Numerical Linear Algebra
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Lecture - 40
Nearness to Singularity

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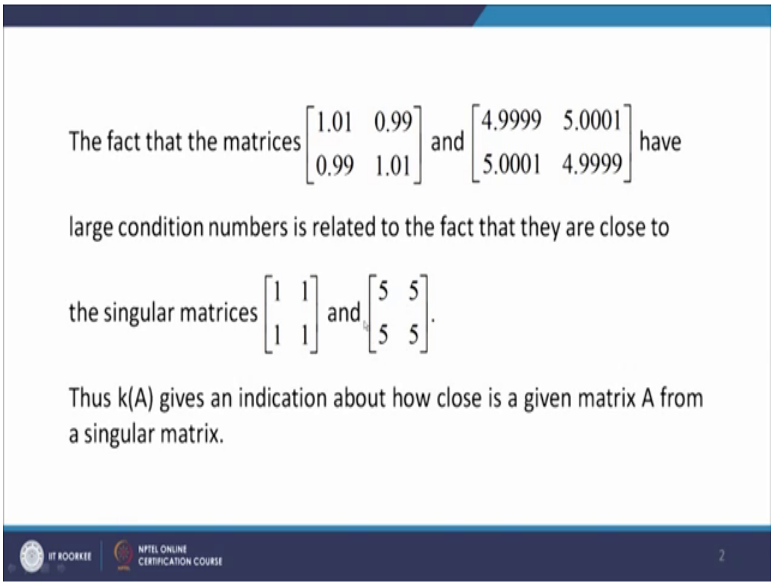

Hello friends, I welcome you to my lecture on nearness to singularity. We have seen that the matrices say 1.01 0.99 0.99 and 1.01, and the other 2 by 2 matrix 4.9999 5.0001 5.0001, 4.9999 have large condition numbers. In our previous lecture, we have seen this that they have large condition numbers; and this they have large condition numbers is related to the fact that they are close to the singular matrices.

You can see 1 1 1 1, this first matrix 1.01 0.99 0.99 0.99 1.01 is close to the matrix 1 1 1 1; and the other matrix 4.9999 5.0001, this matrix is close to the matrix 5 5 5 5 which is also a singular matrix. So, this condition number gives an indication about how close is a given matrix given non-singular matrix A from a singular matrix. We are going to find a relation between the condition number of a matrix and the closeness of the given non-singular matrix from a singular matrix.

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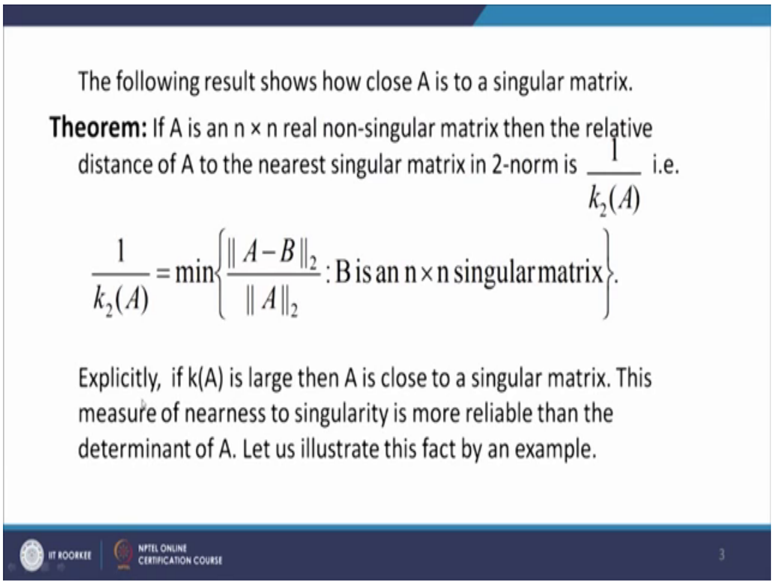

So, the following theorem, the following theorem tells us about this closes if A is an n by n real non-singular matrix then the relative distance of A to the nearest singular matrix in 2-norm is 1 by k 2 A. That is 1 by k 2 A equal to minimum of norm of A minus B 2 over norm of A 2 such that B is an n by n singular matrix. So, from here, you can see if k is large, we have taken here 2, we have prove this formula for 2-norm, it can be proved for other norms also.

So, here we have taken 2-norm. So, we can see that the condition number, if the condition number is large, then A is then 1 by k 2 A is very close to 0. So, if k is very large, then A is close to a similar matrix B. This major of nearness to singularity is more reliable than the determinant of A. Let us illustrate this fact by an example.

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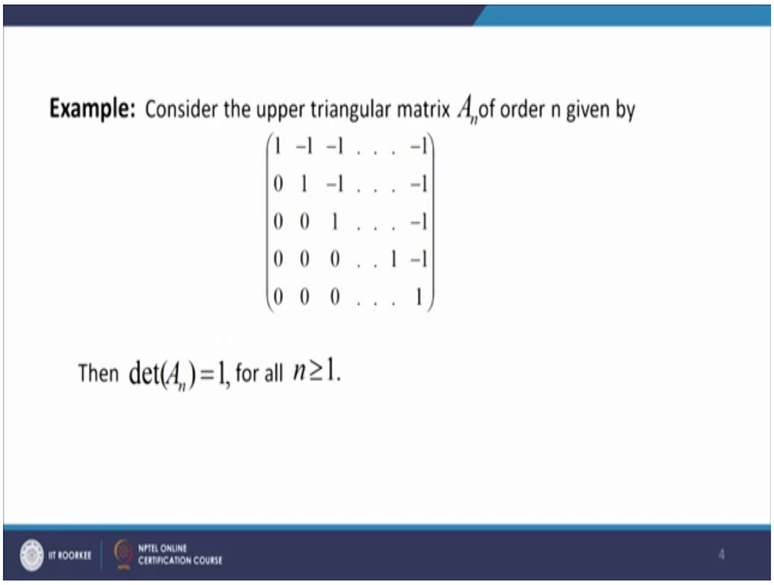

Let us consider this upper triangular matrix A n this upper triangular matrix A n of order n which is given by 1 minus 1 minus 1 and so on minus 1, 0 1 minus 1 and so on minus 1, 0 0 1 and so on minus 1, 0 0 0 and so on 1 and then minus 1, and the last row we have all 0s at the end we have 1. Now, you can see since the diagonal elements here are 1 each and the matrix is upper triangular.

The determinant of the matrix is the product of the Eigen product of the diagonal entries. So, determinant of A n is equal to 1 for all values of n. You can take n equal to 1, 2, 3 and so on. So, for all values of n determinant of A n is the product of the diagonal entries which is 1.

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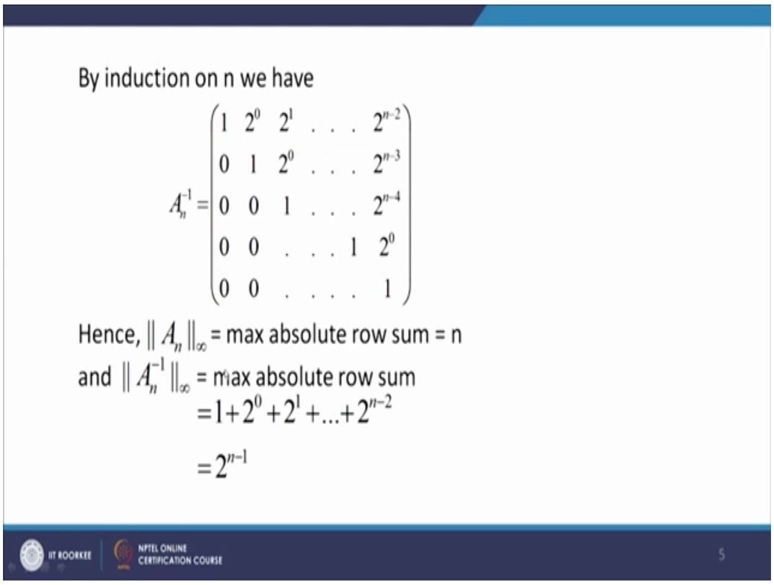

Now, by induction on n, we can easily see that when A n is equal to this then A n inverse A n inverse is given by 1 2 to the power 0 2 to the power 1 2 the power n minus 2. Then the next row is 0 1 2 to the power 0 and so on 2 to the power n minus 3. Third row third row will be 0 0 1 and so on 2 to the power n minus 4, last but 1 row will be 0 0 0 and so on 1 and then 2 to the power 0. And in the last row we shall have 0 0 0 and so on 0 and at the end we will have 1. You can easily verify this for say let us take A 2 by 2 matrix.

