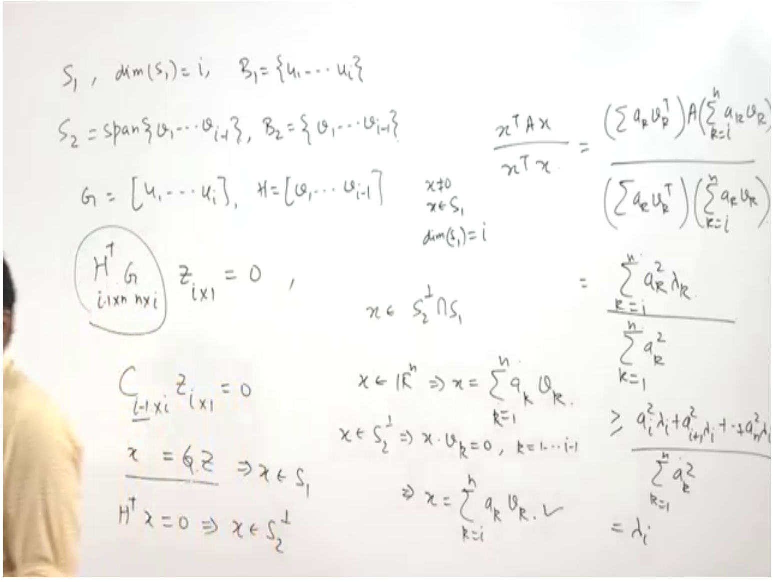

So, let us with the help with the help of this we consider B 2 as a basis of S 2 and consisting v 1 to v i minus 1 as a basis element and then we form a G and then H. If you look at the size of a G and H then it is going to be the size of G is going to be n cross i because all these u i and v i are element of R n. So, the size of the matrix G is going to be n cross i and a size of a H is going to be n cross i minus 1. So, if you form this product H transpose a Gz. So, here this H transpose is i minus 1 cross n G is n cross i and z is i cross 1 equal to 0.

So, if you look at this is going to be what H transpose G is let us say some matrix c whose size is i minus 1 cross i into zi cross 1 equal to 0. If you look at what is this? This is a system of a linear equation having i minus 1 linear equation and variables are i. So, here we have a less number of equation and we have more number of a variable here. So, it means that this always have infinite many solution here. So, if we if we have one, so if we have infinitely many solutions let us call let us take one solution as x. So, let us say with the help of this you call z is one such solution.

So, with the help of z you look at your Gz and call this as x. So, x is Gz means x belongs to the range space of a G or x belongs to range space of G means x belongs to your S 1. So, this implies that x belongs to your S 1. Now, we already know that this H transpose x is going to be 0. So, it means that we say that this implies that your x, x is going to be S 2 perp. So, it means that x is going to be orthogonal to every element of S 2. So, it means that x belongs to S 1 and x belongs to S 2 perp. So, it means that x belongs to S 2 perp intersection S 1. So, S x is an element in S 1 intersection S 2 perp.

Now, we know that this x is element of R n. So, it means that we can write x as linear combination of v i. So, where v k where from k is from 1 to n. So, let us x i write x as k equal to 1 to n a k v k here. Now, we already know that x belongs to S 2 perp. So, it means that what is S 2 perp? S 2 perp here S 2 is the span of v 1 to v i minus 1 and we know that x is belong to S 2 perp means that x is orthogonal to v k for k is from 1 to i minus 1.

So, it means that x has no contribution along v 1 to v i minus 1. So, it means that the constant a 1 to a i minus 1 are all 0. So, here we can say that x can be written as equal to k from i to n a k v k here. Now, with this x we try to find out your x transpose A x divided by x transpose x. So, we are considering that not all a i's are 0. So, it means that here we are considering that x is nonzero, and x belongs to here you can say that in this section.

So, here x belongs to S 1 and dimension of S 1 is basically i. So, here we have consider one S 1 and we have this thing and now, we try to find out this quantity. So, if you look at what is x transpose ax. So, x transpose A x you can calculate from this and you can say that it is going to be what. So, x transpose is a summation a k v k transpose and A and x is basically summation k from 1 to i to n here a k v k. Now, when you operate A on this v k then divided by summation a k v k transpose into k is from i to n a k v k right. And here using the orthogonality orthonormality of these v i we can say that this is going to be and if you operate A on v k then it is going to be lambda k v k. And then operate using the orthogonality we can say that this is going to be a k lambda k a k square lambda k is from 1 to n and here we have summation a k square k is from 1 to n. So, this is going to be this thing, ok.

