Numerical Linear Algebra
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Lecture - 39
Residual Theorem

Hello friends, welcome to my lecture on residual theorem.

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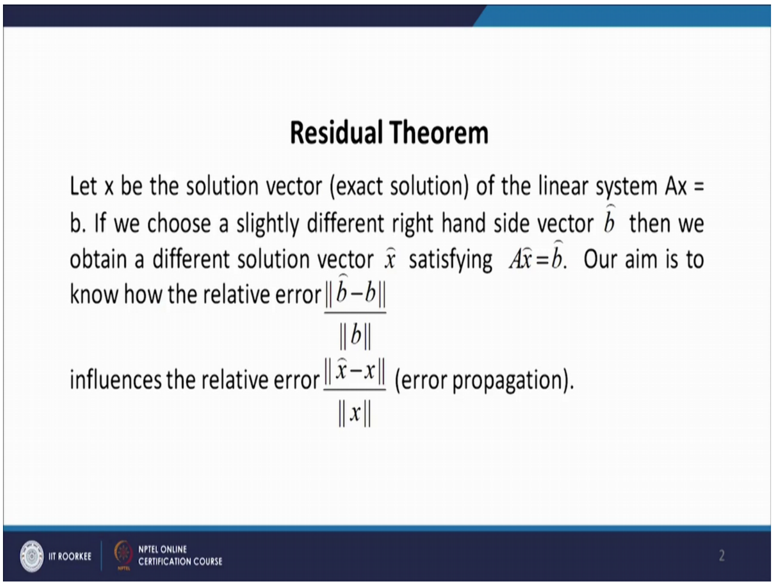

Let x be the solution vector or you can say exact solution of the linear system A x equal to b. Now if you choses slightly different a right hand sent side vector b cap, then we obtain a different solution vector say x cap satisfying A x cap equal to b cap. Our aim is to know how the relative error norm of v cap minus b over norm of b inferences the relative error, norm of x cap minus x over norm of x. So, when there is a slight perturbation in the right hand side vector b, instead of the b b considered b cap, then corresponding 2 x there will be another solution vector x cap satisfying the equation x cap equal to b cap. So, our aim is to know how the relative error in b actually affects the a relative error in x ok.

So, we have A x equal to b which and A x cap equal to b cap.

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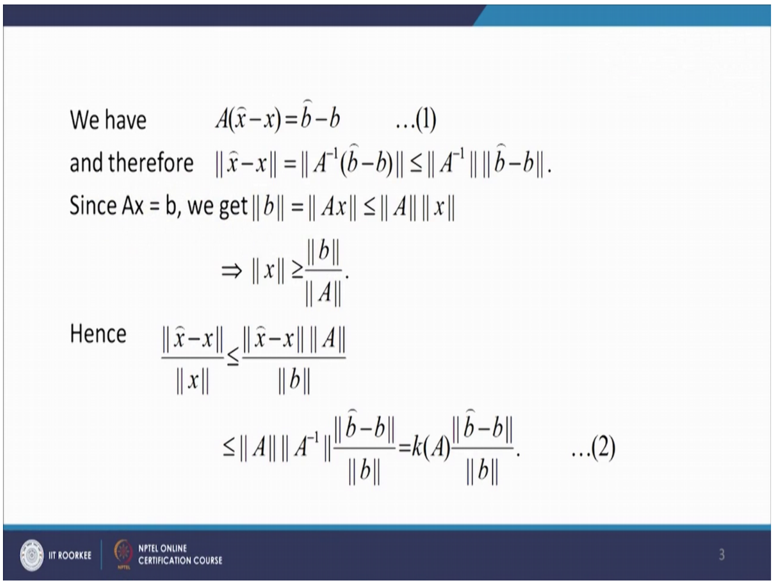

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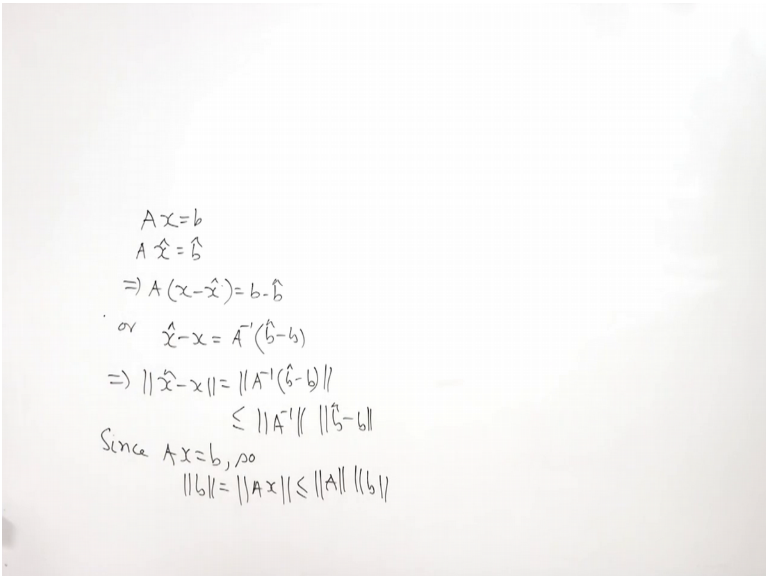