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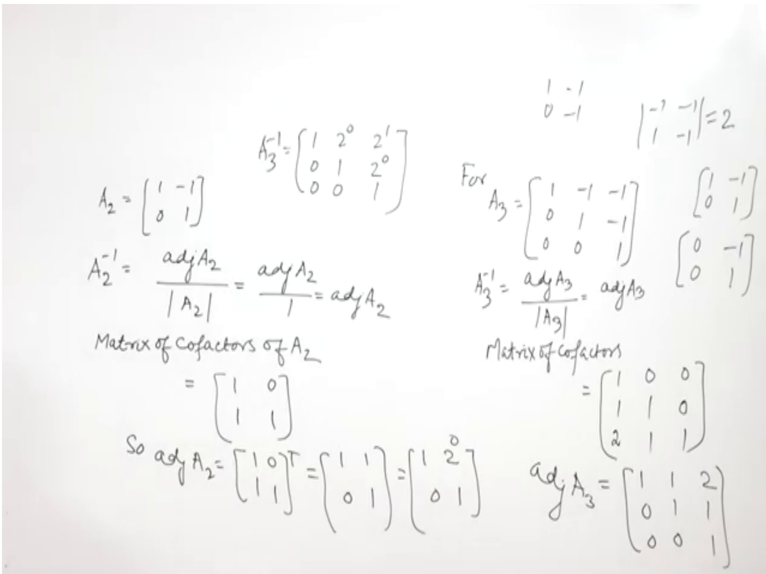

So, when you take a 2 by 2 matrix you have ah A 2 equal to 1 minus 1 and then 0 1. So, when you want to find the A 2 inverse, A 2 inverse is equal to ah adjoint of A 2 divided by determinant of A 2. You can see determinant of A 2 is equal to 1, in fact, we have shown the determinant of A n is equal to 1 for all n greater than or equal to 1. So, this is adjoint of A 2. Now, adjoint of A 2 is the transpose of the matrix of cofactors of A 2 ok. So, let us write the matrix of cofactors of A 2. If you write the cofactors of A 2 then cofactor of 1 is 1; cofactor of minus 1 is 0 into minus 1, so we have 0; cofactor of 0 is plus plus 1; and cofactor of 1 is plus 1. Then we take the transpose of this matrix.

So, transpose of this matrix of so adjoint of A 2 is equal to 1 0 1 1 transpose which we can write as 1 1 0 1. And we can write this as 1 2 to the power 0 0 1. So, you can see the A n inverse when you take n equal to 2, A 2 inverse will be 1 2 to the power 0 0 1. If you take n equal to 3, if you take n equal to 3, you will get this matrix 1. For n equal to 3, A 3 will be 0 sorry 0 3 will be 1 minus 1 minus 1, then we shall have 0 then 1 minus 1 then we shall have 0 0 1. So, here again A 3 inverse is equal to adjoint of A 3 divided by determinant of A 3. And determinant of A 3 is equal to 1, so we get adjoint of A 3.

Adjoint of A 3 let us write the matrix of cofactors here. Cofactor of this 1 is 1 minus 1 0 1 1 minus 1 0 1, 1 minus 1 0 1, which will be equal to when we find determinant of this is 1. So, this is 1. And then you can find cofactor of minus 1 that minus 1 will be you remove this column this row. So, 0 minus 1 0 minus 1 0 1 this is minor. So, 0 into 1, it is 0 with minus sign, so we get 0. And when we take minus 1 we remove this column this row 0 1 0 0. So, again 0 we get. When we take this 0, we delete this row and this column. So, we get minus 1 minus 1 0 1. So, with this is minus 1 and this is at the second first row second first column. So, we associate minus sign with that. So, we get plus 1.

And then when we take 1 here, we delete this column this row. So, we get 1 minus 1 0 1 whose product is 1, so we get 1 here. And when we have minus 1 here we delete this row and this column. So, we have 1 minus 1 0 0 and therefore, we get 0. And then we come here for this cofactor will be we delete this row, this column, and when we multiply this here, so we get 1 here, and then we get determinant of minus 1 minus 1 and 1 minus 1. So, we get determine this is 1 and then we get what will be the determinant. So, this determinant will be 2, 1 and then plus 1, so 2 we will get, this is the cofactor of this is first column third row. So, we get 2 here. And then the cofactor of this we will be getting, so this we delete this row this column and this row, so 1 minus 1 0 minus 1. So, we get 1 0 1 minus 1 0 minus 1 ok.

So, we get minus 1 here minus 1 and this is in the second column third row. So, we associate minus sign with that and we write 1 here and lastly we have cofactor of one. So, we delete this column this row and then we have 1 minus 1 0 1. So, we get 1 here. So, this is what we get, this is the matrix of vectors. And then adjoint matrix will be adjoint of A 3, adjoint of A 3 will be transpose of this matrix.

So, we have 1 0 0 then we have 1 1 1 2 1 1 0 and then we have 2 1 1. Now, we can this is A 3 inverse adjoint of A 3 is A 3 inverse. So, A 3 inverse is equal to 1 2 to the power 0 2 to the power 1 then 0 1 2 to the power 0 then 0 0 1. And you can see for n equal to 3, the A 3 inverse is 1 2 to the power 0 2 to the power 1 then 0 1 2 to the power 0 then 0 0 1. So, we can by induction we can write this form of A n inverse. And now let us find norm of A n infinity.

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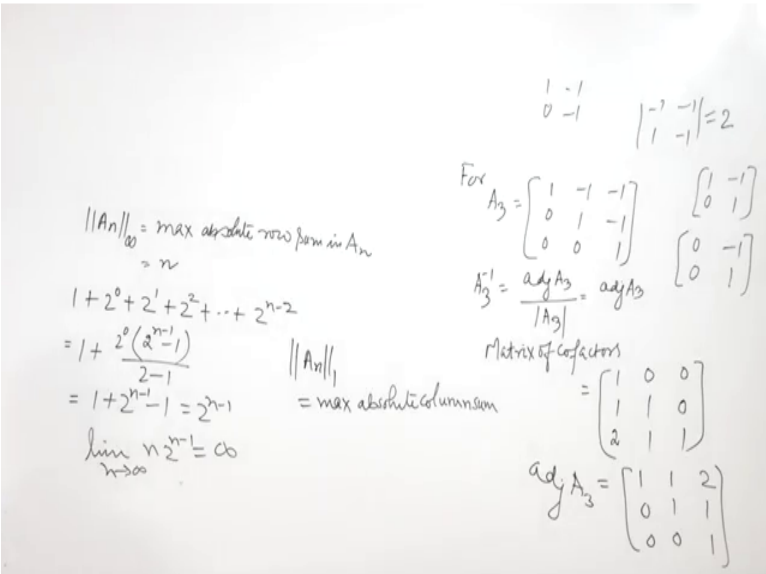

So, norm of A n infinity. This is maximum of absolute row sum in the matrix A n. Let us see what is A n and find the maximum absolute row sum there. So, here you can see in the first row, the entries when you take the absolute values of the entries, they are all 1 1 1 1 and they are n in number. So, we get their total absolute sum equal to n, and then in the second row the sum becomes n minus 1 in the third row sum becomes n minus 2 and in the at the end the last row the sum is 1. So, maximum absolute row sum is equal to n.