Now, we know that this is going to be a greater than or equal to say you take it like lambda 1. So, here you can say that it is going to be bigger than or equal to. Now, here it is it is starting from i. So, here we have. So, it is going to be bigger than a lambda i here, right. So, here we say that once we have your x then you can say that x transpose A x divided by x transpose x is going to be k from i to n a k square lambda k and here it is k is from i to n a k square.

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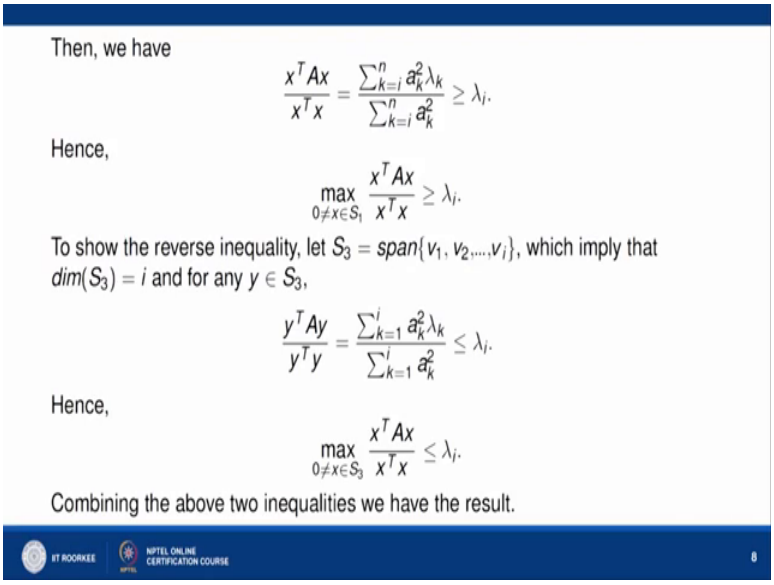

Then if you look at out of these lambda i to lambda n only lambda i is smaller than lambda i plus 1 to lambda n. So, we are simply taking that this is going to be what, you can write it like this that here this is going to be bigger than or equal to we can write it a i square lambda i and a i plus 1 square lambda i and so on, a n square lambda i divided by summation a k square k is from 1 to n.

So, you can take lambda i out and you can say that it is going to be this lambda i here. So, it means that this x transpose A x divided by x transpose x is greater than equal to lambda i. Now, what is this x? x is nonzero and x belongs to S 1 and what is S 1 here done S 1 is any subspace whose dimension is i. So, it means that this is going to be bigger than lambda i. So, maximum is also going to bigger than this lambda i. So, here we can say that hence a maximum over x nonzero x belonging to S 1 x transpose A x divided by x transpose x is going to be bigger than lambda i.

Now, we want to show that this inequality is actually a equality. So, here we want to show that this maximum is going to be less than lambda i for some the subspace of dimension i. So, let us say show the reverse inequality for that let S 3 is span of v 1 to v i, which imply that dimension of S 3 is i. So, here S 3 is basically the subspace spanned by this first i eigenvectors of A.

So, here we say that, now, for any y in this subspace you calculate y transpose a y divided by y transpose y and if you follow the same thing then it is going to be i equal to summation k from 1 to i, a k square lambda k divided by k from 1 to i, a k square and using the same thing here this lambda i is bigger than a lambda i minus 1 and lambda 1. So, it means that this whole quantity is bigger than a less than or equal to lambda i. So, here lambda i is the bigger than lambda i minus 1 up to lambda 1. So, it means that this quantity is going to be less than or equal to lambda i. So, it means that this is for every y in S 3. So, it means the maximum of these quantities again going to be less than lambda i.

So, if you combine that they exist some subspace of dimension S i a dimension i or for which this is less than or equal to lambda i. So, if we look at the combined thing then we can say that a a minimum of a minimum over this a minimum taken over a dimension S equal to i and maximum of A x transpose A x upon x transpose x where x is nonzero element of this a subspace S is going to be lambda i.

So, this proves the mean maximum for any real symmetric matrices whose eigenvalues are lambda 1 to lambda n. And with the help of this minimax theorem we can find out any eigenvalue any eigen value of this matrix A.

