Numerical Linear Algebra

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Lecture - 45
Algebraic and geometric properties of matrices using SVD

Hello friends, welcome to this lecture and in this lecture, we continue our study of, say, algebraic and geometric property with the help of, say, SVD. So, with the help of SVD, we are going to discuss the algebraic and geometric properties about the vector subspaces of, say, matrix say, range space of a null space of a transverse and so on. So, in last lecture, we have seen that how with the help of orthonormal basis of a subspace S, we are able to find out say orthogonal projection on to that vector subspace S. Now, we are going to utilize this concept and try to find out, say, orthogonal projection for range space of a null space of a transverse null space of a and range space of a transverse.

So, let us look at this theorem. So, let A is USV transverse, be the singular value decomposition of a real m cross matrix A, where m is greater than or equal to n and let the rank of A be r. So here, we know what is the rank of A, is r. So, partition U and V as U1 and U2 and V as V1 and V2, where U1 consist of first R columns of U and U2 represent U2, is a matrix of remaining M minus R columns of U. Similarly, V. This V1, represent the first R columns of V and V2 represent the remaining N minus R columns of V.

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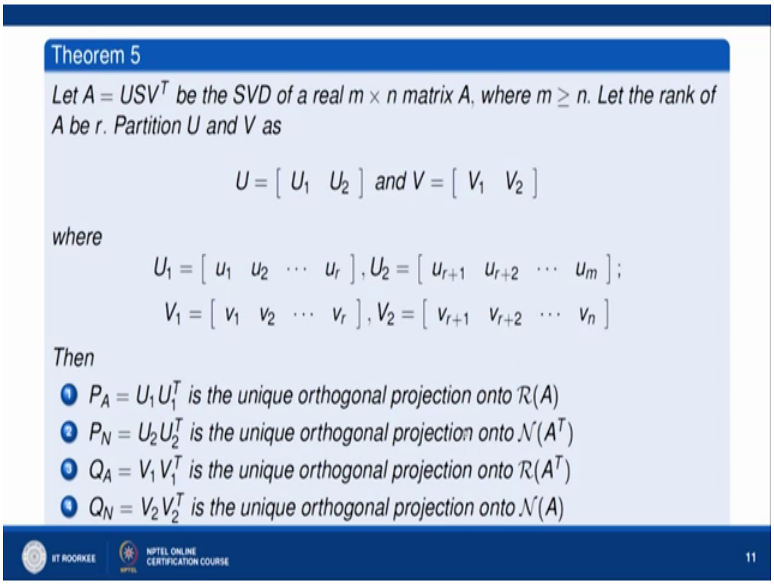

Now, with the help of, we have already seen that this U1, the columns of U1 forms an orthonormal basis of range space of A and similarly, columns of U2 forms a orthonormal basis for null space of a transverse and we have also seen that,V1 and V2 generate the corresponding orthonormal basis. Now, with the help of this and the previous theorem, we try to prove that that PA, which is given as U1 – U1 transverse, is a unique orthogonal projection on to R of A. So here, since this U1 is a basically what U1 consist of column vectors, which generate aorthonormal basis for range space of A.

So, it means that U1 – U1 transverse is a unique orthogonal projection on to R of A, that we can prove with the help of previous theorem and similarly, we can prove that PN, where PN is U2 – U2 transverse, is a unique orthogonal projection on to null space of a transverse. So here, this can be easily done with the help of previous theorem and the previous theorem, is the theorem last theorem which we have proved in previous lecture and recall that the columns of U2 forms an orthonormal basis for null space of a transverse. So, by previous result, we can say that PN which is nothing, but U2 – U2 transverse is a unique orthogonal projection on to null space of a transverse.

Similarly, Q of A, which is given as V1 – V1 transverses a unique orthogonal projection of range space of a transverse. So here also, you have to, please observe that this V1 columns of V1, that is V small V1 to VR, forms an orthonormal basis for range space of a transverse. So, that can be shown with the help of the same singular value decomposition of A. So here, A can be written as USV transverse and let us find out a transverse as VS transverse U. Now, S transverse is same as SS. So, A can be written as VSU transverse. Now see the role of U and V are change. So, we can say that for A, the first R columns are going to be orthonormal basis for say, range space of A. So, the first R columns of V1 are going to be orthonormal basis for A transverse.

So we, knowing this fact, that the first R columns of V forms an orthonormal basis for range space of A transverse and utilizing result of previous lecture, we say that V1 – V1 transverse is a unique orthogonal projection on to range space of A transverse. Now, with the same result, we can say that the remaining N minus R element is going to be orthonormal basis for null space of A. So, it means that this QN, which is nothing, but V2 – V2 transverse is a unique orthogonal projection on to null space of A. So, this theorem is proved with the observation that, what this U1 – U2 – V1 V2 represent and the previous theorem, which says that if we know the orthonormal basis, then we know how to define orthogonal projection on to that subspace S here.

So here, we again, we give the phases that U1 represent the orthonormal basis columns of U1, represent the orthogonal basis on range space of A. U2 represent the orthonormal basis on null space of a transverse and this, V1 columns of V1, represent the orthonormal basis for range space of a transverse and this,V2 columns of V2 represent an orthonormal basis for null space of A. So, using this fact and the previous theorem, we have shown that PA PN QN and QN represent the unique orthogonal projection on to the respective subspaces of RN.