Now, norm of A n inverse infinity. Again norm of A n inverse infinity will be maximum absolute row sum and the first row again we have 1 2 to the power 0 2 to the power 1 to the power n minus 2. And then when we go to the second row first element becomes 0 then we have 1 to the power 0 2 to the power 1 and so on 2 to the power n minus 3. So, sum goes on decreasing therefore, the maximum absolute row sum we shall get when we considered the first row. So, 1 plus 2 to the power 0 plus 2 to the power 1 and so on 2 to the power n minus 2. And you can see that this is a 2 to the power 0 plus 2 to power 1 and so on 2 to the power n minus 2 is a geometric series.

So, we have 1 plus 2 to the power 0 2 to the power 1 is equal to 1 plus this 2 to the power 0 is 1. So, 1 and then we have 2 then we 2 square. So, we have A, A is equal to 2 to the power 0, and then we have how many terms are here we have n minus 1 terms. So, 2 to the power n minus 1 minus 1 divided by 2 to the power divided by 2 minus 1. Let us apply the formula for the sum of a geometric series, here ratio is 2. So, this is 1 plus 2 to the power n minus 1 minus 1 which is 2 to the power n minus 1. So, we get the maximum absolute row sum for the matrix A n inverse.

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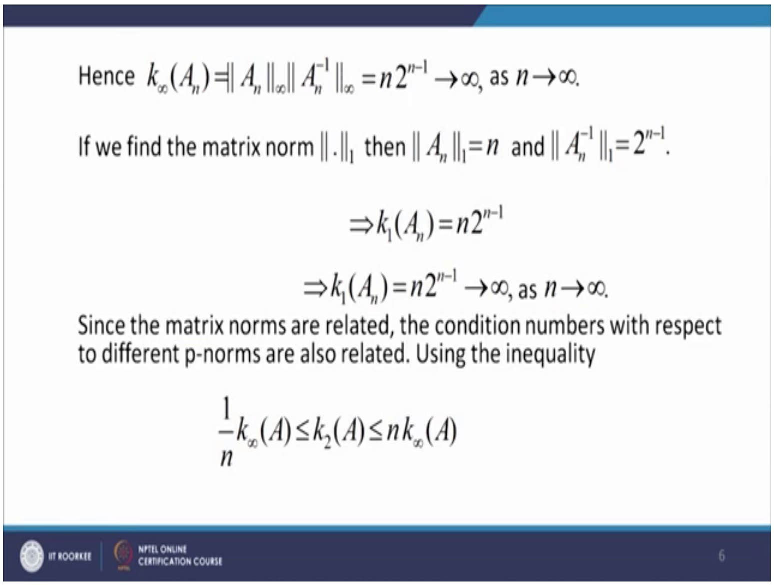

Now hence k infinity A n that is the condition number of A n in the infinity norm which is equal to norm of A n infinity into norm of A n infinity in A n inverse infinity is equal to n into 2 to the power n minus 1. And we can see that l limit n tends to infinity n goes to infinity 2 to the power n also goes to infinity. So, so as n goes to infinity and 2 to the power n minus 1 goes to infinity that is the infinity condition norm condition number of A n in the infinity norm as n goes to infinity goes to infinity. If we find the matrix norm in the 1 norm, then we can see norm of A n 1 norm of this is maximum absolute row column sum, maximum absolute column sum. So, if you see the matrix A n, then you can see the last column gives the maximum absolute column sum because the absolute values of all the entries will be 1 each. So, 1 plus 1 plus 1 and they are n in number. So, it will be n.

So, maximum absolute column sum is equal to n and in the case of A n inverse in the case of A n inverse, the maximum absolute column sum we shall get from the last column. So, we have 1 plus 2 to the power 0, and then 2 to the power n minus 4, this is 2 to power n minus 2 to the power n minus 3 2 to the power n minus 4 and so on, so 2 the power 0 plus and 1. So, this we have already calculated here has 2 to the power n minus 1. So, norm of A n inverse in the 1 norm is 2 to the power n minus 1, and therefore, the condition number of A n in norm in 1 in 1 norm is equal to an into 2 to the power n minus 1. And so again when n goes to infinity condition number of A n goes to infinity.

Now, we know that the matrix norms are related. So, the condition numbers with respect to different three norms are also related. If we can use this inequality to get the condition number in 2-norm; 1 by n k infinity A less than or equal to k to A less than or equal to n times k infinity A; If we know the condition number of A in the infinity norm, we can find the condition number of A in 2-norm.

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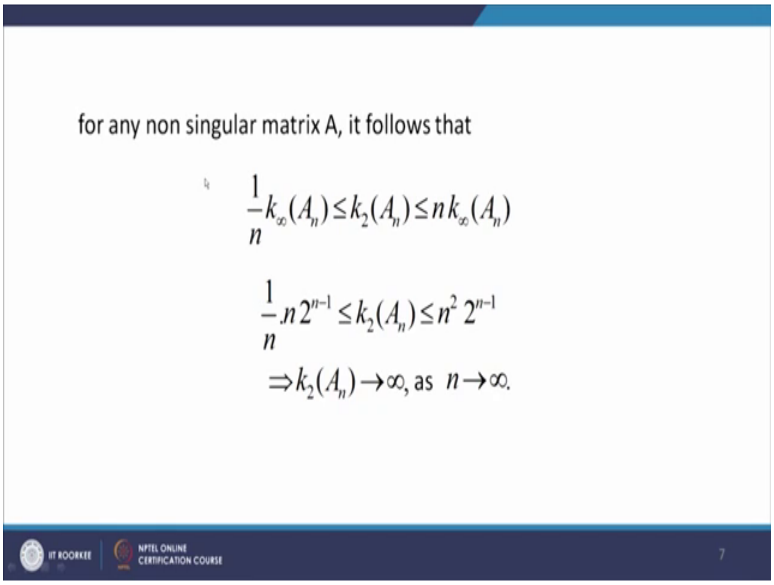

So, let us see from this inequality, we can say that that inequality is valid for any non-singular matrix A. So, here we will get 1 by n k n k infinity A n, we are replacing A by A n matrix less than or equal to k 2 A n less than or equal to n times k infinity n. And we have just now got k infinity A n as n into 2 to the power n minus 1. So, 1 by n into n times 2 power n minus 1 less than or equal to k 2 n less than or equal to n into n times 2 to the power n minus 1.

So, we get n square 2 to the power. So, lower bound is 2 to the power n minus 1; the upper bound is n square 2 to the power n minus 1. So, when n goes to infinity, lower bound goes to infinity, upper bound also goes infinity, so we have k 2 n tends to infinity as n tends to infinity. So, the condition number of the matrix A n as n goes to infinity in 2-norm is also infinity.

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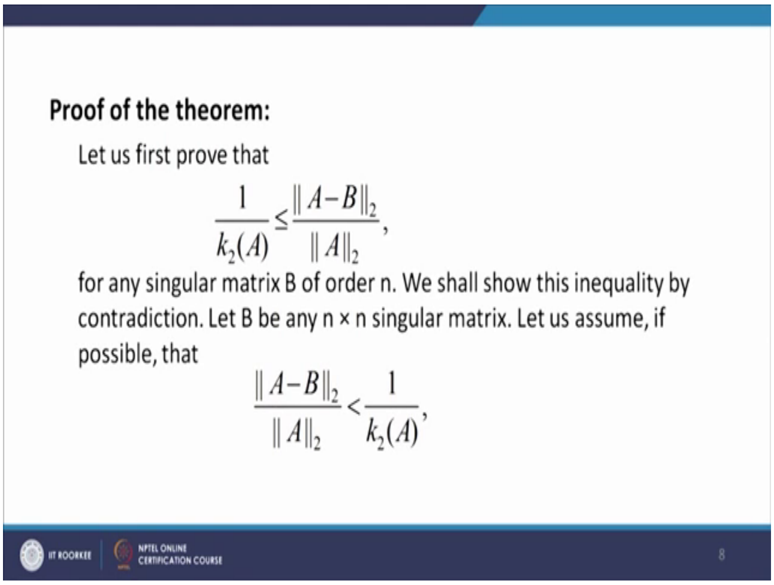

Now, now let us prove the theorem which we have stated earlier. The theorem which gives us a relation between condition number and the ah distance of the matrix A to the nearest singular matrix that this result 1 over k 2 A is equal to minimum of norm of A minus B 2 over norm of A 2 where B is an n by n singular matrix. So, let us prove this result. So, in order to prove this, what we will do is the following.

