Numerical Linear Algebra
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Lecture – 04
Vector Space- I

Hello friends, I welcome you to my lecture on Vector Space. So, there will be two lectures on vector spaces. This is first of those two lectures. Let us see what do we mean by a vector space.

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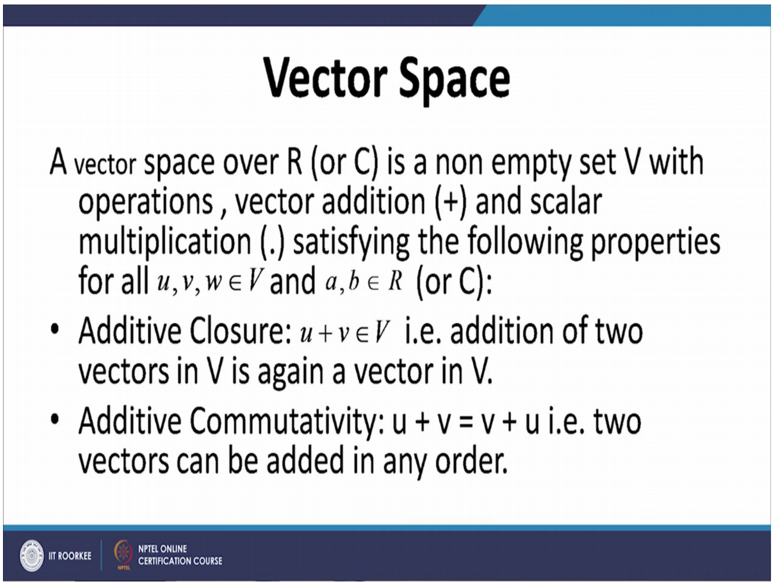

We will be taking here the field of a scalars to be either R or C. So, a vector space over R or C is a nonempty set V with two operations. One operation is that of vector addition and the other one is that of a scalar multiplication. Vector addition operation be denoted by plus and a scalar multiplication operation be denoted by dot ah. And with respect to these two operations the following conditions or the following regimes have to be satisfied for V to be a vector space over R or C.

The first one is additive closure if u and v belong to V then u plus v belongs to V that is the addition of two vectors in V. The elements of V are called vectors. So, the addition of two vectors in V is again a vector in V. Then additive commutatively if you take any two vectors u and v in V then u plus v is equal to v plus u that is the vectors can be added in any order ah. Then associative property with respect to addition that is u plus v plus w is equal to u plus v plus w.

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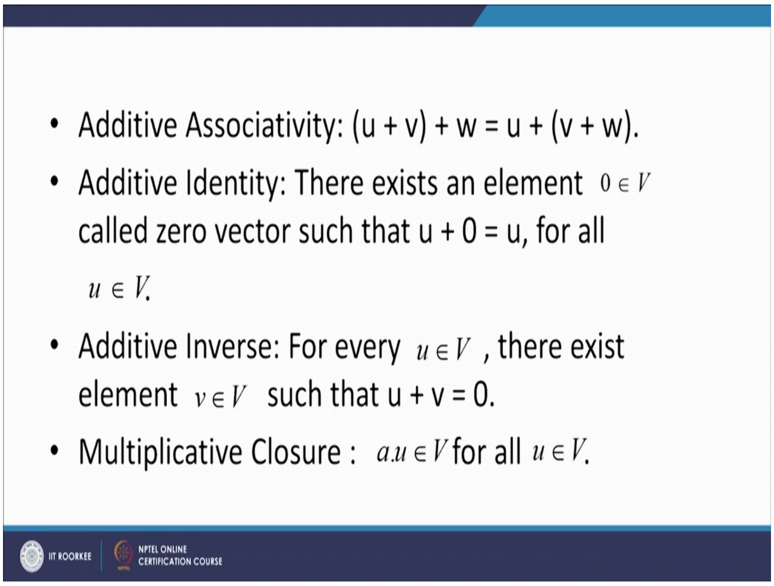

And then comes the additive identity so there must exist in V an element which will be denote by 0. Such that and called it call it a zero vector such that when it is added to any vector of the set V say you take the vector to be u then u plus zero is equal to u for all u belongs to V. So, the there must exist a one such vector which we called as zero vector.

