Numerical Linear Algebra
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Lecture - 41
Estimation of Condition Number

Hello friends. Welcome to my lecture on estimation of condition number. We know that the condition number is very useful.

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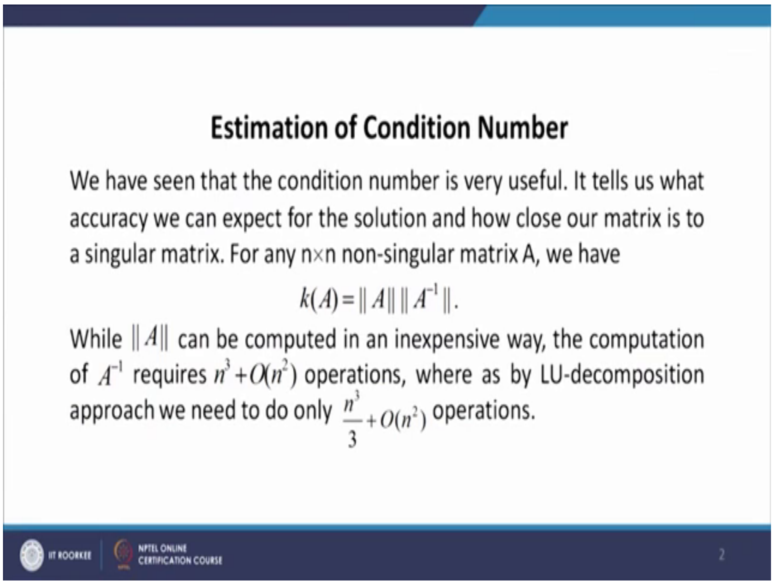

It tells us the accuracy which we can expect for the solution, and how close our matrix is to a similar matrix. And we also know that for n y n non-singular matrix A, the condition number is given by norm of A into norm of A inverse. Now while norm of A can be computed in an expensive way, the computation of A inverse requires n q plus capital order n square operations. If you apply LU-decomposition approach, it will need only n q by 3 plus capital order n square operations.

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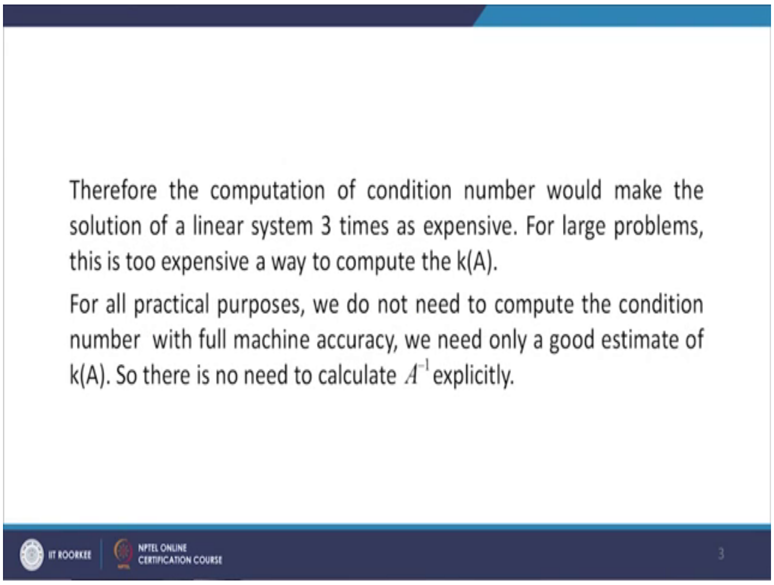

So, the computation of condition number would make the solution of a linear system 3 times as expensive. For large problems this is too expensive a way to compute the condition number. Now for all practical purposes, we do not need to compute the condition number with full machine accuracy. We only need a good estimate of the condition number. So, there is no need to determine the inverse of the matrix A explicitly. What we are going to do is, we are going to discuss a method of the estimation of norm of A inverse.

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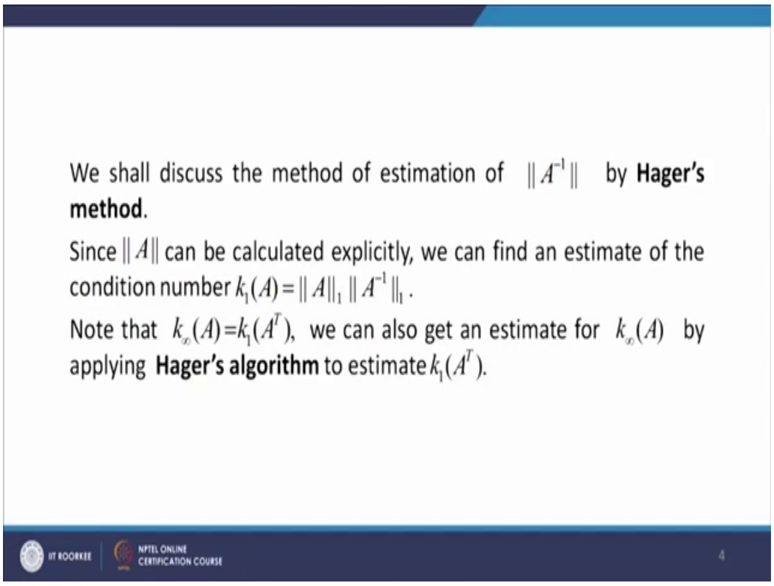

And the method which we will use is the method given by Hager's. So, Hager's method we are going to discuss. since norm of A can be calculated explicitly, once we have an estimate of norm of A inverse by Hager's method, we shall be able to estimate the condition number k A.

Now, here we shall be using the norm 1 norm. So, norm of it is condition number with one norm is equal to norm of A 1 into norm of A inverse 1. And you we know that k infinity a is equal to k 1 A transpose A transpose.