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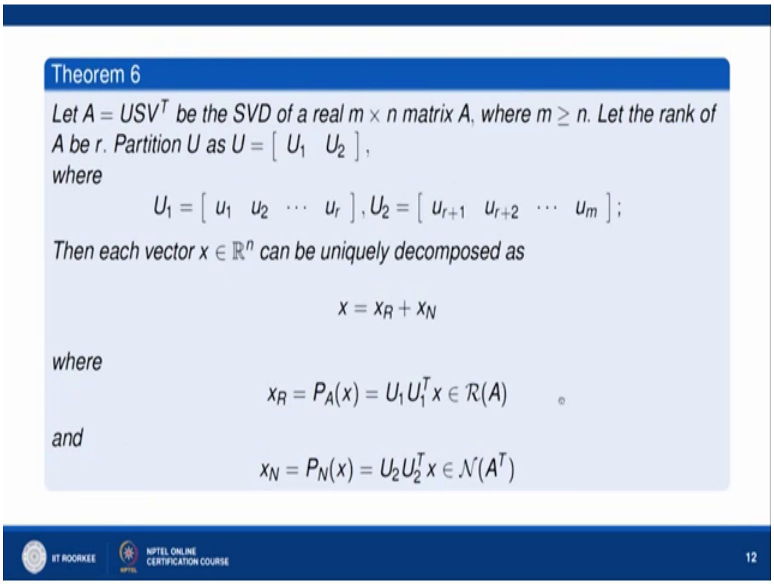

So now, moving on next result, which says that, let A be as USV transverse, be the singular value decomposition of A real m cross n matrix A, where m is greater than or equal to n and let the rank of A be r. So, it means that these information is already given to us and again, we partitioned U and V as this, U as U1 and U2, where U1 represent the first R columns of U and U2 represent the remaining M minus R columns of U.

Now, we say that with the help of this, we can decompose any member x in Rn as xR plus xN, where xR represent the range space of A, where range space of A, means A is already given. So, xR is a member of range space of A and xN is a members of null space of A transverse and xr and xN can be given as P of Ax and xN is given as PNx. Now,PAx, we have already seen that PAx is what,U1 – U1 transverse, x and PNx is nothing, but U2 – U2 transverse x, so, this is the content of this theorem.

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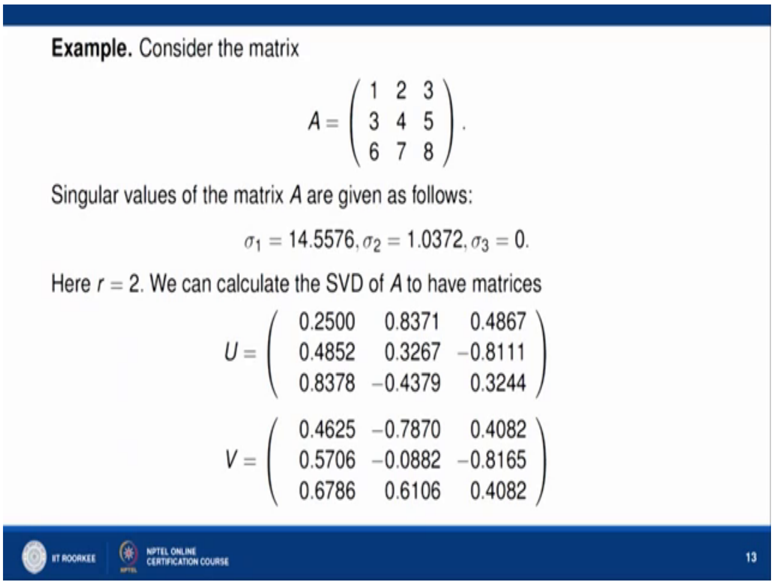

And let us again, this can be easily proved why. So, how we can easily prove? So here, first of all, please observe that this PAX is a orthogonal projection on to range space of A. So, it means that every element can be written as x equal to xR plus, let me write it here.

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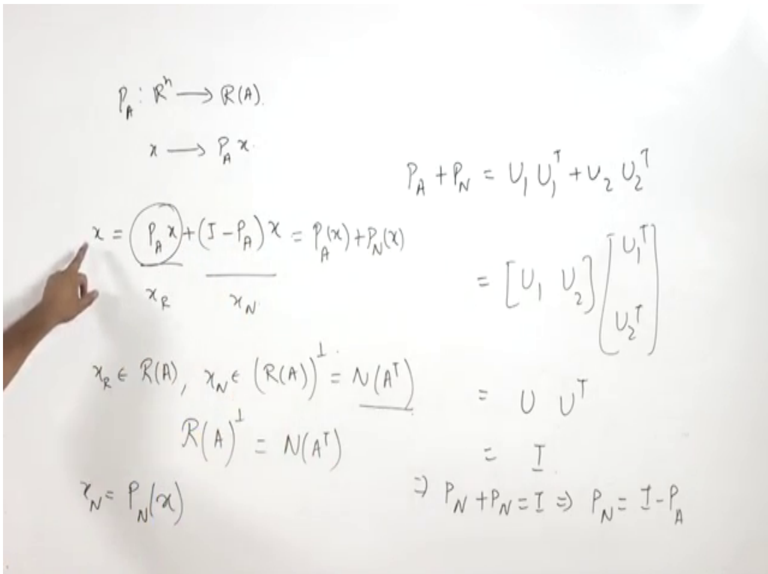