So, we will have A times x minus x cap equal to b minus b cap or a times x cap minus x equal to b cap minus b. And therefore, norm of. So, or I can write it as x cap minus x equal to A inverse b cap minus b and this implies that norm of x cap minus x which is equal to norm of A inverse b cap minus b and this implies that norm of x cap minus x which is equal to norm of A inverse, b cap minus b since this norm is subordinate matrix norm I can write it as less than or equal to norm of A inverse into norm of b cap minus b. So, again we have A x equal to b since A x equal to b. So, norm of b which is equal to norm of A x, and norm A x less than or equal to norm of A into norm of x. So, we have this again by using the fact that, we have subordinate matrix norm. So, this second in inequality norm of b less than or equal to norm of A into norm of x gives you norm of x greater than or equal to norm of b over norm of A.

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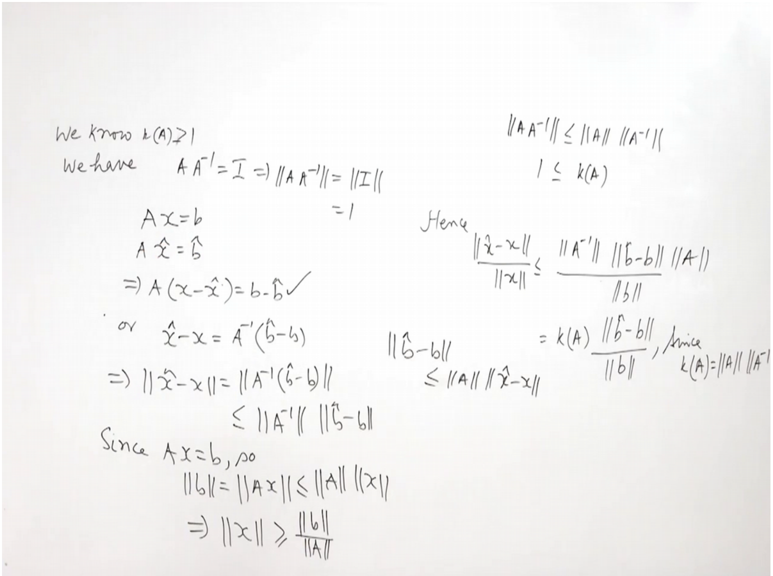

So, now from here we can say norm of x cap over norm of x, hence norm of x cap minus x norm of x over norm of x is less than or equal to norm of A inverse into norm of b cap minus b, divided by norm of b into norm of A. So, this is what we have, this is norm of A we will come here now we know that for any norm k is the condition number of a is equal to norm of A into norm of A inverse we will have, k A times norm of b cap minus b divided by norm of b where k denotes the condition number of A. So, we can see that the relative error in x is less than or equal to condition number of A into relative error in b.

Now let us again notice that norm of b cap minus b is less than or equal to norm of b cap minus b from this equation. From this equation norm of b cap minus b is less than or equal to norm of A into norm of x cap minus x again using the property of matrix norm and x is equal to A inverse b because A x is equal to b. So, norm of x is less than or equal to norm of A inverse into norm of b.

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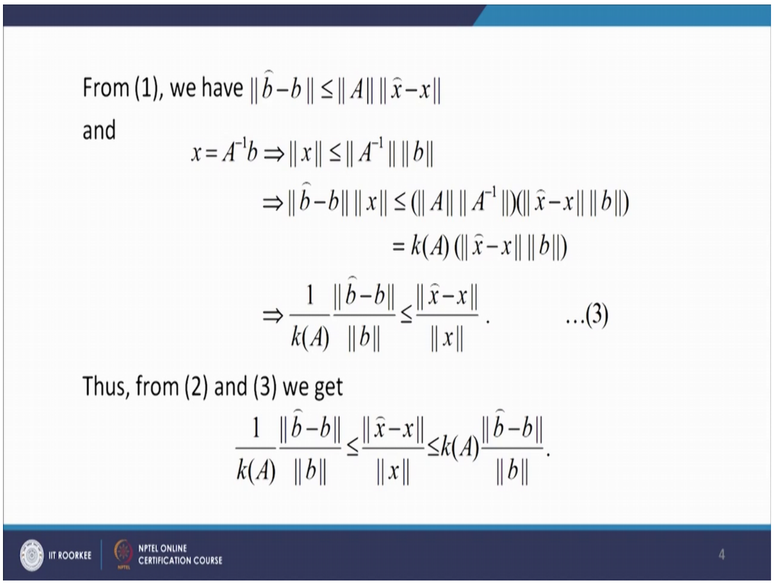