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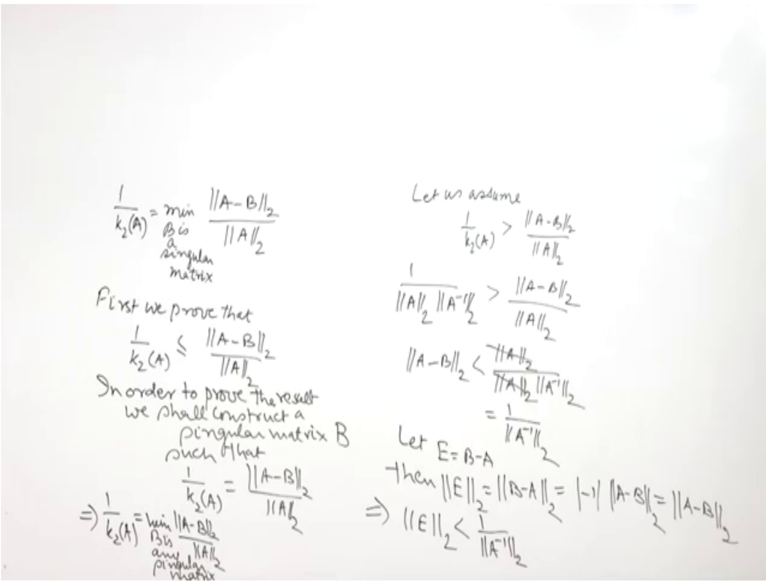

We have to show that 1 over k 2 A is equal to minimum of norm of A minus B 2 divided by norm of A 2 where B is a singular matrix singular n by n singular matrix. So, we will prove this result in two stages. In the first result, we shall show in the firstly, we shall show that 1 over k first we prove that 1 over k 2 A is less than or equal to norm of A minus B 2 upon norm of A 2 where B is any singular matrix. And then in order to say that 1 over k 2 is actually the minimum of norm of A minus B 2 upon norm of A 2 we shall construct a singular matrix B such that 1 over k 2 A is equal to norm of A minus B 2 A minus B 2 over norm of A 2. So, in order to prove the result, we shall construct a singular matrix B such that 1 over k 2 A is equal to norm of A minus B 2 over A 2.

So, this result will be proved in two parts. In the first part, we show that 1 over k 2 A is always less than or equal to norm of A minus B 2 over norm of A 2. And then in order to prove say that 1 over k 2 A is actually the minimum of this, we shall construct a matrix B which is singular such that 1 over k 2 is equal to norm of A minus B 2 over norm A 2. So, two parts together will then prove the required result. When this inequality together with this construction of the matrix B will imply that 1 over k 2 A is equal to minimum of norm of A minus B 2 divided by norm of B 2, where B is any singular matrix B is any singular matrix.

So, let us first prove that 1 over k 2 A is less than or equal to norm of A minus B 2 over norm of A 2. And to establish this inequality, we go by the method of construction. So, let us assume the contrary. Let us assume that 1 over k 2 A is bigger than norm of A minus B 2 over it is not less than or equal to it is greater than. So, so let B be any n by n singular matrix, let us assume if possible that norm of A minus B 2 over norm of A 2 is less than 1 over k 2 A.

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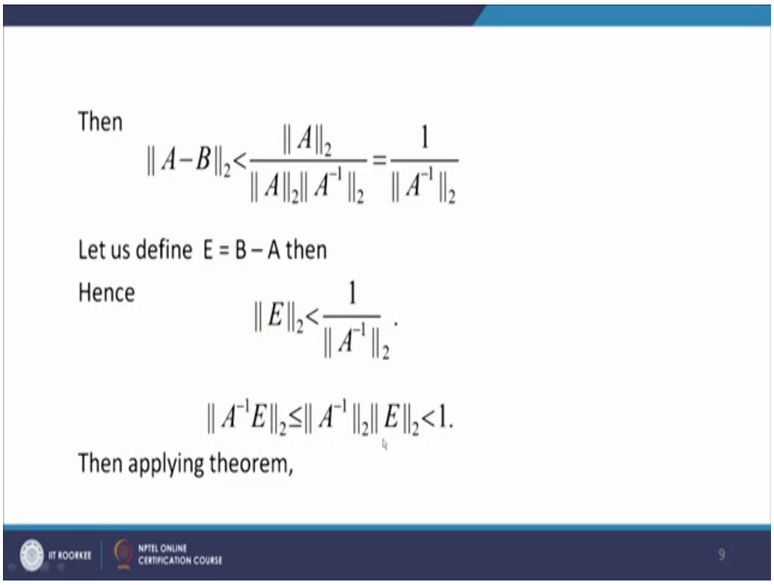

Then norm of A minus B 2, norm of A minus B 2 is less than norm of A 2 over norm of A 2 into norm of A inverse 2. From here, we can see this is norm of condition number of A into norm. So, norm of A 2, we can bring here because it is non-singular matrix. So, it is always positive. So, norm of A minus B 2 is less than norm of A 2 divided by so this will cancel and we get 1 over norm of A inverse 2. So, norm of A minus B 2 is less than 1 over norm of A inverse 2.

Now, let us define this matrix. Let say E is equal to B minus A. So, when we take E equal to let us define E is equal to B minus A, then norm of E 2 is equal to norm of B minus A 2, which is equal to mod of minus 1 times norm of A minus 1 2. So, this is equal to norm of. So, we then have norm of E 2 strictly less than 1 over norm of A inverse 2. So, this we will get then norm of so norm of E 2 less than 1 over norm of A inverse ok.

Now, norm of A inverse E 2 is because we have we are dealing with subordinate matrix norm. So, norm of A inverse E 2 is less than or equal to norm of A inverse 2 into norm of E 2, but from this inequality, we see that norm of A inverse 2 into norm of E 2 is less than 1. So, using this result, we have norm of A inverse E 2 less than 1.

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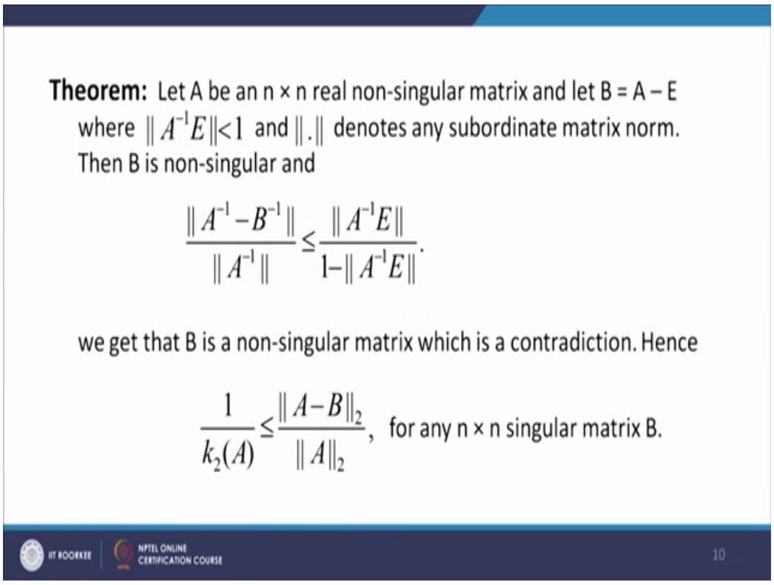

Now let us apply a theorem on matrix norms, this theorem. Let us recall this theorem. Let A B an n by n real non-singular matrix and B equal to A minus E, where norm of A inverse E is less than 1 and norm denotes any subordinate matrix norm. Then the matrix B is non-singular n we have norm of A inverse minus B inverse divided by norm of A inverse less than or equal to norm of A inverse E upon 1 minus norm of A inverse E.