Now, additive inverse for every u belonging to V there must exist an element v belonging to V such that u plus v is equal to 0. Then v is called the additive inverse of u and the zero is the additive identity here ah. Then multiplicative closure corresponding to the multiplicative multiplication operation if u belongs to V and a belongs to the field that is R or C then a into u must belong to V.

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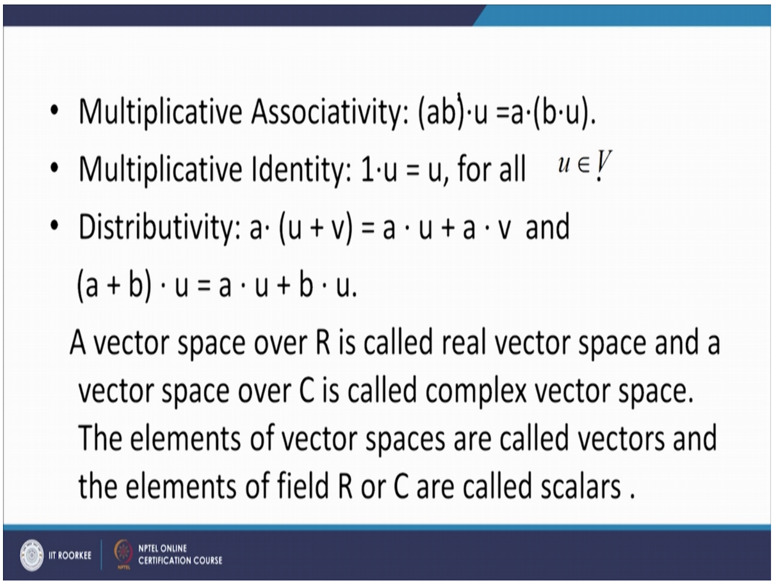

And then multiplicative associativity ab into u equal to a into b into u and multiplicative identity. So, there must exist element 1 in their exist element 1 in a R or C. So, 1 into u is equal to u for all u belonging to V.

Ah then we should be have distributed law that is a left distributive law a into u plus v equal to a into u plus a into v and write distributive law that is a plus b into u is equal to a into u plus b into u. A vector space over R we will call as a real vector space and a vector space over C will be called as a complex vector space. As I said earlier the elements of the vector space V are called vectors and the elements of the field R or C will be called as a scalars.

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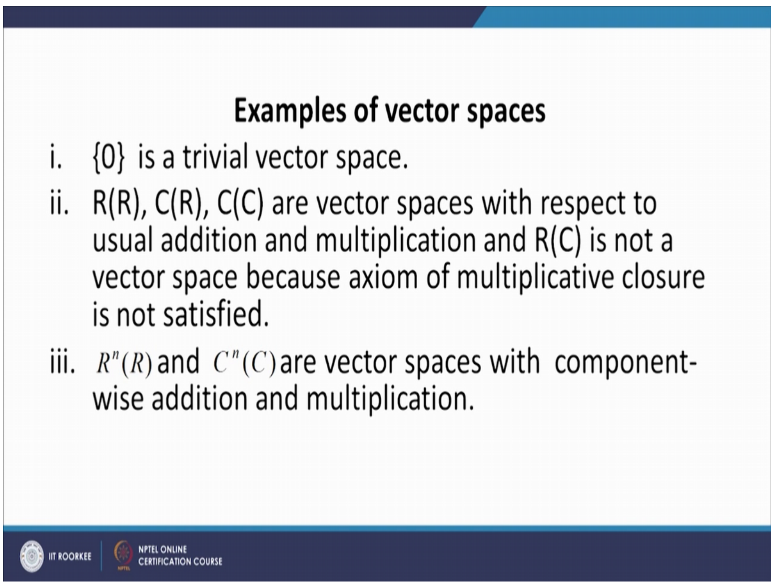

Now, if you take v to be 0 the just descendants at 0 and field you can take R or C.