So here, we know that P of Ax is an orthogonal projection of Rn to say range space of A here. So, here x goes to say P of Ax here. So, x can be written as P of Ax plus I minus P of Ax here, and this, we are calling as xR and this; we are calling as x of N here.Now,XR is a member of what? xR is a member of range space of A and xN is going to be the member of range space of A perpendicular. Now, we already know that range space of a perpendicular is nothing, but null space of A transverse. So, it means that, this is same as null space of a transverse. So, it means that you can write x as element xR, which is the member of range space of A and xN, which is going to be the, remember, of null space of A transverse. Now here, we know that this xR can be written as P of Ax and we know that null space of a transverse, if xN is a element, here, we know that, xN can be written as say, your P of N of x here.

So, we have to show only this thing, that this I minus PAx is nothing, but PNx. So, here to show that PA plus PN is going to be I, that we are going to show, because we want to show that x can be written as Px plus I minus PAx. So, if you can show that I minus PA is PN, then we can write this as PAx plus PNx. So, for that we have to look at this PA plus PN. Now, PA is an orthonormal, orthogonal projection on range space of A. So, it is given as U1 – U1 transverse and P-N is orthogonal projection on to null space of A transverse. So, it is given as, U2 – U2 transverse and if you look at this sum ,this sum can be written as product of U1, U2 into U1 transverse U2 and this is nothing, but U and this is nothing, but U transverse and we already know that, U is an orthogonal matrix. So, it means that this is going to be I. So, it means that PA plus PN is I. So, it means that PN is nothing, but I minus PA.

So, it means that since, x can be written as PAx plus I minus PAx. Now, we already know that I minus A is nothing, but A. So, it can be written as PAx plus PNx. So, it means that, we are able to prove this theorem, that x can be written as xR plus xN, where xR is P of Ax and xN is I minus PAx, which is nothing, but PNx and we know that PAx is nothing, but U1 – U1 transverse x and of course, and this belongs to the range space of A and PNI minus PA which is nothing, but PN is nothing, but U2 – U2 transverse x and this is going to be member of null space of a transverse. Now, look at the numerical verification of these two theorems.

So here, let us consider this matrix A which is as 1, 2, 3, 4, 5, 6, 7, 8.We have just taken matrix here. Now, we can calculate the singular values of the matrix A and it is given as follows: - sigma 1 equal to 14.5,5576 sigma equal to 1.0372 and sigma 3 is equal to 0. So here, we have 1 1 0 vector. So, it means that rank of A is going to be 2 here. So, R is going to be two and we can calculate the singular value decomposition of A and we can have U and V. So, U is given as 0.500 and so on. This, you can easily calculate from mat lab, similarly, you can calculate this V, where V consist of UM Eigenvector corresponding to a transverse A.

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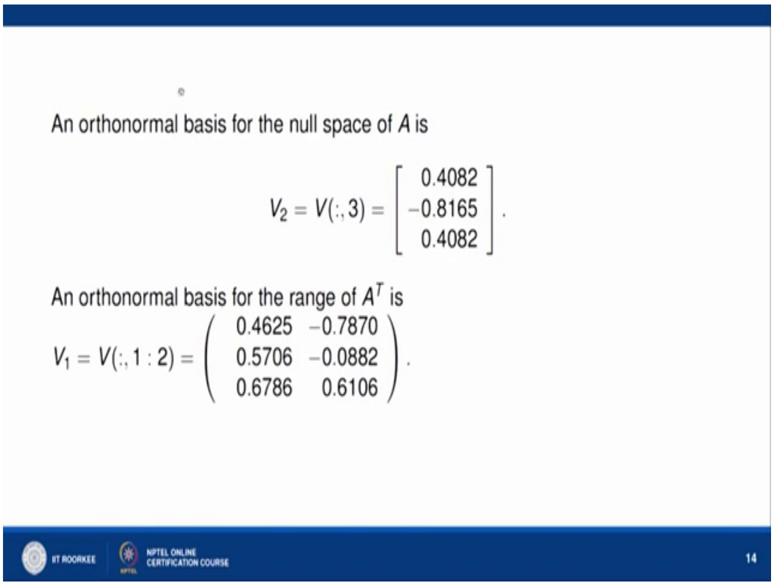

And we can say that, an orthonormal basis for the null space of A is going to be V2, yeah! So, what is the null space of A? It is going to be V2 – V2. What is V2 here? The third column of your, in fact, it is if you look at since R is 2. So, it means that first two columns of U is going to represent the range space of A. So, it means that first two column generate the range space of a last column, represent the span, the null space of a transverse. The first two column is going to be span range of a transverse, last column is going to span the null space of A here.

So, that is what is given here, that orthonormal basis for the null space of a, is the last column of matrix V, which is nothing, but this V column three. So, that is your mat lab command for finding the third column of vector V. So, it is 0.4082 minus 0.8, 165.480. And similarly, orthonormal basis for range space of A transverse, that is given with the help of V1 here. So, that is what we have seen that the first two columns are going to span the range space of a transverse. So, first two columns are going to be orthonormal basis for the range of a transverse.