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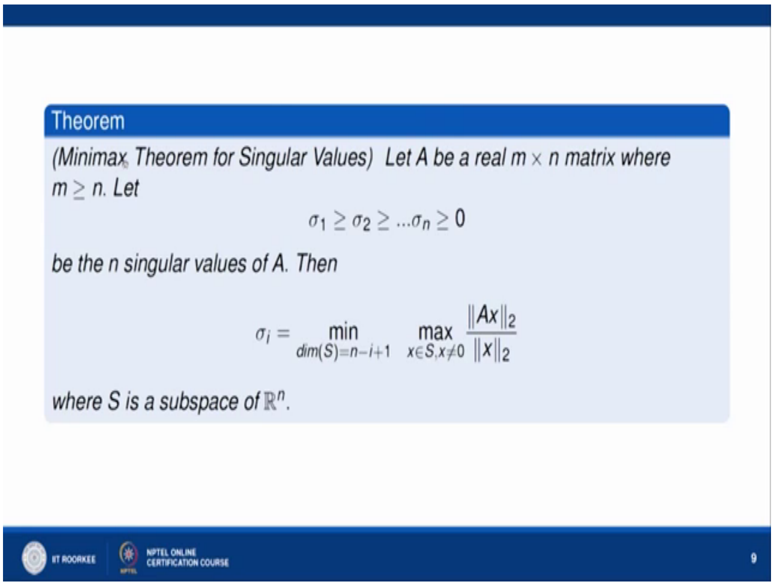

So, let us utilize this minimax theorem for singular values. So, let A be a real m cross n matrix, where m is greater than or equal to n and let sigma 1 is greater than sigma 2 and a sigma n greater than equal to 0 be n singular values of A, then you can calculate your singular values sigma i as maximum or a maximum over a maximum of a 2 norm of A x divided by 2 norm of x, where x is a nonzero and x belongs to S, where S is some vector subspace whose dimension is n minus i plus 1.

The only difference between this theorem and the previous theorem that here your eigenvalues of this matrix is arrange in this order lambda 1 less than or equal to lambda 2 less than or equal to lambda n. But if you look at the singular values singular values are just arranged in a different order just a reverse order that sigma 1 is greater than equal to sigma 2 and greater than equal to sigma n. So, to apply the minimax theorem here we have to in place of dimension of S equal to i we have to go from the right hand side.

So, here we can say the ith element of a transpose a is nothing, but the n minus i plus one th element of A, A transpose A. So, here we can use the previous results the only difference here is that now, because of this order we have to write a dimension of S as n minus i plus 1, where S is a subspace of R n.

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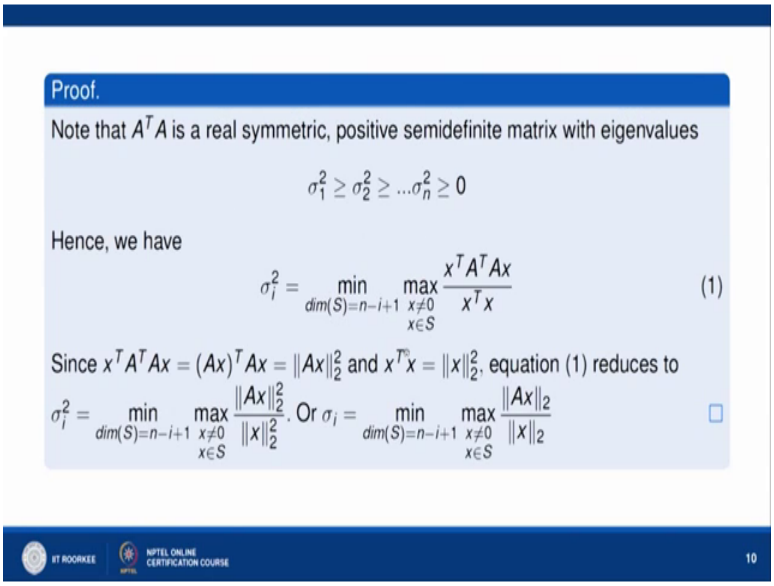

So, let us consider the proof of this A transpose is a real symmetric positive semidefinite a matrices with eigenvalues, sigma 1 square plus greater than equal to sigma 2 square sigma n square greater than equal to 0.

So, using minimax theorem for this matrix A transpose A which is now, real symmetric positive semi definite we can say that sigma i square is equal to dimension of a minimum over dimension of S equal to n minus i plus 1 and maximum over all those nonzero vectors of a S x transpose A transpose A x, here A is replaced by A transpose A. So, x transpose A transpose A x divided by x transpose x and if you look at the numerator and a numerator is going to be a 2 norm of A x whole square, so a square of 2 norm of Ax.

So, this is going to be 2 norm of A x and this is given by 2 norm of x. So, this equation 1 is going to be reduced by this sigma i square equal to minimum dimension of S equal to n minus i plus 1 and maximum over a x nonzero and x belongs to S, 2 norm of A x square and 2 divided by 2 norm of x square. And taking the square root on both the side we can say that singular value sigma i is given by this formula.