So, let us recall this result. So, what we have got with us says that is any matrix B minus A such that norm of E 2 is less than norm of 1 over norm of A inverse 2. Now, so here we are given that A is an n by n non-singular matrix.

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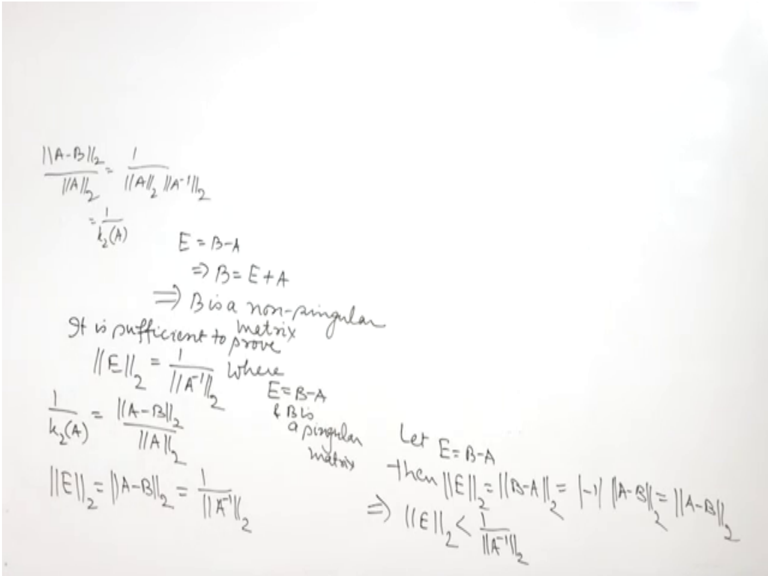

And B is B from here is what from here B is equal to, B is equal to E plus A. So, here what is happening is in this theorem B is equal to A minus E. So, here in this theorem you can take E as minus E. If you take a E as minus E then B is equal to A plus E and when you take E as minus E norm of A inverse minus E will be same as norm of A inverse E and so this inequality will hold, norm of A inverse E less than 1. So, replacing E by minus E in this theorem, we get B equal to A plus E and norm of A inverse E is less than 1, norm of A inverse E 2 is as less than 1 as we have seen just now in our case norm of A inverse E is less than 1. So, we can apply this result.

In this result, we replace E by minus E and then we what we have is a is any real n by n non-singular matrix B is equal to A plus E and norm of A inverse E less than 1, where norm is the any subordinate matrix norm then B is non-singular. So, what we have got is that we must be a non-singular matrix using this result. So, we get that B is a non-singular matrix which is a contradiction, because initially we assume that B is any singular matrix.

So, we get that B at a contradiction, hence we can say that 1 over k 2 A is always less than or equal to norm of A minus B 2 over norm of A 2 for any n by n singular matrix B. Now, we shall construct m singular matrix B such that here we have the equality 1 over k 2 A equal to norm of A minus B 2 over norm of A 2.

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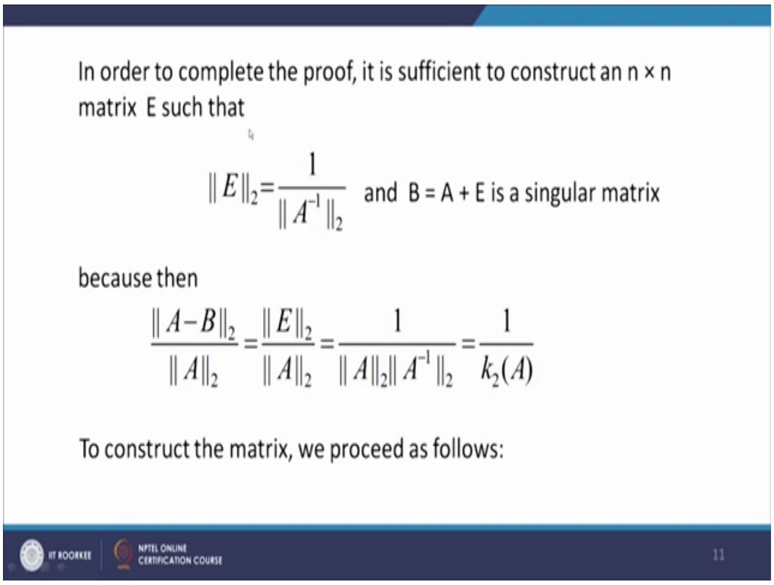

So, let us see how we construct such a matrix. In order complete the proof, it is sufficient to construct an n by n matrix E such that norm of E 2 equal to 1 over norm of A inverse 2. And B is a singular matrix because ah what will happen if we can if we can just prove this we are saying that it is sufficient to prove norm of E 2, norm of E 2 equal to 1 over a inverse 2. Why it is sufficient, because what we want to show we want to show that 1 over k 2 A is equal to norm of A minus B 2 divided by norm of A 2 this is what going to show. Here this E is equal to B minus A where E is equal to B minus A, and B is a singular matrix. So, if so what will mean, but what it will give thus norm of E 2-norm of E 2 is equal to norm of E 2-norm of E 2 will be equal to norm of B minus A 2 that is a minus B 2 ok.

So, this is equal to 1 over norm of A inverse 2 if we if we prove this actually we are proving this. Now, you can see if you divide by norm of A 2. So, norm of divide by norm of A 2 on both sides. This is nothing but the condition number of A into norm. So, we are coming we are getting this. Therefore, it is sufficient to prove that norm of E 2 is equal to 1 over norm of A inverse 2, where E is given by B minus A, and B is singular matrix. Now, let us see how we construct such a matrix.

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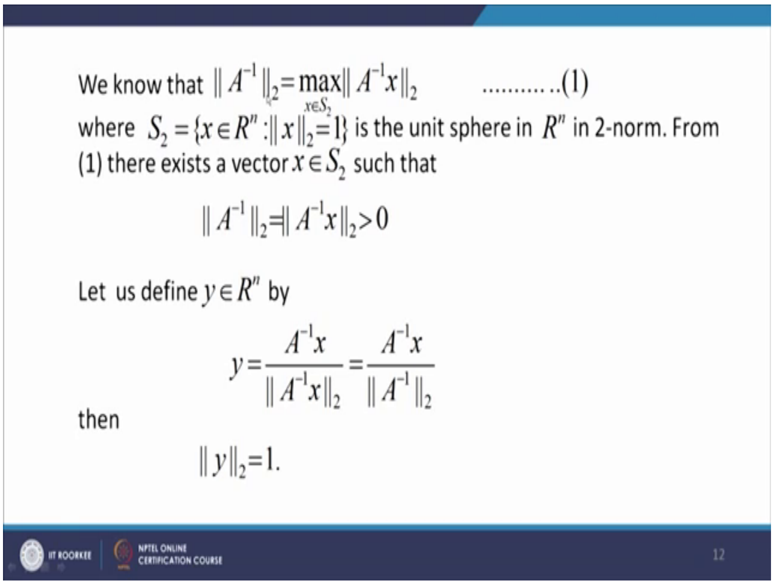

We know that norm of A inverse norm of A inverse in 2-norms is maximum of norm of A inverse x 2, where x is any element in these set S 2 and S 2 set is all those vectors x belonging to r n such that norm of x 2 equal to 1. This is this set S 2 is actually the unit sphere in r n in the 2-norm. And from 1 from this equation, it follows that there exists a vector x belonging to S 2 such that norm of A inverse 2 is equal to norm of A inverse x 2 which is greater than 0. Why is it greater than 0?