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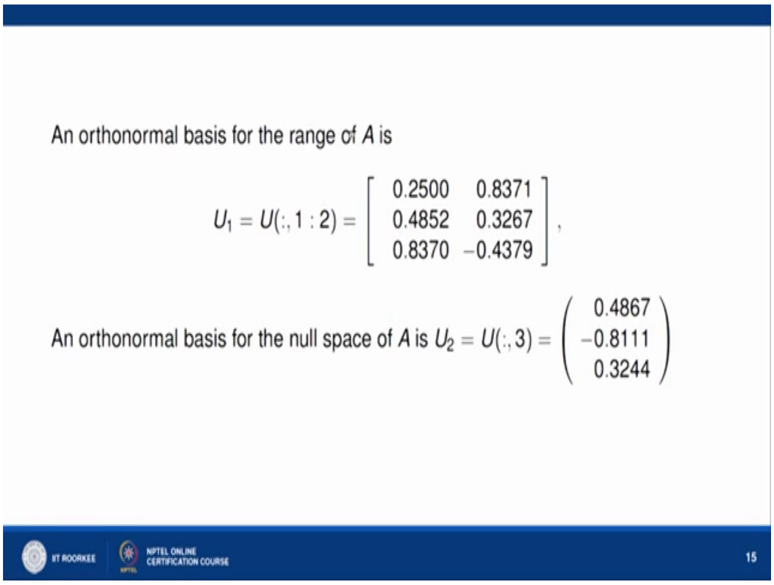

And range space of A, we know that the first two column of U is going to be orthonormal basis for the range of A and the last column generate the null space of a transverse and it is given as 0.4867 minus 0.81111, .3244. So, that is what we can say here, and with the help of this, we can also generate unique orthogonal projection on your A range space of A range space of A transverse null space of A and null space of A transverse.

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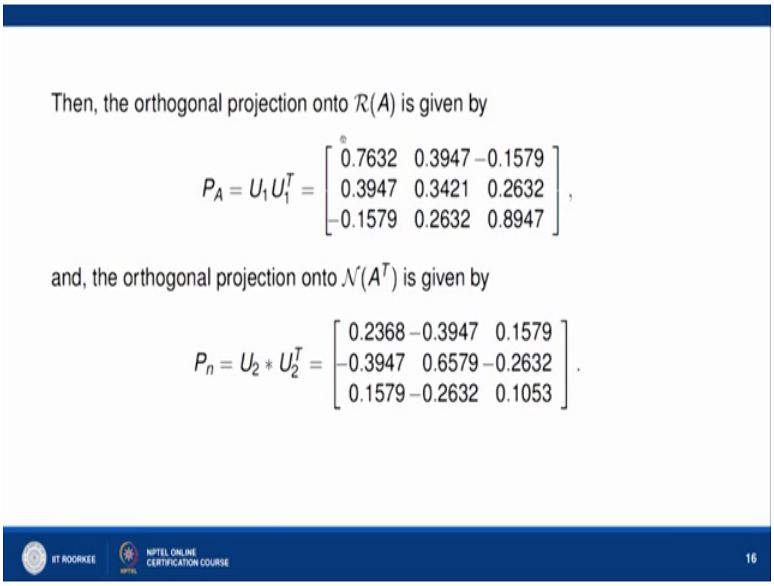

And we can calculate like this, the orthogonal projection on to range space of A is given as U 1 U 1 transverse where columns of U 1 represent the orthonormal basis of range space of A. So, here if you look at U1,it will generate the range space of A. So, these columns are orthonormal basis of range space of A. So, U1 – U1 transverse is going to be orthogonal projection on range space of A. So, that is what is written here. So, PA can be written as U1 – U1 transverse, as this an orthonormal projection on to null space of A transverse.

So, just to look at what is the orthonormal basis of null space of A transverse orthonormal basis for null space of A transverse, is going to be this 0.4867 minus point 0.8111 and 0.344. So, with the help of this U2, you calculate U2 – U2 transverse and it is given as PN. So, PN is going to be orthogonal projection on to null space of a transverse.

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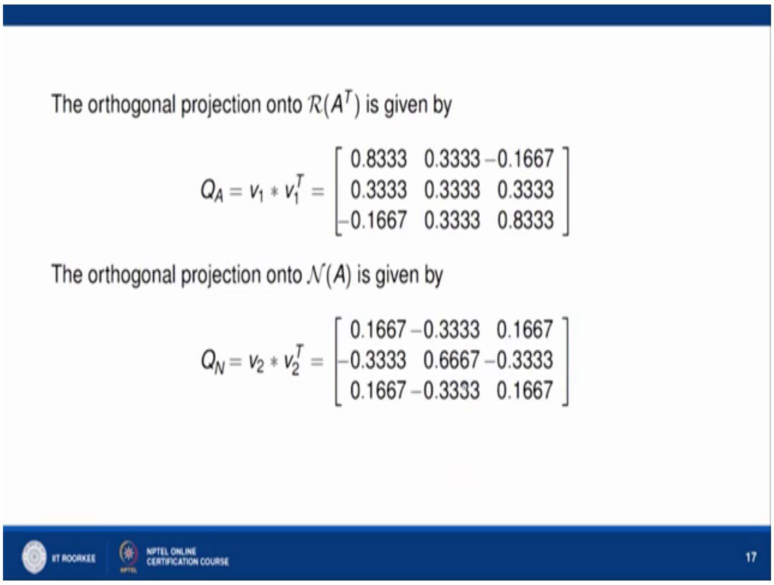

Similarly, we can calculate the orthogonal projection on range space of a transverse for that, please recall that orthonormal basis for range space of a transverse is the first two column of the matrix V and we call that matrix V as V1.V1 represent the first two columns of matrix V.