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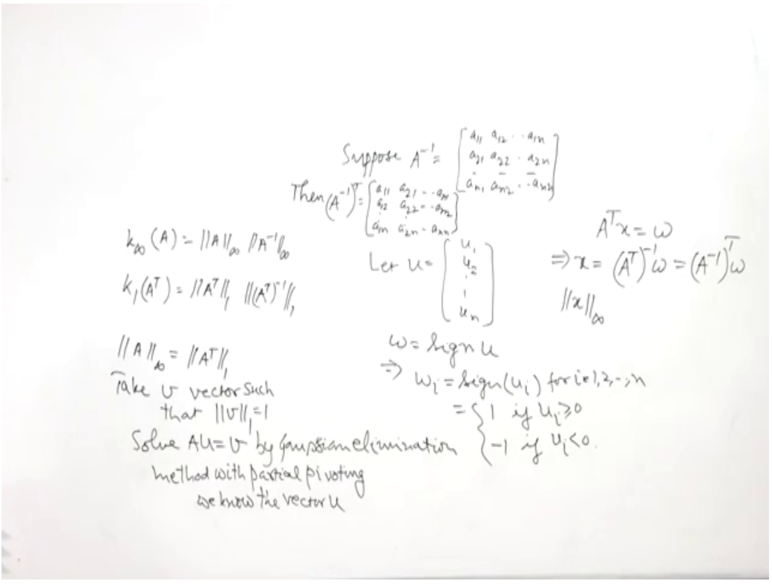

K infinity a is equal to norm of A infinity into norm of A inverse infinity, and k 1 A transpose will be equal to norm of A transpose, one norm of A inverse A transpose inverse 1 ok. And norm of it is norm of A infinity is equal to norm of A transpose 1. Because norm of A transpose one is the absolute column maximum absolute column sum, and this is maximum absolute row sum.

So, here norm of A infinity is the maximum absolute row some while when you take the transpose of the matrix A ok, the rows becomes rows becomes columns. So, norm of A transpose one is the maximum absolute column. So, sum so, they are both equal. So, once we have the once we know the norm of A inverse 1 ah, and norm of A 1 we can determine k 1 A. Hence once we have the estimate of k 1 A, we can similarly get the estimate of k infinity A, by finding the condition number of A 1 transpose in the one norm.

So, k infinity a can also be estimated by this Hager's algorithm.

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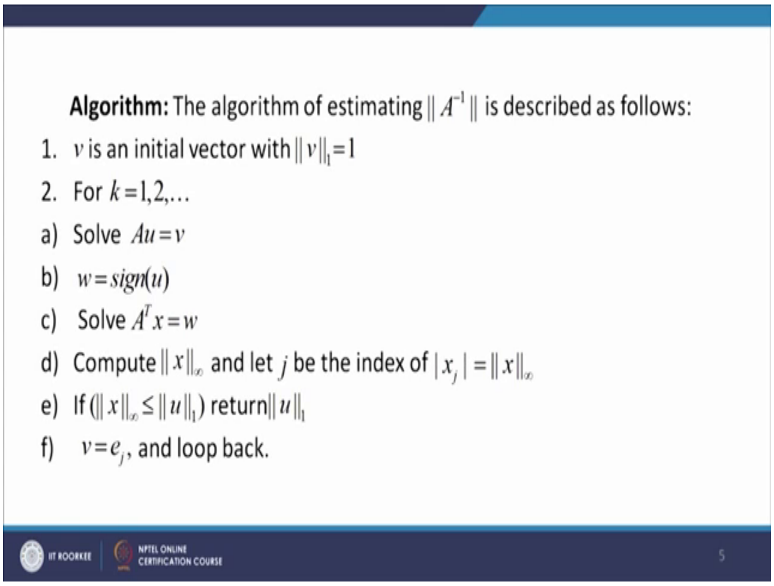

Now the algorithm goes like this, to estimate norm of A inverse, we begin with an initial vector, say v with norm of v in one norm equal to 1. So, let us take an initial vector v with norm 1 in the one norm. And then let us take k equal to 1. So, we are doing first iteration. So, k equal to 1, then solve Au equal to v. So, we what we do? We take v vector such that norm of v 1 equal to 1. And then solve the equation Au equal to v.

We can solve this equation by Gaussian elimination method with partial pivoting. So, when you solve Au equal to v, by the Gaussian elimination method with partial pivoting. We know the vector u; let us say u equal to u 1 u 2 and so on; U n, then we define w equal to sign of u, w equal to sign of u. So, this sign of u means, the r that is to say we get w i equal to sign of u i for i equal to 1 to n. So, on up to n sign of u i is equal to 1, if u i is greater than or equal to 0. And minus 1, this is defined as one if u i is greater than or equal to 0, and minus 1 if u i is strictly less than 0.

So, we get w i the value of w i corresponding to the component u i of u. So, this w i will be having values plus minus 1. Now what we do is we solve A transpose x equals to w equation. A transpose x equals to w gives what? We have w vector with us w is plus minus; where w is the vector signed of u, where components of w are plus minus plus 1 minus 1 like that. So, A transpose x we solve this equation, again by Gaussian elimination method with partial pivoting. So, this gives you x equal to A transpose inverse w ok. We can solve this equation, A transpose A inverse. Now what we do let us find the norm of x infinity norm of x infinity gives what? this is A transpose inverse let us say, I can also write this as A inverse transpose w; now suppose A inverse matrix is like this.

Say A 1 1; A 1 2 A 1 n; A 2 1 A 2 2 A 2 n and so on. A n 1, A n 2, A n n ok. Suppose A A inverse matrix is this; then A inverse transpose will be A 1 1, A 1 2, A 1 n, A 2 1, A 2 2, A 2 n and so on. A n 1, A n 2, A n n ok. When A inverse transpose is multiplied by w ok, when A inverse transpose is multiplied by w, the components the ws are plus 1 or minus 1 like that ok. So, norm of x infinity will be equal to norm of x infinity we are multiplying A inverse w inverse 1 transpose by w.