So, it means that if you know the 2 norm of A x and 2 norm of x and you can calculate the sigma i by this minimax result. So, it means that every singular value can be given by this.

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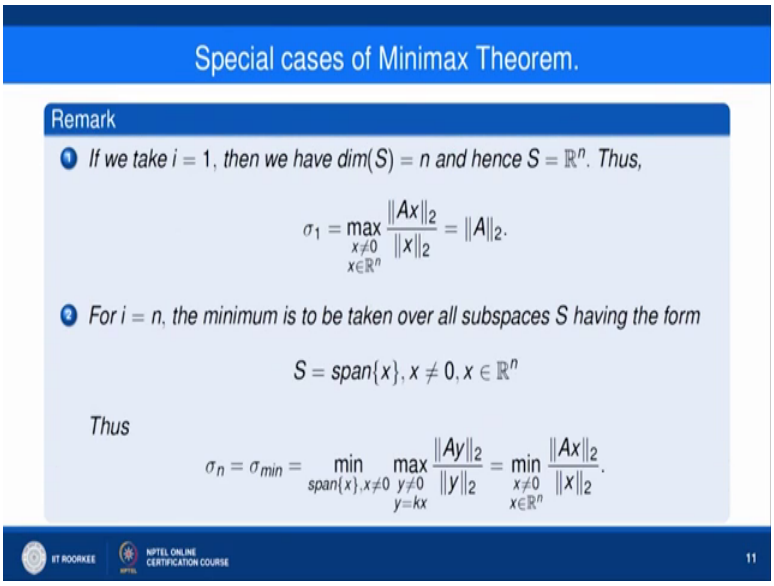

So, now, let us case some a special case of minimax theorem. So, let us take i equal to 1. So, if you take i equal to 1 then this i is equal to 1. So, it means that your a sigma 1 you can calculate at minimum over dimension of S equal to n maximum over x nonzero and x belongs to S.

So, if you look at dimension of S is going to be n here. So, dimension of S equal to n means we are taking the minimum over say R n here. So, here you can say that sigma 1 is going to be maximum over a nonzero vector of R n here and 2 norm of A x divided by

2 norm of x. If you look at this is nothing, but 2 norm of A matrix norm of A here. So, sigma 1 is going to be matrix norm of A here. Here matrix norm means 2 norm of matrix A here.

Similarly if we take i equal to n then the minimum is taken to a minimum is to be taken over all subspace S having the singleton element that is S is span of x, where x is nonzero and x belongs to R n. So, in this case your sigma n which is going to be a sigma min can be calculated by minimum of a span of x x is not equal to 0 maximum y nonzero and y equal to kx 2 norm of Ay divided by 2 norm of y which is nothing, but minimum of non 0 vector of a R n a 2 norm of A x divided by 2 norm of x. So, this is the special case of minimax theorem.

And with the help of a minimax theorem we wanted to consider the this result which is important result and it says that the finding the singular values of matrix A is basically a well conditioned a problem.

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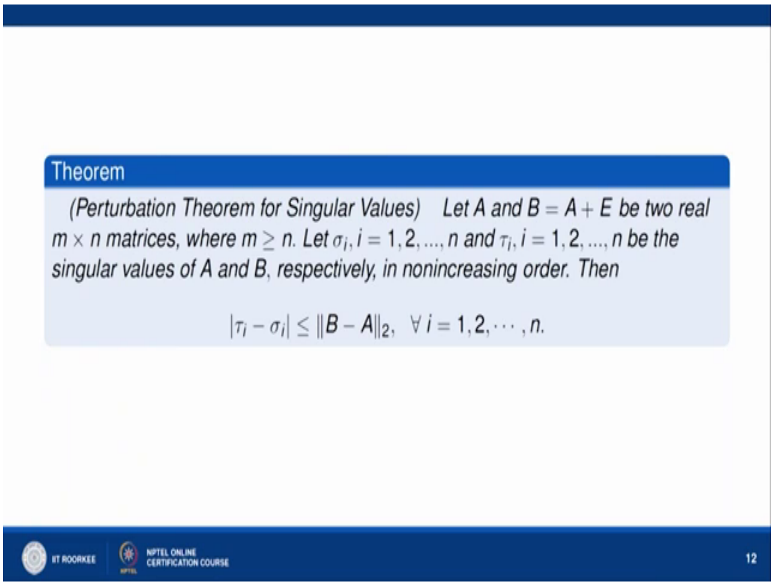

So, just consider the theorem here. So, let A and B, where B is defined as a a plus E be 2 real m cross n matrices where m is greater than equal to n and let us sigma i or n sigma values of A matrix A and tau i are n singular values of this perturb matrix A plus E be the singular values of A and B respectively in non increasing order, so order is maintained. And then a modulus of tau i minus sigma i is bounded by 2 norm of B minus A.