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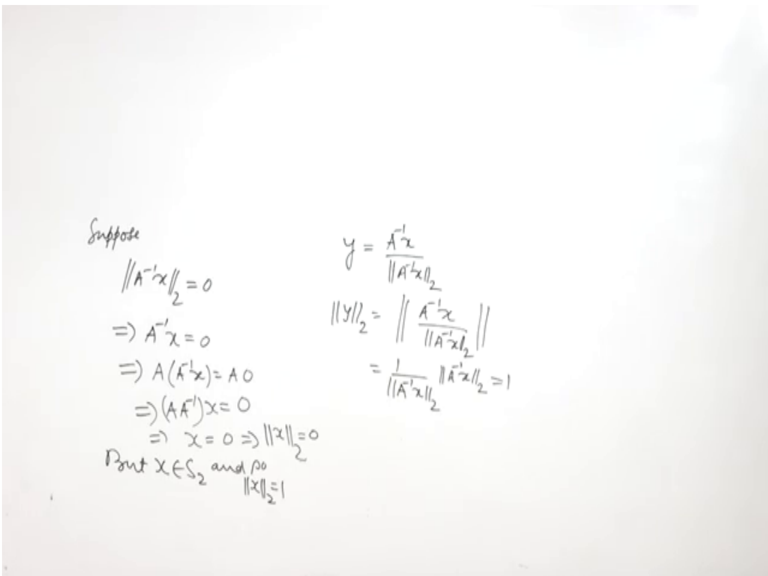

Suppose norm of A inverse x 2 is not greater than 0, then it must be 0. Suppose, suppose norm of A inverse x 2 equal to 0, then by the property of norm A inverse x must be a 0 vector which will imply that you can post multiply by pre multiply by A. So, A A inverse x equal to A into 0 vector or we can say A A inverse x equal to 0 or x equal to 0. So, x will be a 0 vector, but x is a vector in the sphere S 2 whose norm is 1. So, x is not a 0 vector, but x belongs to S 2 and so norm of x is equal to 1 not 0, this will give you norm of x as 0. So, norm of A inverse x 2 is strictly positive.

Now, let us define a vector y belonging to R n. We define it as y equal to A inverse x divided by norm of A inverse x 2. So, if we define it like this then it is actually equal to a inverse x divided by norm of A inverse 2 because here we have norm of A inverse x 2 equal to norm of A inverse 2. So, y is then equal to A inverse x divided by norm of A inverse 2. And then if you find the norm of y then what will get y is equal to y is equal to this. So, norm of y this is a positive quantity. So, we will get 1 by property of norm, we get norm of y 2 equal to 1.

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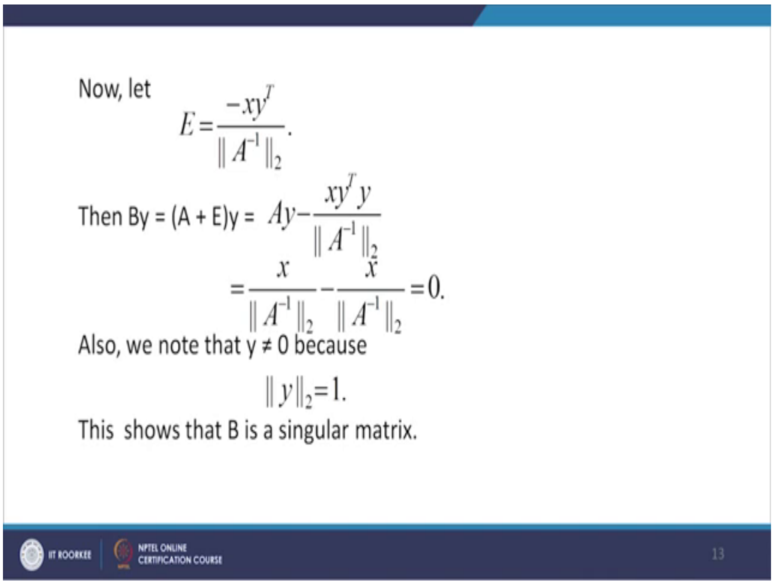

Now, let us define the matrix say E as minus x by transpose over norm of A inverse 2. This matrix is n by n matrix. How it is n by n matrix?

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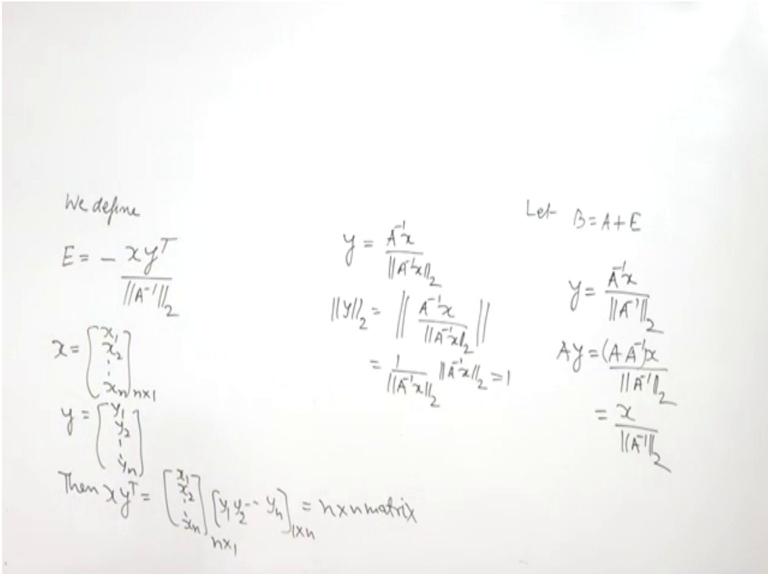

We define E as minus x y transpose divided by norm of A inverse 2; x is n by 1 matrix let us say like this x 1, x 2 and so on x n. So, this is n by 1 matrix. And y, y is also an element of R n y is y 1, y 2, y n again n by n n by 1 matrix. Then x y by x into y transpose will be so this is n by 1 matrix, then we have 1 by n matrix. And so we shall get n by n matrix. So, E is minus x into y transpose divided by norm of A inverse, E is an n by n matrix ok.

Now, let us find B y let us say B is equal to A plus E. Let us say let us say B is equal to A plus E. We have to show that when we define E like this, this B comes out to be a singular matrix. So, what we doing is let us find B y, B y equal to A plus E operating on y; A plus E when operates on y what we get A y plus E y. And E y is minus x y transpose y divided by norm of A inverse 2. A y, we can go to we can go back what is what is y, y is equal to A inverse x divided by norm of A inverse 2; y is equal to, so A y is equal to A A inverse x divided by divided by norm of, but A inverse is identity matrix identity into x is x. So, x divided by norm of A inverse 2. So, what we get is A y equal to x upon norm of A inverse 2. Now, let us see how this x y transpose y upon norm of A inverse 2 becomes x over norm of A inverse 2.