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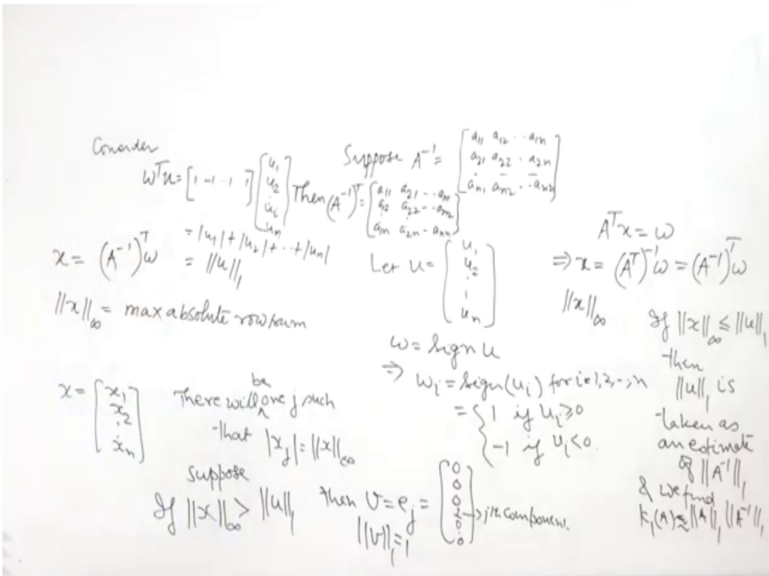

W means one or minus 1 like that. So, w is equal to x ok, this x equal to A inverse transpose the x norm of x infinity will be equal to maximum absolute maximum absolute row sum ok. This maximum absolute row sum of A inverse transpose into w.

Now, let us say since it is maximum absolute row sum, there will be one a for which suppose norm of x ah, suppose x is x 1 x 2 x n suppose x x is x 1 x 2 x n. Then norm of x infinity will be equal to maximum absolute row sum. So, there will be one a such that norm such that norm of x infinity will be equal to mod of x j. So, there will be one a such that mod of x j equal to norm of x infinity.

Now, if norm of x infinity is less than or equal to norm of u 1 ok. Suppose, and what is norm of u 1 let us see. W transpose u ok, w transpose u will be ok, now here we will have corresponding to A u i which is positive we will have w i equal to 1 and if ui is negative we will have minus 1 ok.

So, w transpose u will give you suppose this is u 1 u 2 u i u n. If i if u 1 is positive I shall write one here, if u 2 is negative I shall write minus 1 here, and so on if u i is say positive positive value I will write one here if u 1 is positive I will write one here. So, when you multiply them ok, this row vector with this column vector, what you will get you will get u 1 plus. So, mod of u 1, plus mod of u 2 and so on mod of u n that is what is. So, if u is u 1 u 2 u n, you get the norm of u 1 ok. So, w transpose u is norm of u 1. So, what happens be be compare norm of x infinity with norm of u 1 ok. Now if norm of x infinity is less than or equal to norm of u 1 ok.

Then norm of u 1 will be an estimate of the condition number. We written norm of u 1 as the value of the condition number, if norm of x infinity is strictly greater than, it is greater than norm of u 1 ok, norm of x infinity is greater than is strictly greater than norm of u 1, then we will take v equal to e j, e j is the vector belonging to r r to the power n whose j th component is one other component are 0. So, this is equal to 0, 0, 0 and so on 1, 0, 0, 0. And this is the j th component. And this j is the same value for which mod of x j becomes norm of x infinity ok. The component of x such that, mod of x j becomes equal to norm of x infinity.

So, we will take with we we equal to e j, when you take v equal to e j norm of v becomes norm of v 1 becomes 1, and we again go back, and start the process with k equal to 2. The first time it is k equal to 1, when we go back k will be 2. Again, we will find Au equal to v with this vector v ok. We will find u, and then we will find w equal to sign u, we will solve the equation A transpose x equal to w, again compute norm of x infinity, and find the value of j such that mod of x j equal to norm of x infinity if norm of x infinity is less than or equal to norm of u 1. Then norm of u 1 will be the condition number estimate of the estimate of norm of A inverse 1.

So, if norm of x infinity less than or equal to norm of u 1 ok. Then norm of u 1 is taken as an estimate of norm of A inverse 1 ok. And we find k 1 A equal to norm of A 1 into norm of A inverse 1, an approximate value of the condition number. So, as so, long as norm of x infinity is greater than norm of u 1, we we look back ok. Now let us see let us explain this algorithm further. So, in the Hager's algorithm what we are doing we are initially picking an arbitrary vector with norm v 1 equal to 1.

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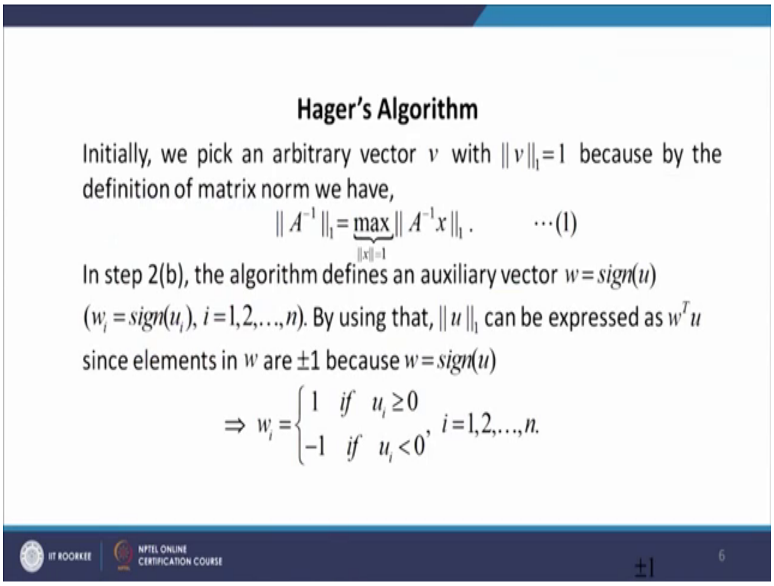

Why we are doing this? Because norm of A inverse 1 is equal to maximum of norm of A inverse x 1 with norm of x 1 equal to 1. So, see we this is the definition of matrix now by definition of matrix now A inverse 1 is equal to maximum of norm of A inverse x 1 over all vectors x, such that norm of x 1 equal to 1.