Now, if you look at 3 norm of B minus A it means that 2 norm of E. So, it means that your singular values are well conditions. So, it is insensitive to perturbation. So, if a is perturbed by this a matrix E then their singular values are also perturbed by the maximum perturbation is up to the 2 norm of matrix A here the similar result we can prove in terms of Frobenius norm.

So, this is that this if you perturb our matrix then their singular values are perturbed at most the size of the 2 norm of the perturb matrix matrix that is E. So, let us prove this theorem and say that since we know that by minimax theorem tau i can be calculated by the following result.

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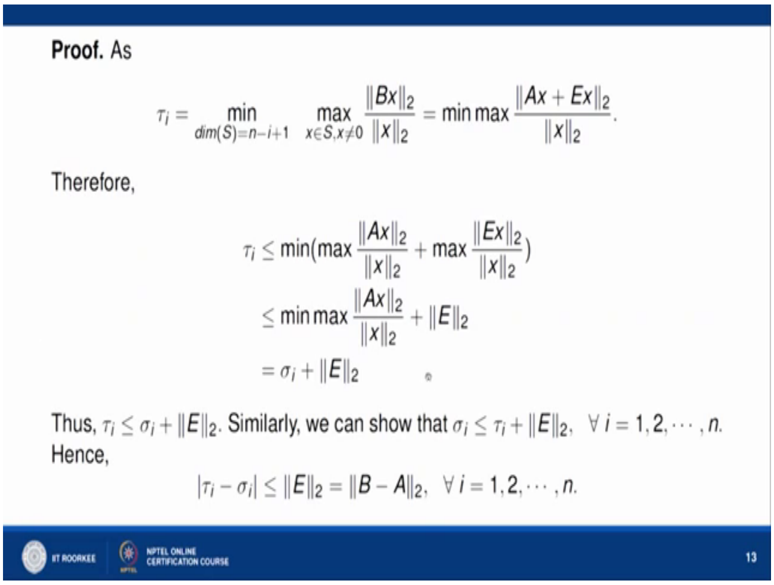

So, tau i is the singular value of matrix p. So, we can say that tau i is given by minimum over all those subspaces whose dimension is n minus i plus 1 and maximum over this maximum of this quantity. So, be a 2 norm of B x divided by 2 norm of x and B x is what? A x plus E x.

So, here we can say that this is minimax of 2 norm of A x plus E x divided by 2 norm of x and if you simplify the using triangle inequality over a 2 norm it is given by a minimum of maximum of 2 norm of A x divided by 2 norm of x plus maximum to a 2 norm of E x divided by 2 norm of x. And if you look at this can be written as minimax of a 2 norm of A x divided by 2 norm of x and this is nothing, but 2 norm of E. And if you look at this is the formula by which we can calculate the singular value of a matrix A. So, here this is nothing, but sigma i. So, it means that tau i is less than or equal to sigma i plus 2 norm of E and that is true for every i, i is from 1 to n. So, it means that we can show that similarly we can show that sigma i is less than tau i plus 2 norm of E for every i from 1 to n. So, it means that if you simplify the this can be written as sigma tau i minus sigma i is less than or equal to 2 2 norm of E which is equal to 2 norm of B minus A.

So, here what we have done here? We started with tau i and then we have shown that there is going to be less than or equal to sigma i plus 2 norm of E which is given here. Now, we can start with the sigma i and we can show that sigma is going to be less than or equal to tau i plus 2 norm of E and if which is given by this. So, it means that if you combine this and this then we can say that modulus value of tau i minus sigma i is going to be bounded by a 2 norm of E and 2 norm of E is going to be a norm of B minus A.

So, it means that and this is true for every i. So, it means that that if you perturb your matrix A then your singular values of matrix is going to be perturbed by some quantity whose value whose modulus value is not bigger than 2 norm of perturb matrix E. So, it means that your singular values are going to be insensitive to the perturbation. So, it is a well conditioned problem, so singular values of A given any matrix A is a well conditions here well conditions.

Now, the singular kind of proof we can give for a Frobenius norm so that we will discuss in next lecture. Here we stop our lecture. So, in this lecture we have utilized the a singular value decomposition to find out say a algebraic property of the matrix A and the last result shows the that the singular values a singular values of A given matrix A is a insensitive to do perturbation.

So, with this we stop and thank you very much for listening us and we will meet in next lecture.