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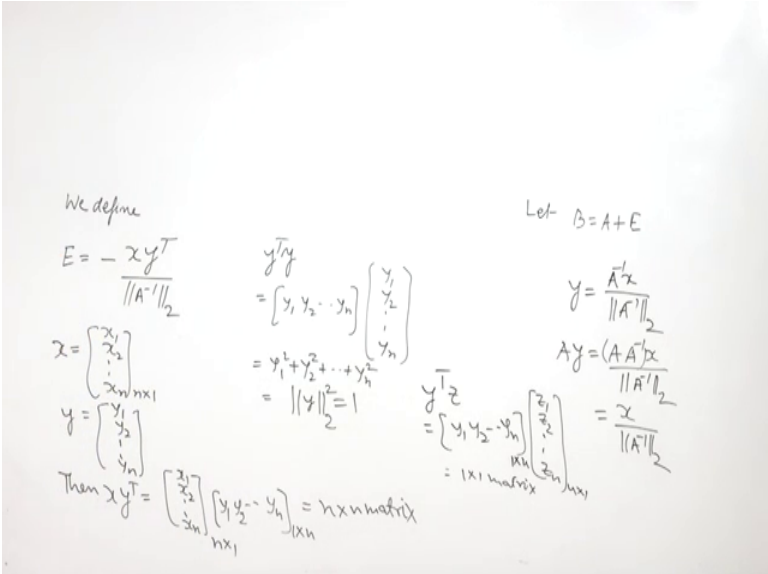

See y transpose y is what, y transpose y is equal to y 1, y 2, y n, and then y 1, y 2, y n. So, this is 1 by n matrix; this is n by 1 matrix. So, when we multiply we get 1 by 1 matrix, and that 1 by 1 matrix is y 1 square plus y 2 square and so on y n square. And this y 1 square plus y 2 square and so on y n square is nothing but norm of y norm of y 2 square but we have seen earlier that norm of y norm of y 2 is equal to 1, this we have seen.

So, this we let us go to back, here, here you see norm of y 2 equal to 1, this we have seen earlier. So, norm of y 2 equal to 1 means y transpose y equal to 1. When y transpose y equal to 1, we will get x upon norm of A inverse 2. So, this is this is equal to 0. Now, so what we get B y is equal to 0, and y is a nonzero vector. We can see here because norm of y 2 equal to 1, so B is a nonzero vector. And therefore, the matrix B must be a singular matrix, because B is not singular it is non-singular then you pre multiply by B inverse. B inverse B y equal to 0 will give you y equal to 0. So, since y is not equal to 0, it follows that B is a singular matrix. So, we have reached this stage.

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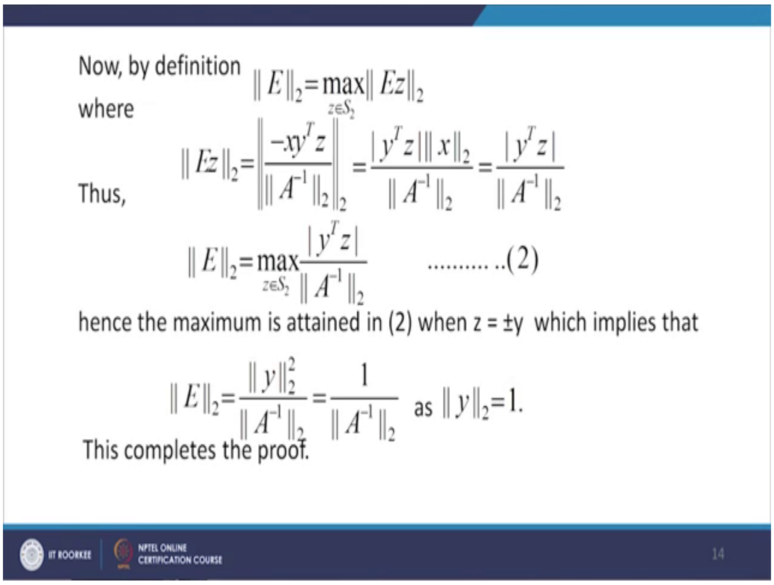

Now, let us show that norm of E 2 is 1 upon norm of A inverse 2 that is that is what we want to show now. Now, by definition, norm of E 2, E is a matrix norm of E 2 into norm it is maximum of norm of E z 2 where z belongs to x 2. Norm of E z 2 is what let us replace the value of E here, the value of E is minus x y transpose divided by norm of A inverse 2. So, we have just put the value of E here, and what we what we got is minus x y transpose z upon norm of A inverse 2, and then whole of nor in norm inverse 2. This is norm 2, it should be it is norm 2. Now, y transpose z, y transpose z is a scalar quantity. Remember y transpose z, because y transpose is row vector and z is a column vector.

So, when you multiply this is n by 1 matrix, this is 1 by n matrix and this is n by 1 matrix. So, what we will get 1 by 1 matrix meaning thereby we are getting a scalar y 1 into z 1 plus y 2 into z 2 and so on y n into z n. So, y transpose z is a scalar inside here. So, when it comes out of the norm, it comes out as a mod of y transpose z. And then what are we left with norm of x 2 divided by norm of A inverse 2, but x is a vector belonging to x 2. So, norm of x 2 equal to 1, and therefore, we have mod of y transpose z divided by norm of A inverse 2. And thus norm of E 2 is equal to maximum of mod of y transpose z divided by norm of A inverse 2 where z is a element of x 2. Now, let us see when the maximum into will be attend for what z.

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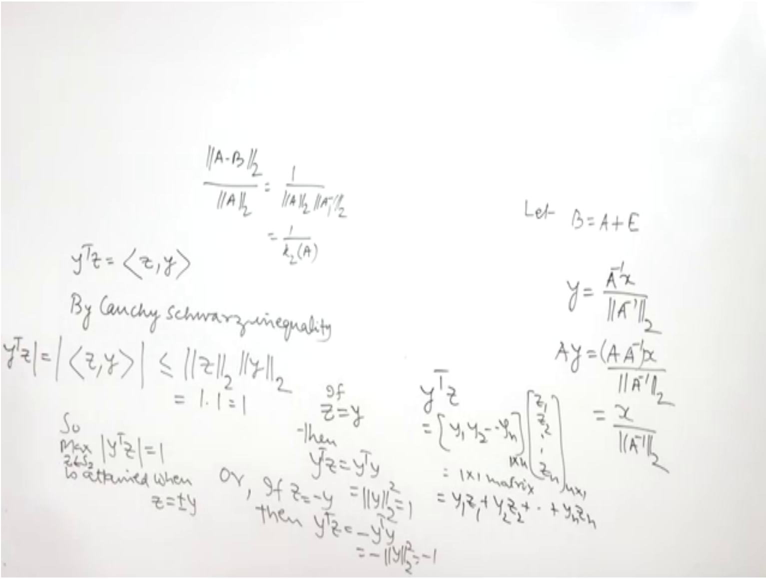

As you see here y transpose z, y transpose z is y 1 z 1, y 2 z 2 and so on y n z n. Let us notice that y transpose z is y 1 z 1 plus y 2 z 2 and so on y n z n. And this y transpose z is nothing but the inner product of z and y inner product of z and y vectors in r n is nothing but y transpose z. And by Cauchy-Schwarz inequality mod of inner product of z and y is less than or equal to norm of z into norm of y.

Since, we are dealing with 2-norm, we have norm of this norm of this. Now, z is an element of x 2 here, z in a element of x 2, here and y has norm 1. So, norm of z 2 is 1, norm of y 2 is 1, so we have 1 1. This means that maximum value of y transpose z this is nothing but mod of y transpose z cannot exceed 1 maximum value that it can attain is 1.

And when this maximum value 1 will be attained if we take z vector equal to y vector as we take z equal to minus y vector; because if you take z is equal to y, if z is equal to y, then y transpose z y transpose z equal to y transpose y equal to norm of y 2 square which is equal to 1. Or if you take or if z is equal to minus y, then y transpose z y transpose z is minus y transpose y which is minus norm of y 2 square which is minus 1 so, maximum value of mod of y transpose z which is 1 is attained when z is equal to plus minus y z is an element of x 2.