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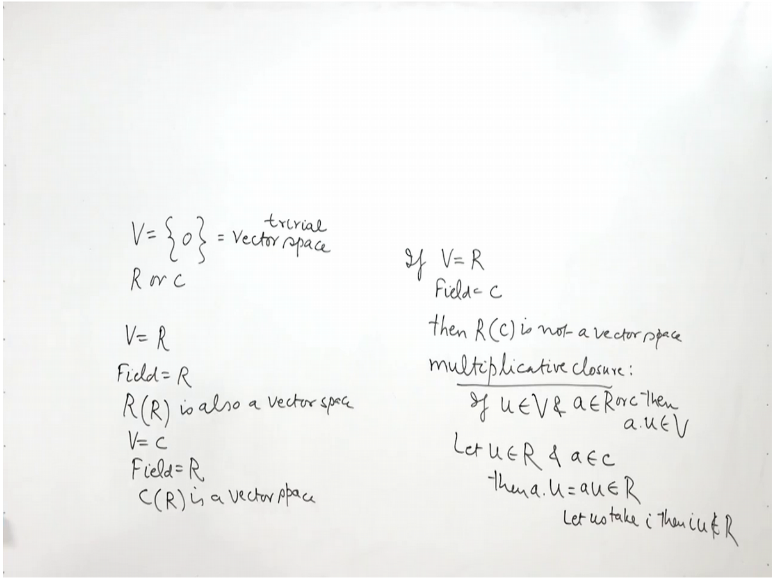

Then V is a vector space over R and V is also vector space over C. We can easily see that all the axiom of the vector space R satisfied for this vector space. So, this V is a vector space ah. So, it is called as a trivial vector space ah. If you take the V to be equal to R if you take V equal to R the set of real numbers and field you take equal to R that is the scalars are taken from R. So, then R over R is also vector space. If you take V equal to C and field to be equal to R then again C over R is a vector space.

But, if you take V equal to R and we take field equal to f a field equal to C then R over C is not a vector space. This is because the multiplicative closure property- axiom will not be satisfied here ah. Multiplicative closure is what? If u is any element belonging to U and a belongs to this R or C then a into u must belong to, let me take it an V then a into u must belong to V. So, this is the multiplicative closer axiom.

Now, here what do we have? So, here our V is R ok. So, let u belong to R and a belong to C; that means, a is a complex number. Then what we want is that a into u which is au must belong to R. Which is not in general true because if I take a is equal to a complex number say probably power complex number say i ok. Let us take a equal to i then i into u will not belong to R. Because iu is not a real number. So, R over C is not a vector space now if you take RNR. Let us say we take to be R to the power RN and field let us take as R.

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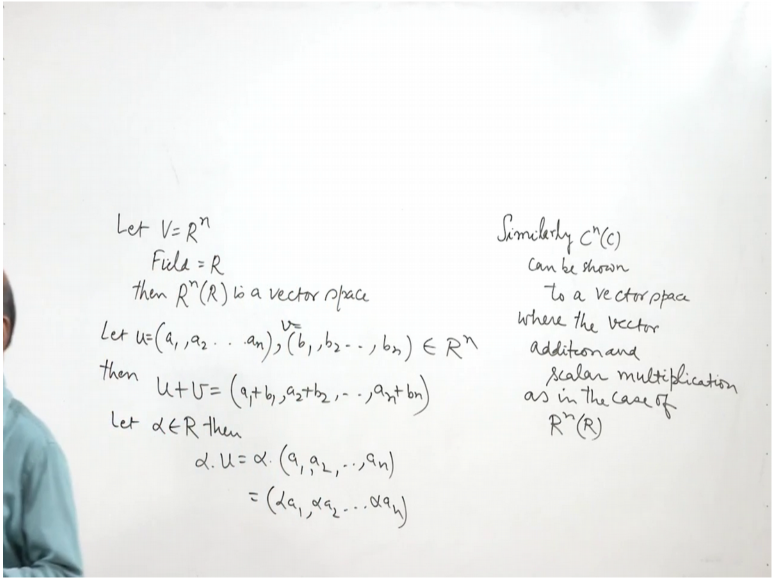

Then RnR is a vector space what we do is let us see how we define the vector addition in a scalar multiplication here.