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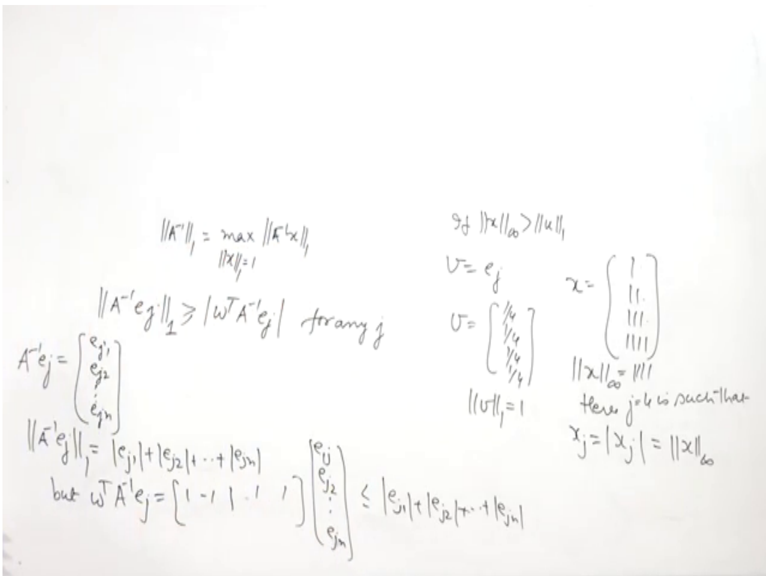

So, we begin with a vector v e, such that norm of v 1 equal to 1. And then we shall try to find the maximum of norm of A inverse x 1 ok. So, that we get an estimate of norm of A inverse 1. Now in this step 2 b. So, let us take k equal to 1 ok. Then in the step b in the in the step 2 a b solve the equation Au equal to v for the value of u. In the step 2 b what we do? In the step 2 b we define an auxiliary vector w equal to sign u. So, that w i as I said w i equal to sign u of u i i equal to 1 2 and so on up to n, by using that as I had told you norm of u 1 equal to w transpose u.

Because the w elements in w are plus minus 1, because w equal to sign u means w i equal to 1 if u i is greater than or equal to 0 minus 1, if u i is less than 0 for i equal to 1 2 3 and so on up to n. So, w transpose u is nothing but norm of u 1.

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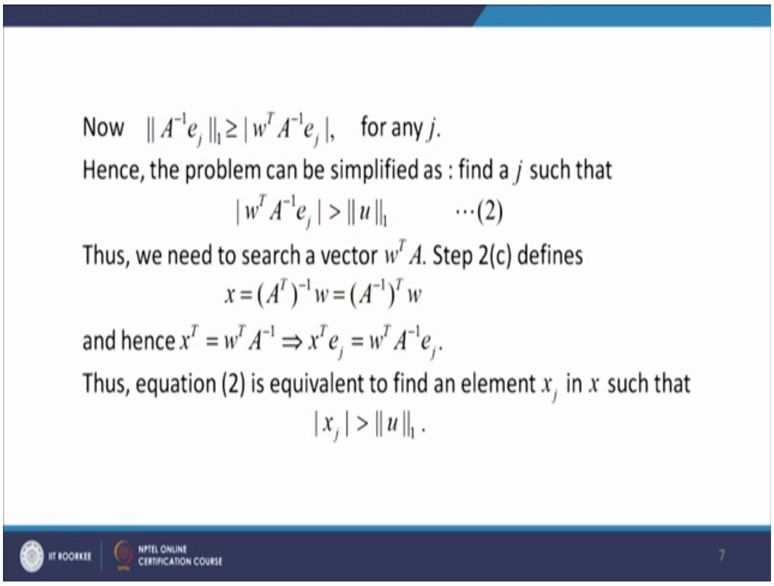

Now, norm of A inverse e j 1 is strictly greater than or equal to norm of A inverse e j 1 is greater than or equal to w mod of w transpose, w transpose A inverse e j e j for any j. Why because w transpose the components of w transpose are 1 minus 1 plus minus 1, and you are multiplying that by A inverse e j ok.

So, what you are doing actually you are you are A inverse e j is a column vector ok. When you multiply A inverse e j by a multiply to this row vector, then what you get is you get some value with that value will always be less than or equal to this ok. Because here you are multiplying say, say suppose A inverse e j is this. Say I write e j 1 e j 2 and so on, e j n ok. As such norm of if you calculate norm of A inverse e j is this equal to modulus of e j 1 plus modulus of e j 2 and so on modulus of e j n ok. It will be like this. But if you multiply w transpose, but w transpose A inverse e j ok. This will be here you will have 1 minus 1 1 minus 1 like this ok. Or even you will have so, components are 1 minus 1 like that, you are multiplying them by e 1 j e 2 j e n j e j n.

If these components are the same sign as the sign of suppose, you take e j I here, e j I has the same sign as the sign of I th component here. Then only you will get mod of e j 1 mod of e j 2 mod of e j n. Otherwise the value when you multiply this this row vector by this column vector will always be less than or equal to this ok. So, this is always less than or equal to norm of modulus of e 1 j e j 1 plus mod of e j 2 and so on mod of e j n is always than this. So, that is why we write norm of A inverse e j is always greater than or equal to modulus of w transpose A inverse e j for n a hence, the problem can be simplified as, we have to find a j such that mod of w transpose A inverse e j is strictly greater than norm of u 1.