So, what we have maximum is attend in this equation when z takes value plus minus y and when z takes plus minus y and when z takes plus minus y value mod of y transpose y as we have just now seen is norm of y 2 E square. So, norm of y 2 E square here we have and then norm of y 2 is equal to 1. So, we will have 1 over norm of A inverse 2. And thus we are able to construct a matrix B such that norm of E, E, E is equal to, E is equal ah E is equal to B minus A ok. So, such that norm of B minus A 2 or norm of A minus B 2 is equal to 1 over norm of A inverse 2 which is equal to 1 over. So, we have we have prove the result that is norm of A minus B 2 over norm of A inverse A 2 is equal to 1 over this which is equal to 1 over k 2 A. So, we have prove this result; this complete the proof of the theorem. Now, let us take an example on this.

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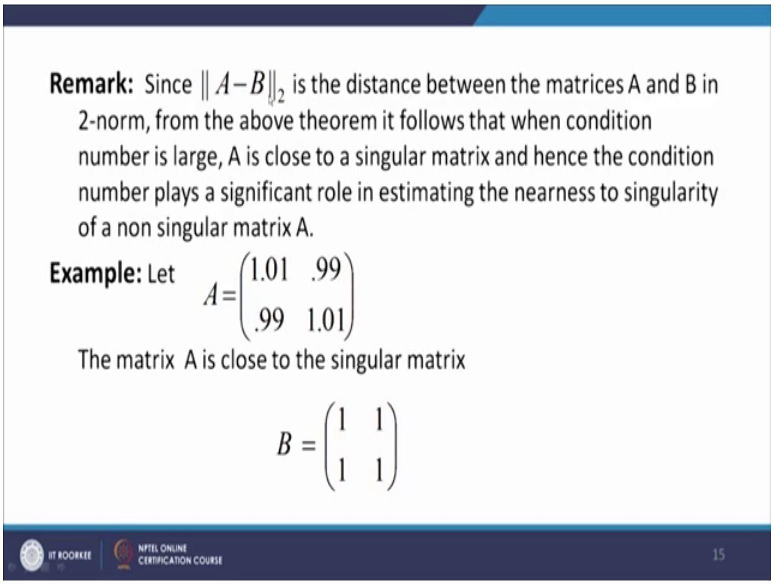

Before that let us take a remark, let us do this. Since, norm of A minus B 2 is the distance between the matrices A and B in 2-norm from the above theorem it follows that when the condition number is large is close to A singular matrix, because when the condition number is large 1 over k 2 able become smaller. And so the distance between the matrices A and B will become small. So, A will be close to a singular matrix and hence the condition number plays a significant role in estimating the nearness to singularity of a non-singular matrix A.

For example, let us look at this matrix A equal to 1.01 0.99 0.99 1.01. This matrix is a non-singular matrix. And you can see that this matrix is close to the singular matrix B which is 1 1 1 1 if you compare this entries 1.01 is close to 1, 0.99 is close to 1. So, this matrix non-singular matrix is close to the singular matrix B equal to 1 1 1 1. Now, let us find the condition number here and see the closeness of the matrix A 2 the matrix B.

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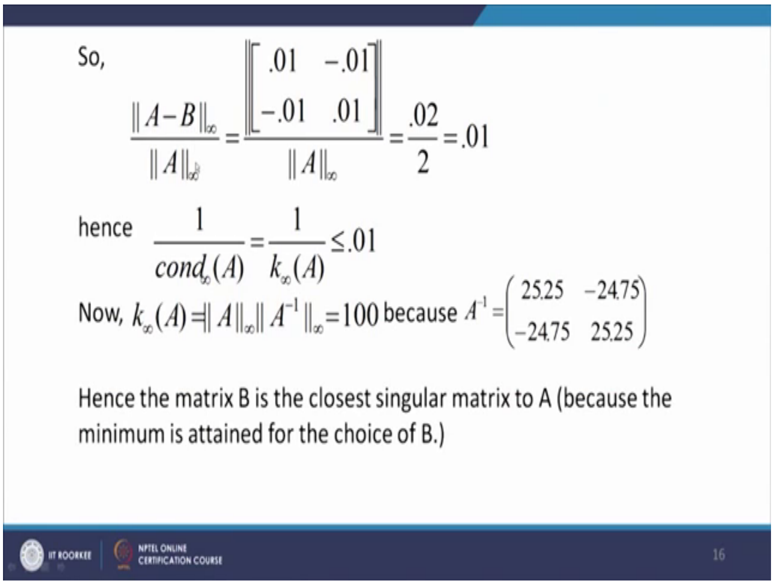

So, now let us find norm of A minus B infinity over norm of A infinity that is the relative distance of A from the singular matrix B. So, A minus B here is 0.01, minus 0.01, minus 0.01, 0.01, and therefore, the infinity norm of this A minus B matrices maximum absolute row sum. So, we can see maximum absolute row sum is 0.02 in both the rows, so norm is 0.02. And when we find norm of A infinity, A is this matrix. So, maximum absolute row sum here is 2, 1.01 plus 0.99, here it is 0.99 plus 1.01. So, maximum absolute row is sum is 2. So, norm of A infinity is 2. So, 0.02 divided by 2 that is 0.01.

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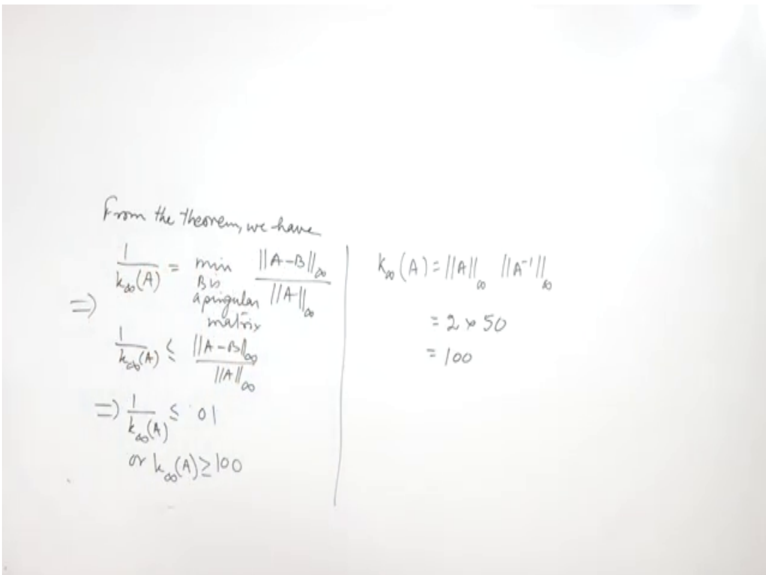

Now, we have the from the theorem, we have 1 over condition number of the matrix A in the infinity norm equal to minimum of norm of A minus B infinity divided by norm of A infinity, where B is a singular matrix. If instead of 2-norm in the theorem, we use infinity norm then we will have this result. So, 1 over k infinity A is less than or equal to this where B is any singular matrix. So, for the matrix B, which we have considered here which is a singular matrix. What we have 1 over k infinity A is less than or equal to 0.01 or k infinity A is greater than or equal to 100. So, k infinity A must be at least 100, this is what we get.

And when we actually calculate k infinity A, what do we get k infinity A by the formula is norm of A infinity into norm of A inverse infinity. And norm of A infinity we have seen this is equal to 2, norm of A inverse infinity we can find for that we will need to find the inverse matrix A inverse. And when you find the A inverse matrix, it is 25.25, minus 24.75, minus 24.75, 25.25. So, norm of A inverse infinity will be maximum absolute row sum here. So, 25.25 plus 24.75 which will give you 50 ok.

So, what we have 2 into 2 into 50, so 100. So, k infinity the condition number of A is equal to 100. And here what we get condition number of A is greater than or equal to 100. This means that the matrix B, this matrix B is the nearest to closest singular matrix to A, because we are getting k infinity A 2 B equal to 100. So, B matrix B is the closest singular matrix to A, because minimum is attained, minimum this minimum, attained is attend for the choice of B. So, with this, I would conclude my lecture.

Thank you very much for your attention.