So, let us say a 1, a 2 and so on a n b 1, b 2 and so on b n belong to R to the power n ok. So, this is my u and this is my v the element v in Rn. So, then u plus v the vector addition u plus v in Rn will be defined as, component wise addition. So, a 1 plus b 1 a 2 plus b 2 and so on an plus bn. And the scalar multiplication let us see how we define this scalar manipulation. So, let alpha be a scalar in R field, R then alpha into u. This defined as alpha into a 1 a 2 an. So, it is component by scalar multiplication.

So, alpha a 1 alpha a 2 and so on alpha an so, when we define the vector addition like this and a scalar multiplication in this manner. Then we can easily verify that all the axioms of the vector space are satisfied. So, RnR is the vector space. Similarly CnC can be shown to be a vector space, where the vector addition and scalar multiplication are defined as above. as in the case of RnR.

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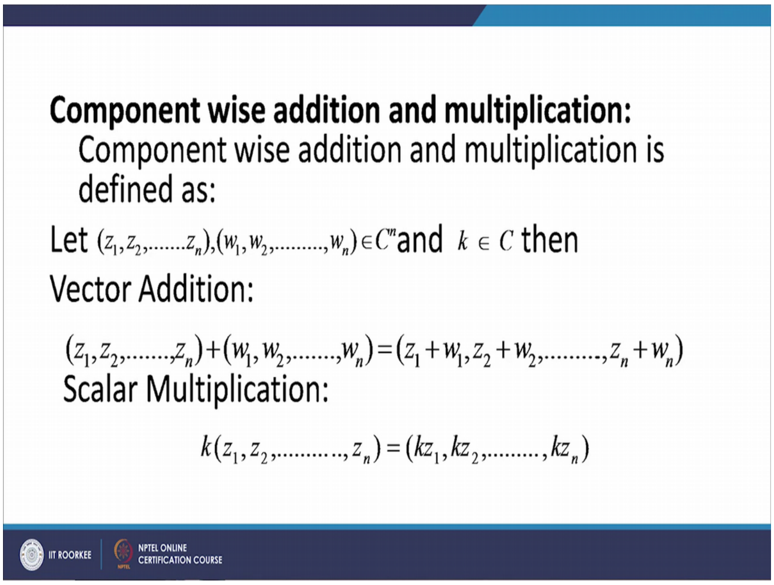

Now so this slide now tells us how we define the components by addition and scalar multiplication which we have just now discussed.

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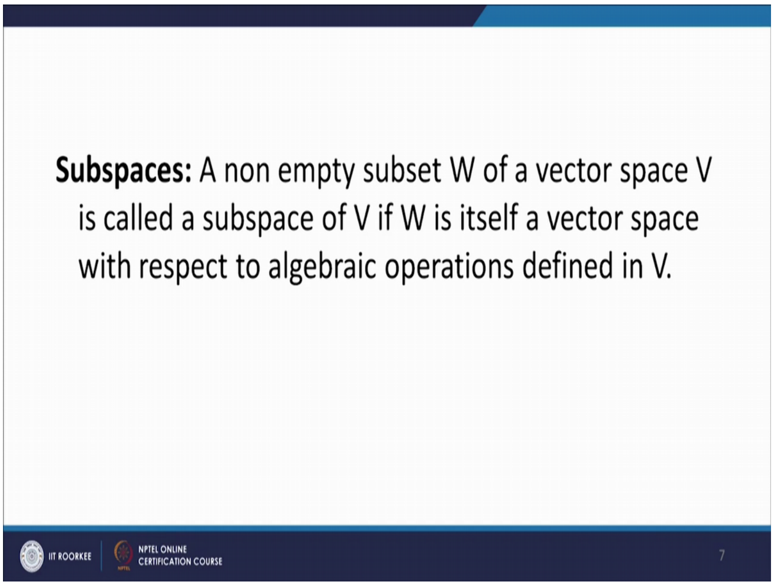