And so, we need to search a vector w transpose A step 2 c defines c, step 2 c step 2 c defines A transpose x equal to w ok. And what it does? So, x equal to A transpose inverse w, or it is equal to A inverse transpose w, and hence x transpose equal to w transpose A inverse ok. And x transpose e j equal to w transpose A inverse e j. Thus, equation 2 is equivalent to find an element x j in x see equation 2 this equation.

This equation 2, mod of w transpose A inverse e j greater than greater than norm of greater than norm of u 1 is equivalent to finding an element x j, in x such that mod of x j is greater than norm of u 1 ok. Because you can see x transpose is equal to w transpose A inverse e j vector ok.

So, we this is equivalent to finding an element x j in x such that modulus of x j is greater than norm of u 1.

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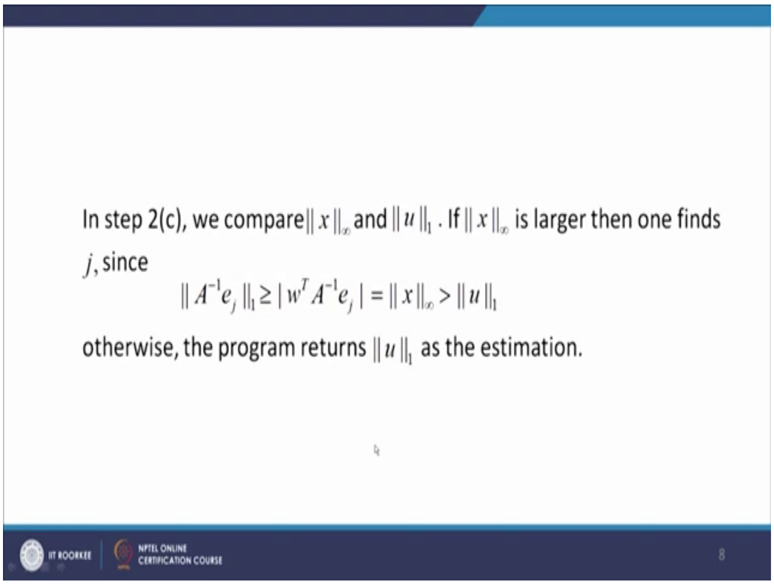

Now, in step 2 c we compare norm of x infinity, and norm of u 1, if norm of x infinity is larger than larger, then one finds j ah, such that mod of x j is equal to norm of x infinity. And since norm of A inverse e j 1 is greater than or equal to modulus of f w transpose A inverse e j, which is equal to norm of x infinity. We have just now seen see norm of x infinity is equal to this one, norm of x infinity equal to modulus of x w transpose A inverse e j for every j.

So, norm of A inverse e j 1 is greater than or equal to this ok. And this is equal to norm of x infinity and norm of x infinity is greater than norm of u 1. Now if norm of x t infinity is greater than norm of u 1 ok, then we will take v v equal to; as I said v equal to e j. If norm of x infinity is greater than norm of u 1, we take v equal to e j where j th component is one other components are 0. And then norm of v will be equal to 1 and v, again look back if norm of x infinity is less than or equal to norm of u 1, then norm of u 1 is written as the estimate of norm of A inverse 1. And we find the approximate value and approximate value of the condition number.

So, this is algorithm.

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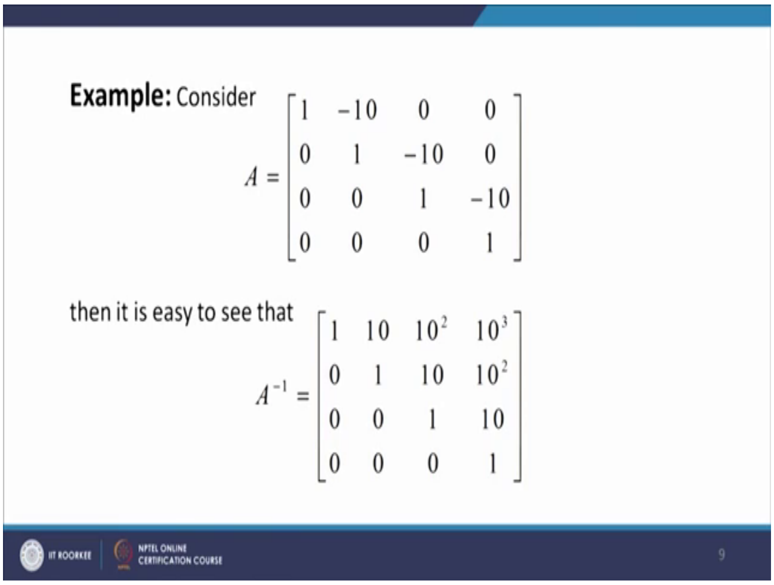

Let us see how we apply this algorithm to solve problems, suppose we take this matrix A equal to a is 1 minus 10, 0, 0, 0, 1 minus 10, 0, 0 and then 0 0 1 minus 10, 0, 0, 0, 1. So, this is 4 by 4 matrix, now for this matrix it is not difficult you can easily find A inverse. And A inverse is 1 10 10 square 10 cube 0, 1, 10, 10 square 0, 0, 1, 10. And 0 0 0 1.