So, here it is V1 here. So here, V1 is the first two column of the matrix V here. So, that will give you the range space of a transverse. So, QA which is given as V1 and V1 transverse is going to be the orthogonal projection on to range space of A transverse. Similarly we can say that, orthogonal projection on to null space of A is given by V2 – V2 transverse and V2 is the last column of the matrix V here and it can be calculated like this and look at. So, this is what this is the content of first theorem.

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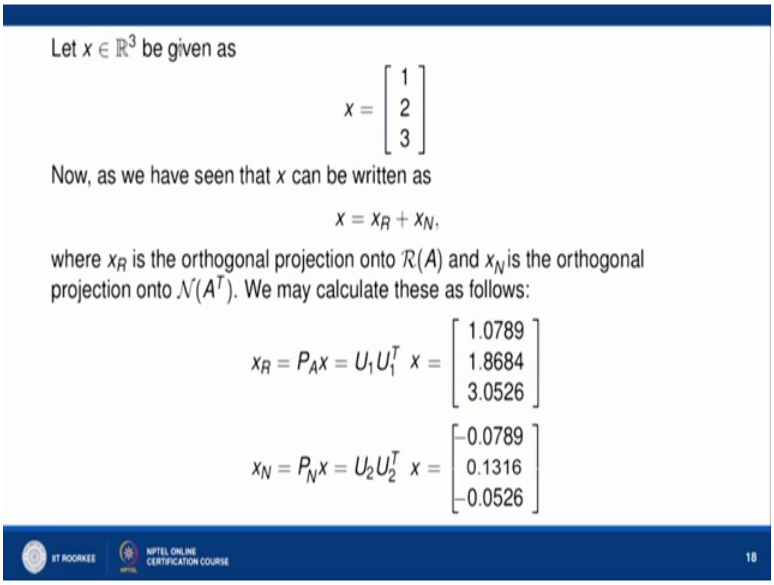

And if you look at the content of last theorem, which says that any vector x can be written as xR plus xN, where xR is PAx and xN is going to be PNx. So, we can calculate for this vector. So, let us say that, take any element in R3. So, let us take 1, 2, 3 and XR and xN. So, xR is PAx, which is nothing, but U1 – U1 transverse x, which is given as 1.0789 and so on. Similarly, you can calculate xN which is nothing, but PNx, which is given as U2 – U2 transverse x and it is minus 0.789 minus 0.0526. So, that is what we can calculate from this.

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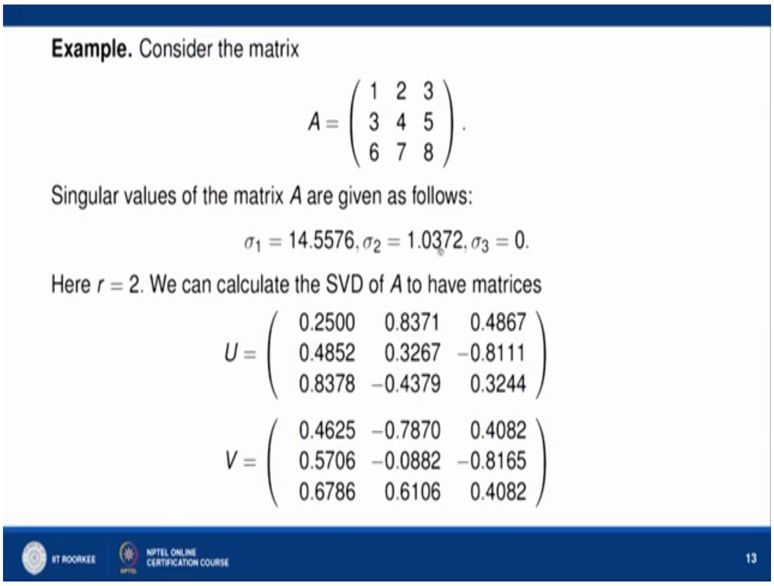

So, we have seen that given a matrix A, we can calculate singular values, we can have singular value decomposition of A and with the help of this, and we are able to find out orthonormal basis for different subspaces of generated through A. So, whether it is null space of A or null space of A transverse range space of A transverse and range space of A with the help of singular value decomposition, matrices U and V and also with the help of these, U1, U2, VI and V2, we are able to find out orthogonal projection on to range space of A, which is given as P of A and range space null space of A transverse.

That is U2 – U2 transverse and range space of A transverse, that is QA and orthogonal projection on to null space of A, which is given as V2 – V2 transverse and also, we have shown that any element of R3 can be written as unique representation in terms of element of range, space and element of null space of a transverse and it is related to the matrices U1 – U1 transverse U2 – U2 transverse, that is what we have seen in this example and we can actually see that, their sum is going to be the 1, 2, 3 if you look at, you can easily see it here. Now, let us do the same thing with the help of mat lab here.