Now, next comes a subspace a nonempty subset W of a vector space V is called a subspace of V. If W is itself a vector space with respect to the algebraic operations defined in V. Let W be a nonempty subset of a vector space V then we will call it as a subspace of V if W is itself a vector space with respect to algebraic operations defined in so, we can put it in mathematically like this.

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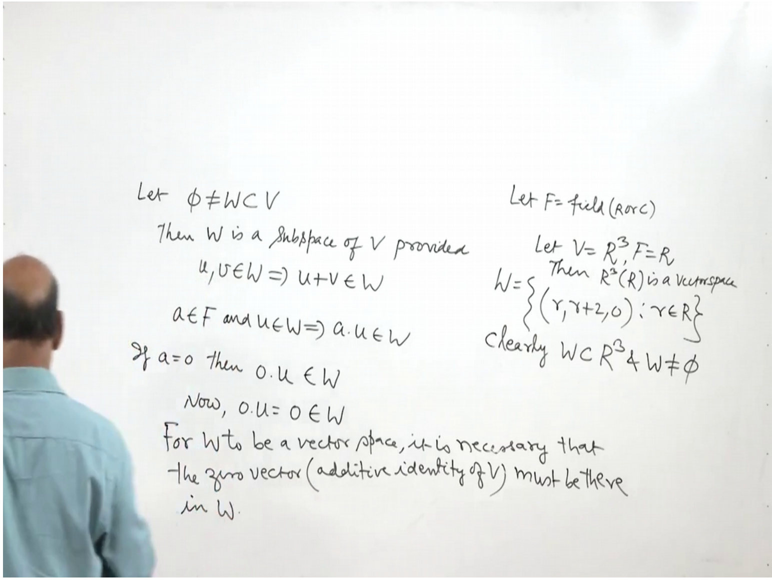

Let phi be not equal to W and W is a subset of V. Then W is a subspace of V provided uv belong to W imply that u plus v belong to W. And a belong to field R or C. So, field i will write as F ok.

Let us say let F be the field which is either R or C. So, to include R or C both we will write F. So, let a belong to F and u belong to W then a into u must belong to W. So, if these two conditions are satisfied then it can be shown that W will be a subspace of V. All other axioms of which are there in the definition of vector space they will all be satisfied we just need to show the following two conditions in order to arrive at the conclusion that W is a vector space subspace of v ok. Now let us look at this second condition ok. Second condition is a belongs to F and u belongs to W implies a into u belongs to W.

So, here if I take a equal to 0 if you take a equal to 0 then 0 into u 0 into u belong to must belong to W. Now 0 into u is equal to zero vector ok; 0 into u zero vector. So, zero vector belongs to W. So, this means that for W to be a subspace it is necessary that the zero vector which is the zero vector of the vector space v must belong to W ok. So, for W so for W to be a vector space it is necessary that the zero vector which is also the additive identity of V; the zero vector which is the additive identity of V must be there in W.

So, whenever we want to check whether a given subset W of V is a subspace of V or not. T he first thing that we do is we check that whether the zero vector of the space b is there in W or not. If it is there in W then we proceed further and check the these conditions. If it is not there in W we say that W is not a subspace. For example, let us consider this example let vv equal to R cube, F equal to R ah. Let us take W to be so we know that R cube R is a vector space. So, let us define W to be the set of all tuples are R plus 2,0, but R is a real number.