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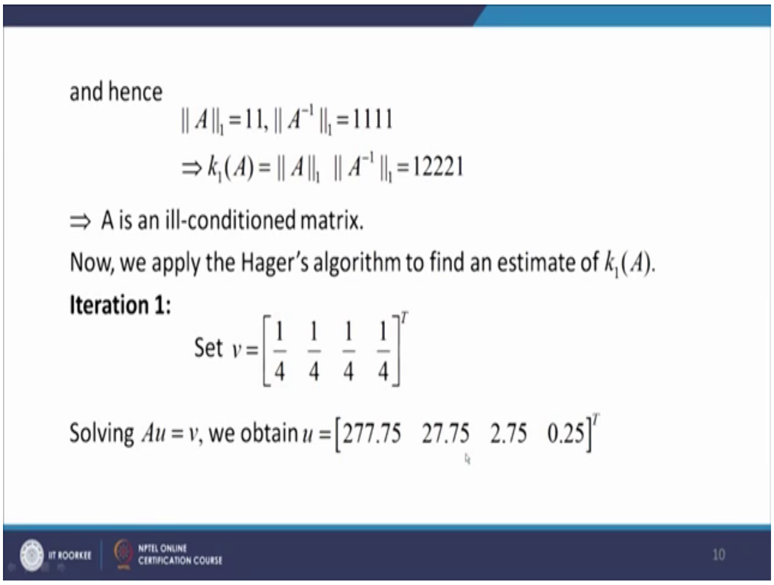

Now so, we can easily see that, norm of A 1 norm of A 1 is 11 norm of A 1 means maximum absolute column sum. So, maximum absolute column sum you can see in the matrix A first column has sum 1, and the second column has absolute sum 1 11 third column has absolute sum 11. 4th column has absolute sum 11. So, maximum absolute column sum is equal to 11. And therefore, norm of A 1 equal to 11. While in the case of A inverse matrix. If we find norm of A inverse 1, then what do we notice here you can see the absolute column sum. Maximum absolute column sum will come from the last column ok. So, 10 cube is 1000; 1000 plus 100, that is 11 100 than 11 10 then 1. So, 11 11. So, we have 1 1 1 1. So, that is norm of A inverse 1. And so, condition number of A is equal to norm of A 1 into norm of A inverse 1; which is one double 2 2 1 ok, which is quite high.

So, we can say that a is an ill conditioned matrix. Now let us apply the Hager's algorithm to find an estimate of the condition number of A in 1 norm ok. So, as we said in the algorithm, we start with a vector ah, say v whose norm is equal to 1. So, let us take the vector v to b 1 by 4 1 by 4 1 by 4 1 by 4, because we are start we have taken the 4 by 4 matrix ok. So, if it if we take a to b n by n matrix, we will take v equal to 1 by n 1 by n 1 by n like that. Since we are taking 4 by 4 matrix we take the vector v to be 1 by 4, 1 by 4, 1 by 4, 1 by 4 transpose of that. So, we get a column vector. And the column sum then is 1 by 4 plus 1 by 4 plus 1 by 4 plus 1 by 4 equal to 1. So, norm of v 1 is equal to 1.

So, norm of v 1 is equal to 1. Now we solve the equation Au equal to v. And it turns out that u comes out to be 277.75, 27.75, 2.75, 0.25. Then you can see, in the in the vector u all the components are positive ok, positive real numbers.

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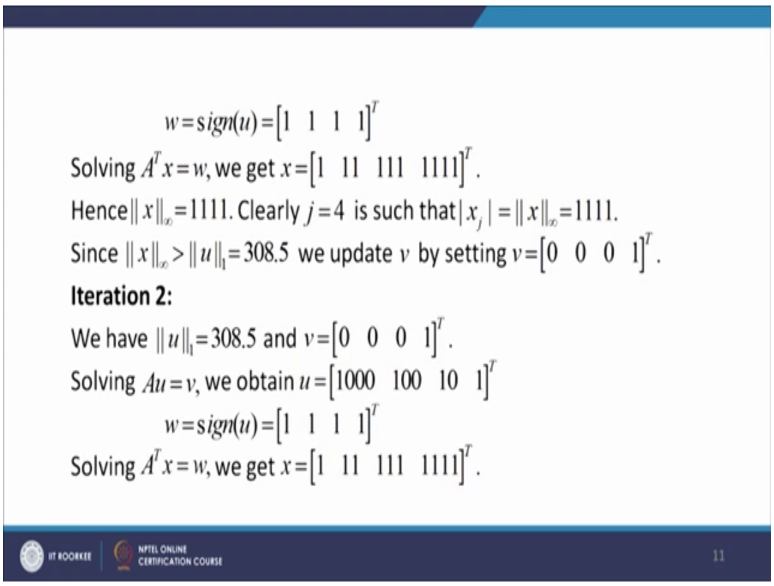

So, when we find w equal to sign u, we have said that sign u is sign u i is equal to one if u i is greater than or equal to 0, and minus 1 if u i is less than 0. So, still here u is are all positive. So, w equal to sign u will be 1 1, 1 1 and we are taking transpose. Now let us solve the equation A transpose x equal to w.

When we solve this equation A transpose x equal to w, it turns out that x is equal to 1 11, 100 11 and then 1 1 1 1 transpose ok. So, here we can see. Norm of x infinity norm of x infinity is the maximum absolute row sum ok. So, first row is of the vector x first row is one second row is 11 third row is 1 1 1 and 4th row is 1 1 1 1. So, norm of x infinity is 1 1 1 1. And further more we have to find the value of j such that mod of x j is equal to norm of x infinity.

So, since norm so, x is equal to 1 11, 1 1, 1 1, 1 1 1 ok. So, norm of x infinity is equal to 1 1, 1 1 and this j equal to 4 ok. J is equal to 4 is the component of x x, x such that mod of x j equal to. So, here j j is equal to 4 is such that mod of x j ok. This x 1 x 2 x 3 x 4 ok. So, mod of x j is equal to norm of x infinity. Here all the components are positive. So, I can write x j. So, x 4 x 4 is norm of x infinity ok. Now what we have? What we notice that norm of x infinity is 1 1, 1 1. While norm of u 1, norm of u 1 is how much? If you find norm of u 1, u is this vector 277.75, 27.75, 2.75, 0.25. If you add all of them norm of u 1 means maximum absolute column sum.

So, you add all of them. What we get is 308.5. So, norm of x infinity is is strictly greater than 308.5. And therefore, we update v by taking v equal to e j vector. E j means e 4 vector. E 4 is equal to 0, 0, 0, 1 ok. 4th component will be one other component will be 0. So, v is now new vector 0, 0, 0. 1 and we do do the iteration 2 ok, k equal to 2.