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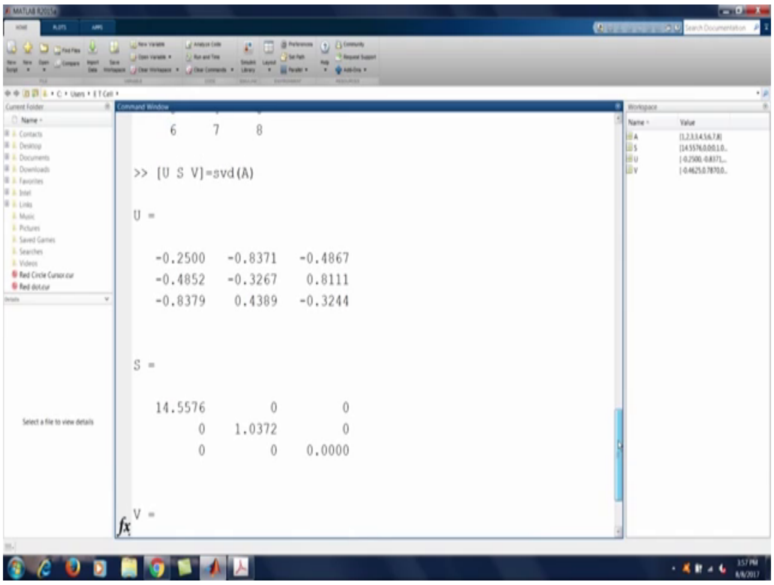

So, for that let us look at what is your matrix A. So, matrix A is going to be 1, 2 and 3, second row is three and 4 and 5 and then, the last row is going to be 6, 7, 8. I think 6, 7, 8. So, let me check it is what, let me check it here, it is 1, 2, 3, 4, 5, 6, 7, 8. So, it is given here. So, A is going to be this matrix and we can find out our matrix U and V as this. So, here we write U, S and V; which will give you the component matrices U, S and V from

SVD of A. So, let us look, write this command and it will give you the singular value decomposition of this matrix A and here is this. So, here if you look at this, first is going to U, is this minus O.2500 and so on. S is going to be this and V is going to be this.

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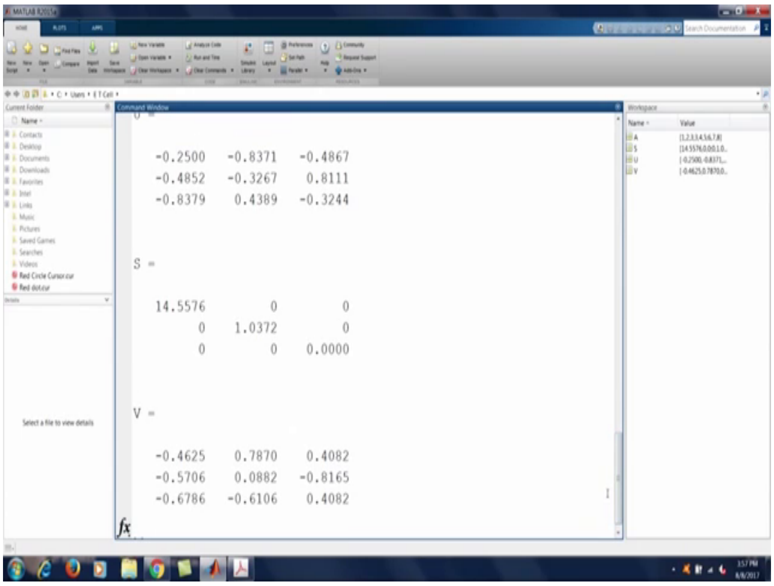

If you look at it is matching with this. So, here U is this, V is this and singular values given by 14.5576 and 1.0372 and sigma 3 is this. So, here if you look at your one of the Eigen value of this singular value of this, A is going to be zero here. So, it means that rank of A is going to be two. So, it means that now your U1 is going to be what. So, let us define your U1 as the first two column of U. So, here to write down the first two columns, we use this notation 1-2-2. So, first two columns are 1 and 2. So, here this is your U1, which is minus 0.2500, 0.4852, 0.8379 and which is listed here. So, here your U1 is this is the first two column of matrix U and that will give you the orthonormal basis for range space of A. So, U1 is this, similarly, you can define the last column of U that is U2. You can call it and U2 is the last column of U. So, that is going to be the range, orthonormal basis for null space of A transverse.

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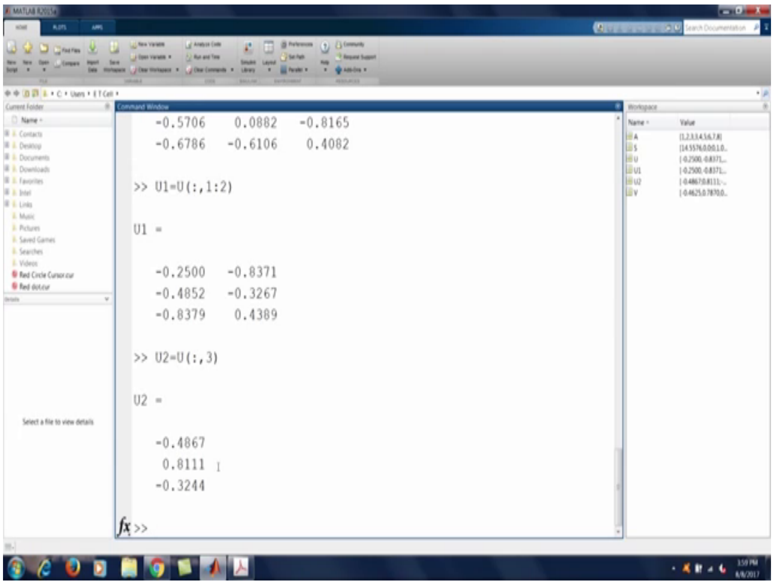