Now, clearly W is a subset of R cube, because the elements of W are tuples 3 tuples are of the type xyz where x is r y is r plus 2 z is 0. So, they are real numbers. So, it is a sub clearly it is a subset of R cube and also it is a non empty subset of R cube because you take r to be say for example, one then 1 3 0 element belongs to W. So, clearly W is a subset of R cube and W is not equal to Phi.

Now, let us see you can see here that zero vector which is the additive identity in R cube does not belong to W because if zero vector belong to W then we note that the zero vector 0 0 0 of R cube does not belong to W because if it belongs to W then r, r plus 2, 0 must be equal to 0 0 0 for some r for some r belonging to 0.

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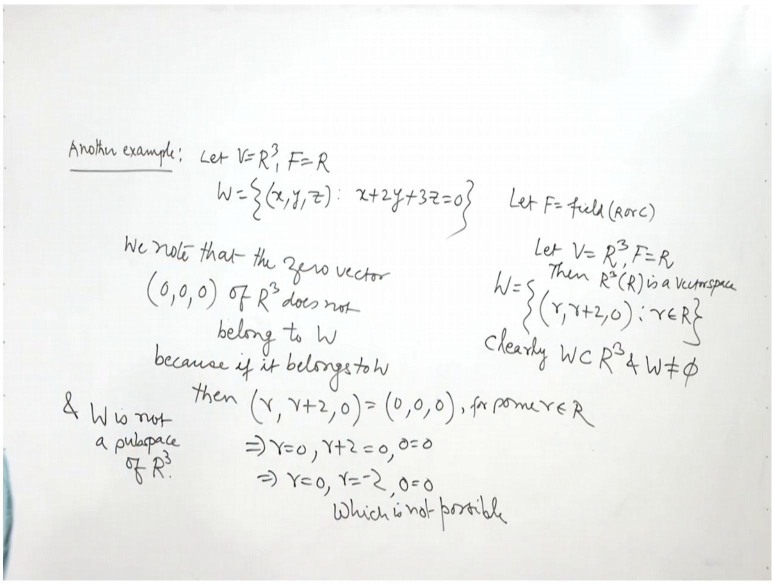

Now r , r plus 2, 0 is equal to 0 0 0 this implies that r is equal to 0, r plus 2 equal to 0, 0 equal to 0.

So, here we get r equal to 0, here we get r equal to minus 2 which is not possible ; which is not possible.. So, 0 0 vector 0 0 0 vector does not belong to R cube does not belong to

W and therefore, W is not a subspace. W is not a subspace of R cube. Again let us take V equal R cube F equal to R field to be R and W to be x y z such that x plus 2 y plus 3 z equal to 0. So, then you shall see that it forms a vector space. It form it is a subspace of R cube.

So, it is clear first of all we see that the zero element of all the additive identity of R cube which is 0 0 0 belongs to W clearly. W subset of R cube and W is not equal to phi.

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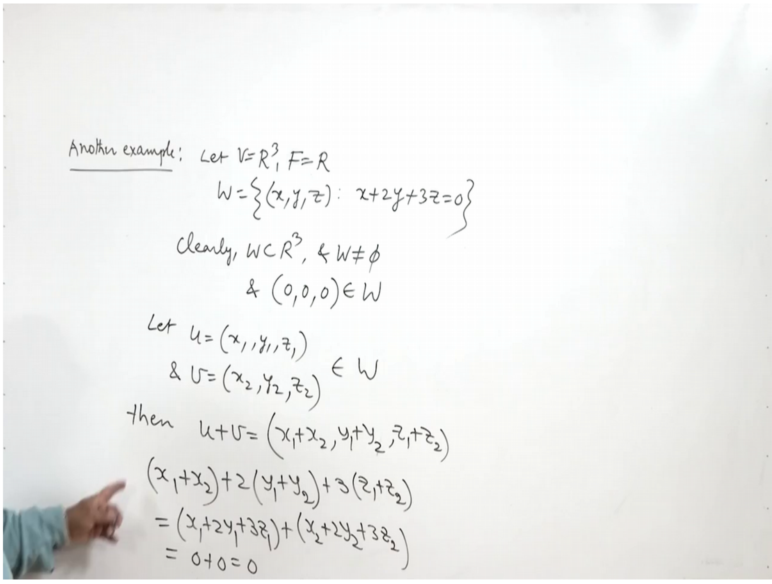