So now we start with this vector v ok, for this vector v ok. We solve again. And this time we have norm of u 1 equal to 308.5. We have with us solving Au equal to v now what we do ok. We solve the equation Au equal to do not confuse this Au equal to v with the u we have earlier ok. So, this is new new u vector ok.

So, solve Au equal to v; v is 0 0 0 1 column vector. And we get u equal to 1 triple 0 1 double 0 1 0 10 that is 10 and one column vector and again all the components of u are positive. So, we will get w equal to 1 1 1 1 transpose. Now we solve then the equation A transpose x equal to w, x comes out to be the column vector 1 11 100 11 and then 1 1 1 1 transpose ok.

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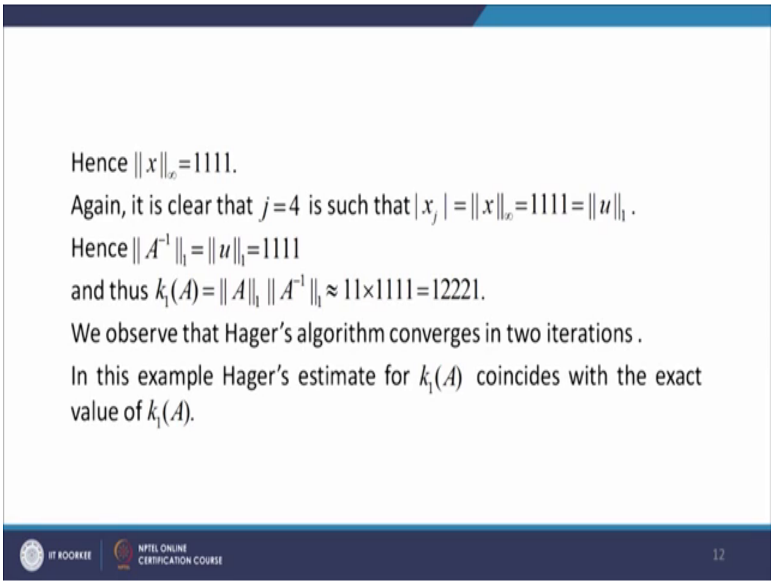

So, now we can again find the norm x infinity. Norm of x infinity is 1 1 1 1, and v is again see that for j equal to 4, mod of x j is equal to norm of x infinity. And norm of x infinity is 1 1 1 1. And norm of u 1, let us see norm of u 1 this time. So, this is our new u, this our new u 1 thousand 100 10 1. And you can add all of them, you get 1 1 1 1. So, norm of new u new vector u has norm 11, 11.

So, norm of x infinity is equal to norm of u 1. And therefore, norm of u 1 is written as an estimate of norm of A inverse 1 ok. So, norm of A inverse 1 is equal to norm of u 1 is equal to 1 1 1 1. And thus an estimate of k 1 A is equal to 11, 11 is norm of A 1 into norm of A inverse 1, that is 1 1 1 1.

So, one double 2 2 1. So, here when we apply the Hager's algorithm, we find that the Hager's algorithm converges in just 2 iterations. And it it is value the estimate that we get of the condition number coincides with the exact value of the condition number in one norm in just 2 iterations ok.

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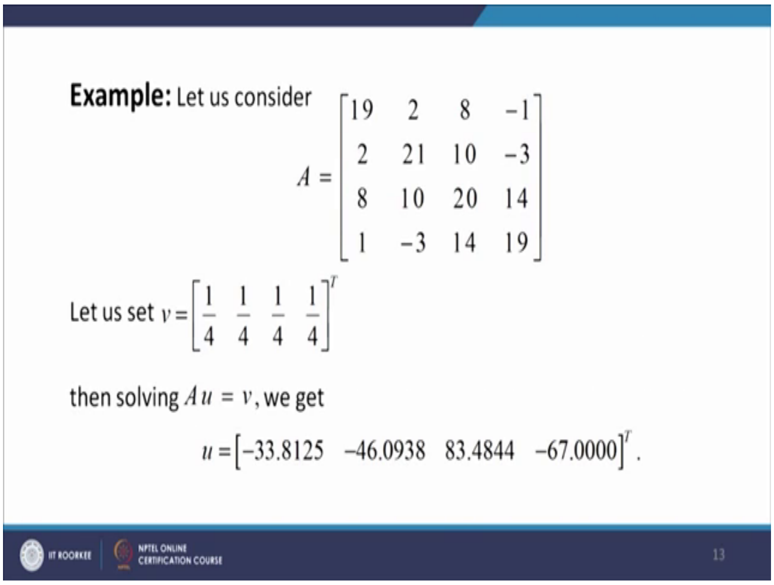

Let us take one more example. So, let us take this time again 4 by 4 matrix A with the entries 1928 minus 1 221 10 minus 3 8 10 20 14 1 minus 3 14 19. Again, we take v to be a vector with norm 1. So, since we are having 4 by 4 matrix, we take the vector v to be 1 by 4, 1 by 4, 1 by 4, 1 by 4. Then we solve the matrix equation Au equal to v. We get u equal to minus 33.8125 minus 46.0938 83.4844 minus 67.0000.

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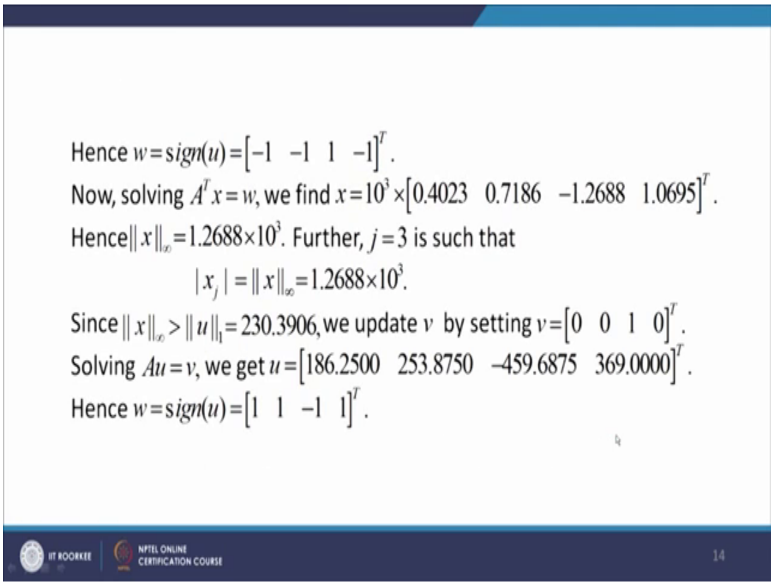

And here we can see some components are positive some components are negative. So, first component is negative in u, and therefore, when you write w equal to sign u ok.