So, that is your U2. So, U2 is minus 0.48367minus 0.8111 minus 0.3244 and that we have seen that, U2 is going to be null space of a transverse. So, that is what we are calling as U2. Similarly, we can calculate V1. So, V1 is basically what? V1 is the first two columns of V. So, that again, we are writing as one to two and this is going to be range span, the range space of a transverse. So, V1 which is minus 0.4625, 0.7870, these two columns represent. I am not saying what is. So, first column and second column, these two column will generate the orthonormal basis for null space of, say, range space of A transverse.

So, here, orthonormal basis for range space of a transverse given by V1, which consist of first two column of matrix V. Similarly, we can say that the last column of V, which is V2 represent the null represent the orthonormal basis for null space of A. So, that we are writing as V2 here, V2 is basically the last column. Last column here is third column. So, third column is going to be V2 here. So now, we are able to find out, say, orthonormal basis for these spaces. So, for range space of A, null space of a transverse, range space of a transverse, null space of A. We have able to find out say orthonormal basis. Now, with the help of orthonormal basis, we are able to, we are going to calculate your PA PN and QA and QN. So, PA is basically what? PA is the orthogonal projection on range space of A.

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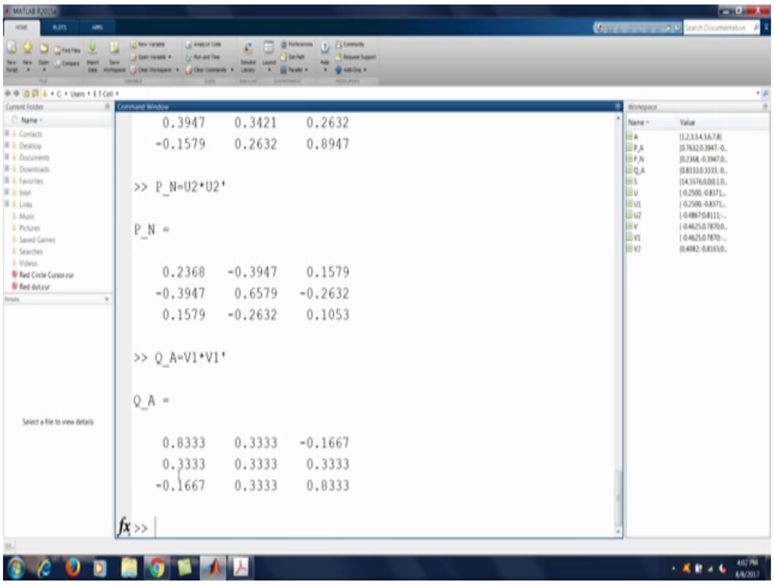

So, what is PA here? So, let us calculate P of A here. So, P of A is going to be U1 andU1 – U1 transverse. So, U1 transverse. So, that is going to be PA here. So, PA is 0.7632, 0.3947 minus 0.1979 and so on. So, PA is going to be the orthogonal projection on to range space of A. So, that is what we have seen here A, 0.7632, 0.3947 minus 0.1979 which is we are listing here.

So, PA is given by this. Now, PN is going to be what? So, PN is going to be, say, orthogonal projector projection on to null space of a transverse. So, that we can write it here as, U2 into U2 transverse. So, that we are listing here. So, that is your P of N. So, PN is U2 – U2transverse here. So, that we are going to write it and it is given by 0.368, 0.39497, 0.197 and if you look at it, it is matching here right. So, PN, we can calculate like this similarly we can calculate Q of A. So, Q of A is going to be what? Q of A is going to be orthogonal projection on range space of a transverse, that is going to be, V1 into V1 transverse. So, we can calculate QA as point 0.8333, 0.333, minus 0.1667 and that we can look at here.

So, you can write it Q of AQ of A is V1 – V1 transverse. So, that we have listed here, similarly, we can calculate Q of N. So, Q of N represent the orthogonal projection on null space of a and it is given by V2 into V2 transverse and that will give you, say, QN here and that is 0.1667 minus 0.1333, 0.1667. So, that is what is given here. So, whatever is the representation of your U and V, but when you calculate your orthogonal projection that is, PA PN QA QN, that is going to be unique orthogonal projection. So, it means that whatever be the representation or whatever be the component, U and V you obtained through your mat lab, but if you calculate your P and PN, it is going to be one and the same thing.

So, it means, it is going to be unique one. So, we have calculated PA PN QA QN that you can see that it is exactly matching with what we have seen here, now, look at the last part. So, for that you calculate x, as you write 1, 2, 3 here.