You can take x equal to 0, y equal to 0, z equal to 0. Then x plus 2 y plus 3 z equal to 0. So, W is not phi and 0 0 0 belong to W ok. Now so what we will do your now so now, let us proceed further to show that W is a subspace. So, let u equal to x 1 , y 1, z 1, and v equal to x 2, y 2, z 2 ok. Be any two elements in W ok. Then u plus v by definition of vector addition will be equal to x 1 plus x 2, y 1 plus y 2, z 1 plus z 2.

We have to show that x 1 plus x2, y 1 plus y 2, z 1 plus z 2 belongs to w. For, but it will belong to w only when it satisfy the property that x 1 plus x 2 plus 2 times y 1 plus y 2 plus 3 times z 1 plus z 2 equal to 0. So, let us see whether it is 0 or not. So, x 1 plus x 2 plus 2 times y 1 plus y 2 plus 3 times z 1 plus z 2. We have to show that it is equal to 0 and it can be written as x 1 plus 2 y 1 plus 3 z 1 plus x 2 2 y 2 plus 3 z 2 since x 1 y 1 z 1 belongs to W. So, x 1 plus 2 y 1 plus 3 z 1 is equal to 0 x 2 plus by x 2 y 2 z 2 belongs to W. So, x 2 x 2 plus 2 y 2 plus 3 z 2 equal to 0.

So, we get 0 plus 0 and so u plus v hence u plus v belongs to W.

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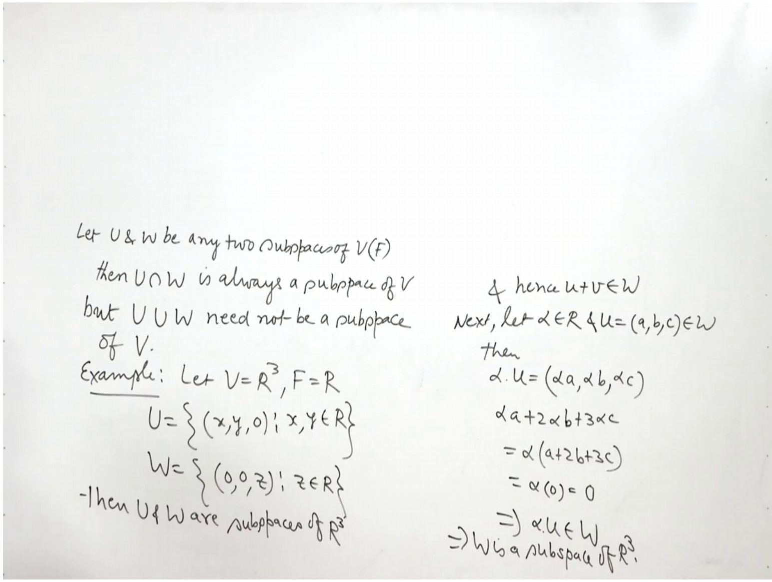

Next let us say, let alpha be a real number here the field F is R. So, alpha be a real number and u equal to a, b, c belongs to W. Then alpha into u is equal to alpha a, alpha b, alpha c ok. But it will belong to W only when alpha a plus 2 alpha b plus 3 alpha c equal to 0. So, alpha a plus two alpha b plus 3 alpha c we can write as alpha times a plus 2 b plus 3 c.

Now, a plus 2 b plus 3 c is 0 because u belongs to W. So, this is alpha into 0. So, it is zero vector so this zero vector ok. A, a b plus 2 b plus 3 c is equal to 0. So, alpha into 0 is equal to zero vector. So, what we will get. So, sorry alpha into 0 is zero scalar. So, a plus 2 b plus 3 c this is not vector this is scalar. So, alpha plus 2 b plus 3 c is equal to this scalar 0. So, this implies that alpha into u belongs to W. So, W is a subspace of R cube.