Since u the component u 1 is negative the component of w. That is w one will be minus 1 corresponding to the second component u 2 which is negative the component w 2 will be minus 1. This is positive. So, w 3 will be plus 1, and this is negative. So, w 4 will be minus 1. So, you can see w equal to sign u is minus 1 minus 1 1 and minus 1. Now we solve the equation A transpose x equal to w. And we find x equal to 10 to the power 3 into 0.4023 0.7186 minus 1.2688 1.0695 transpose.

And so, we can see norm of x infinity, norm of x infinity is again the maximum absolute row sum. So, when you multiply 10 to the power 3 here. And the maximum absolute value is 1.2688 into 10 to the power 3. So, we get 1.2688 into 10 to the power 3. Again, we can see that the third component ok. The third component is such that, it is absolute value equals norm of x infinity ok.

So, modulus of x j is equal to norm of x infinity which is equal to 1.2688 into 10 to the power 3. Now norm of x infinity here is 1.2688 into 10 to the power 3, while norm of u 1 norm of u 1 we can find. So, we can take the absolute sum of all these values ok. because norm of u 1 is absolute column sum absolute. So, so, we can take the absolute values of all these numbers and take their sum. And it turns out to be 230.3906.

Now, you can see that 1.2688 into 10 to the power 3 is much greater than 230.3906. So, we update v by setting v equal to e j. E j means e 3 because j was equal to 3 when mod of x j became equal to norm of x infinity.

So, v is equal to 0, 0, 1, 0. We solve Au equal to v, and you get u equal to 183.2500 253.8750 minus 459.86875, 369.0000 transpose ok. Now first component here is positive. So, w equal to sign u is equal to 1. First component is one second component is 1. Because this is positive, third component is minus 1 here. Because this is negative. And 4th component is one because this is positive 369.0000.

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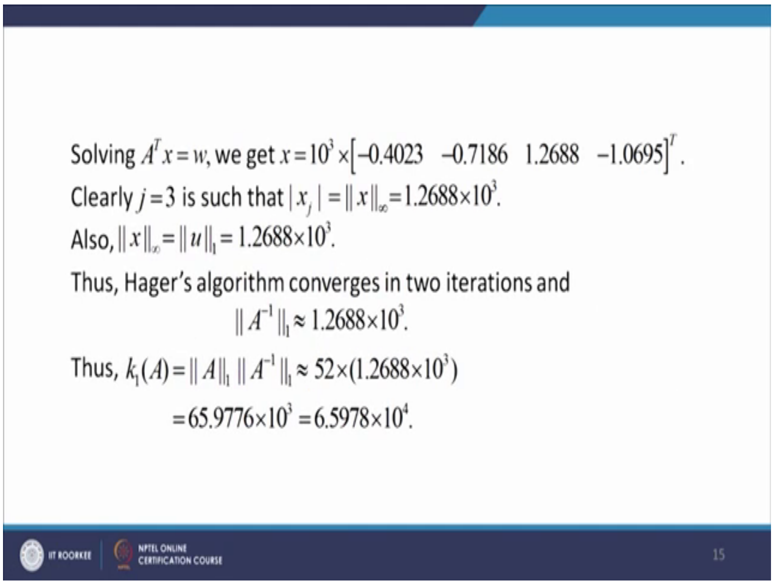

So, we get the vector w like this. And then we solve the equation, A transpose x equal to w ok. We shall get x equal to 10 to the power 3 into minus 0.4023 minus 0.7186 1.2688 minus 1.0695. Now let us see which value here. The absolute value is is is the greatest. So, you can see here 1.2688 ok, 1.2688 is the greatest value multiplied by 10 to the power 3. So, clearly a equal to 3 such that mod of x j is equal to norm of x infinity is equal to 1.2688 into 10 to the power 3. Now norm of u 1, let us find norm of u 1. Here you can consider the absolute values of all these 4 components of u, and add them.

What we get is 1.2688 into 10 to the power 3. So, norm of x infinity equals norm of u 1. So, norm of u 1 is written as the as an estimate of norm of A inverse 1 ok. So, norm of A inverse 1 is approximately taken as 1.2688 into 10 to the power 3. So, an estimate of the condition number k 1. A is norm of A 1 norm of A 1 you can see you can look at the matrix A norm of A 1 here. The first column has sum 19 plus 221 plus 829 plus 130. So, first column has sum 30 absolute sum of absolute values in the first column in the second column 2 plus 21 20 23 then ten. So, 30 3 plus 3 absolute value of minus 3 is 3. So, we get 36 ok.

And here 8 plus 10 18 18 plus 20 38 38 plus 14 is 52 ok. So, and here what we get? 1 plus 3 4 4 plus 14 18 plus 19 37. So, 52 is the maximum maximum absolute column sum. So, norm of A 1 is equal to 52. And we multiply it by an estimate of the norm of A inverse 1. So, 1.2688 into 10 to the power 3. And this is equal to 35.97 76 into 10 to the power 3; which is 6.5978 into 10 to the power 4; which is quite high number ok. Now so, we can say, that this matrix is also an ill conditioned matrix. With that I would like to conclude my lecture.

Thank you very much for your attention.