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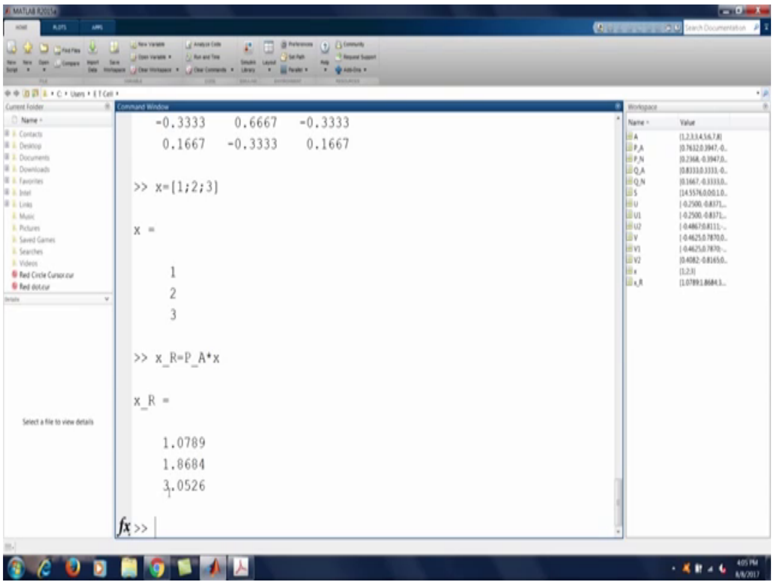

So, X is this vector. Now, you want to calculate what xR is here. So, what is xR here? xR is going to be P of Ax , PA with x right? So, it is same as this 1.07891, 0.86843,and 0.0526. So, that is what we have calculated here. So, xR is nothing, but PAx which is nothing, but U1 – U1 transverse x and that is going to be this. So, now, let look at your xN. So, xN is going to be your PN, xN and it is going to be minus 0.2368, just a minute ok, it is PNx. So, let me write it, this is some problem here, here xN is what? Here, xN is PN into x, not this.

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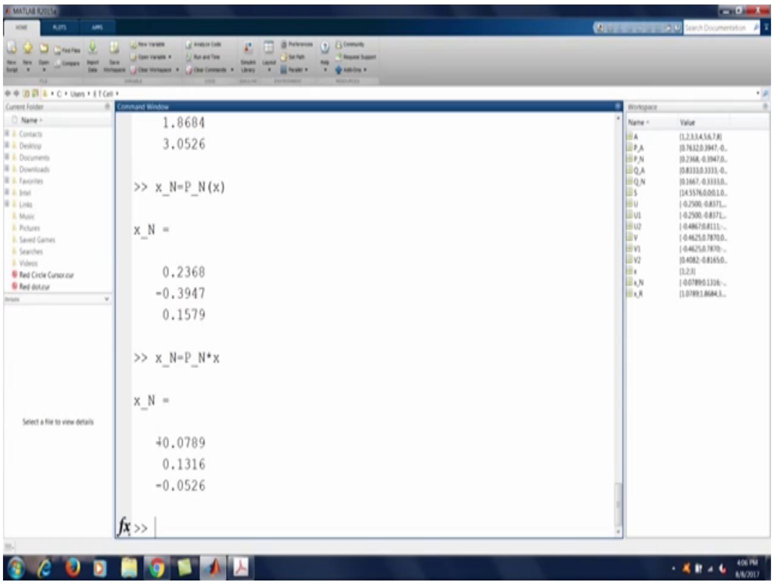

So, here there is small problem here, delete. So, here we have this. So here, look at here now it is ok? So, xN is going to be minus 0.0789, 0.1316, minus 0.0526. So, that is matching here, that xR is this, xN is this and now, if you add these two, you are getting your x here. So, it means that x is going to be uniquely represented as xR and xN, where , xR is PAx and xN is Nx. So, that is what we have done here, is we have discussed the theory part and with the help of mat lab, we have just verified our theory and we say that though your U and V may not be unique, because your orthonormal basis for range space of A or any vector subspace may not be unique, but the orthogonal projection on to the subspace S is going to be unique. So, that we have seen here.

So, what we have seen in today's this lecture, that how to form an orthogonal projection on to a vector space S with the help of orthonormal basis of that subspaces. So here, in this lecture, we have considered the important vector subspace, that is range space of A, null space of A range, space of A transverse, null space of A transverse and we have already seen that in previous lectures of S, singular value decomposition theorems that we know, what is the orthonormal basis for range space of A, it is the first R columns of U and similarly, the remaining M minus R element of U is going to be orthonormal basis of null space of a transverse. Similarly, the first R columns of V going to generate the range space of a transverse and the remaining N minus R columns generate [vocalized- will form an orthonormal basis for null space of A.

So, using this information, we have calculated the orthogonal projection which is nothing, but form a matrix. So, you have U1. So now, if you calculate U1 – U1 transverse, that is going to be orthogonal projection on to range space of A.

So similarly, in a same manner, we have find out say orthogonal projection on to range space of a null space, of a range space, of a transverse null space, of a transverse and also we have seen one example, where we utilizing the mat lab, we have shown that this actually work out and here, we want to give one more phase, that though your U and V, may not be unique, but these orthogonal projection going to be unique orthogonal projection.

So here, we stop and next class, next lecture, we will see how we utilize the singular value decomposition theorem to find out, say, other algebraic properties of matrix A. So, here, we stop thank you for listening us.

Thank you.