Now, if you take two subspaces, any two subspaces of a vector space, say u and W then their intersection is always a subspace of W. So, if u and W let u and W be any two subspaces of V vector space V over field F ok. Then u intersection W is always a subspace of V, but it is not in general true when you configure u union W ok. So, so let U and v be any two subspaces of a vector space V over the field F then u intersection W is always a subspace of v.

But U union W need not be a subspace of subspace of V. For example, let us say let V be equal to R cube F be equal to the field R u we take as these set of all points x by 0 where x, y belong to R ok. That is all points of the xy plane ok. R cube is nothing, but these set of points in the space ah. So, here when we take the z component 0 it means that we are considering all points in the xy plane. So, it is two dimensional subspace.

Now, then W is we take as 0 0 z; that means, all points lying on the z axis ok. You can easily verify that both U and W are subspaces of R cube subspaces of U and W are subspaces of R cube. If you take a and U intersection W here then you get the zero subspace the only zero subspace ok. That is here we get only the zero vector U, U is the xy plane w is the z axis. So, they meet at only origin. So, there is only one point here 0 the zero vector is nothing, but 0 0 0.

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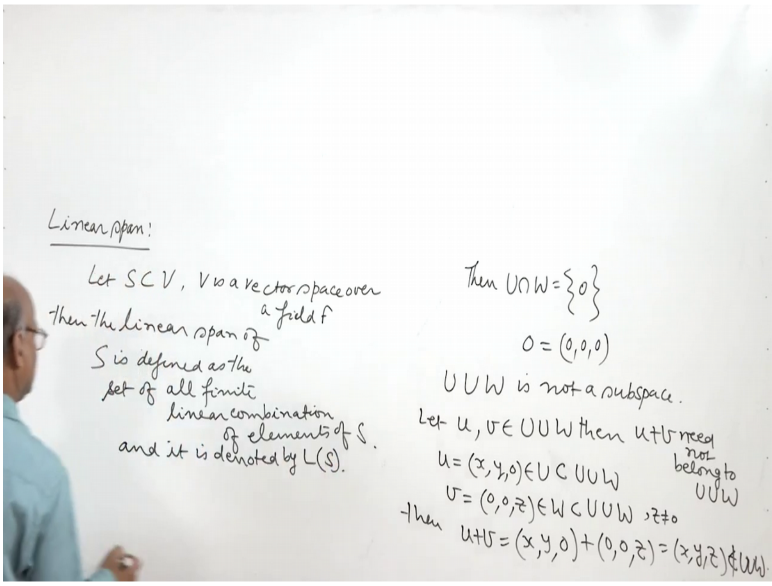

So, there is only one element in this U intersection W. Now U union W here is not a subspace it is not a subspace. So, we can easily check this because if you take any elements say let us say let u belong to u and v belong to W union W union W ok. Then u plus v need not belong to need not belong to union W. What I can do is I can take u equal to x by 0 that is I take an element from U. So, that it also belongs to U union W. U is subset of union W so I can take u to be x by 0 and I take v equal to 0, 0, z which is which is an element of W. So, it also belongs to U union W. So, I take I mean to say in U union W I take an element from u and I take an element of W ok. Then u plus v u plus v is equal to x by 0 plus 0, 0, z where z I am taking to be nonzero; z component I am taking to be nonzero. So, this is equal to x, y, z.

Now, this element does not belong to U and this element does not belong to W. So, it does not belong to U union W because this element is neither there in U nor there is nor it is there in W. So, it does not belong to U union W. So, union W is not a subspace. Now let us consider linear span. What we mean by a linear span? Let S be a subset of a vector space V ah. V is a vector space over a field F. So, let s be a subset of V then the linear span of S is defined as the linear span of S is defined to be the set of all linear combinations of elements of S and it is denoted by LS so, that I would like to conclude my lecture.

Thank you very much for your attention.