And therefore, a we can multiply the tool equalities and we get norm of b minus b cap into norm of x less than or equal to norm of A into norm of A inverse, into norm of x cap minus x into norm of b again k A equal to norm of A into norm of A inverse. So, we get k A times norm of x cap minus x into norm of b. And dividing by k A because we know that remember we know that k A is always greater than or equal to 1; see for any matrice subordinate matrix norm we know that, k A is greater than or equal to 1 the condition number of a greater than equal to why because we know that we have A into A inverse equal to identity matrix. So, norm of A into A inverse is equal to norm of I and norm of I we know is equal to 1. p norm of I is equal to 1 and since the we have the subordinate matrix norm.

So, norm of A A inverse is less than or equal to norm of A into norm of A inverse by definition condition number of A is norm of A into norm of inverse and this is norm of I norm of I is equal to 1. So, one is less than or equal to condition number of A. So, condition number of A is always greater than or equal to 1 and therefore, we can divide by k here when we divide by k we get 1 over k norm of b cap minus b upon norm of b less than or equal to norm of x cap minus x divided by norm of x. And now we combine this inequality which is number 3 to with the inequality which is number 2, which we proved earlier and we get one over k A into norm of b cap minus b divided by norm of b less than or equal to norm of x cap minus x divided by norm of norm x, less than or equal to k A times norm of b cap minus b dived by norm b. Now let us see look at the inequality and see other what it means; we have proved that norm of x cap minus x that is relative error in x is less than or equal to condition number of A into relative error in b.

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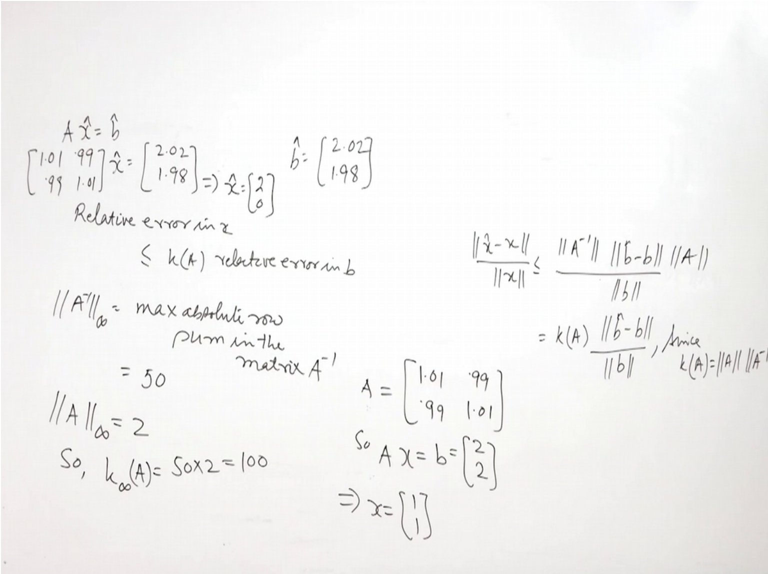

So, the condition number here plays a significant role in getting the relative error in x now.

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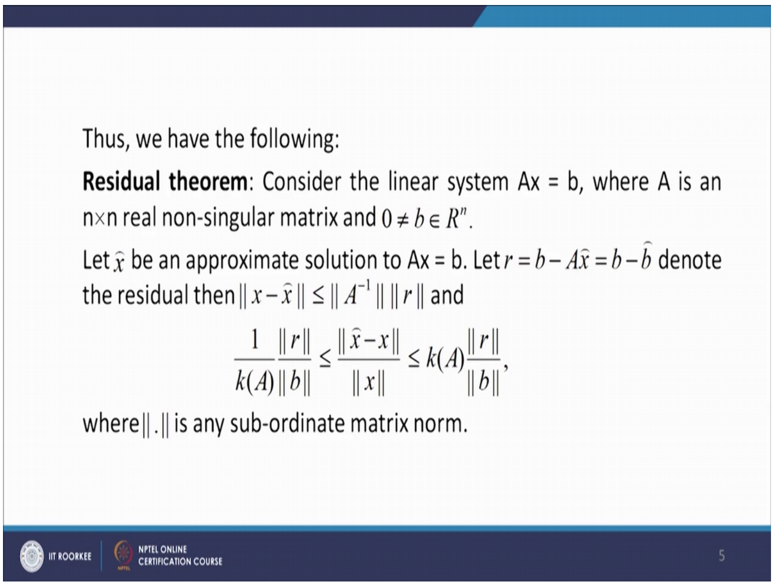

So, actually we have prove the following theorem; what we have proved is we consider any linear system A x equal to b, where A is any real nonsingular n by n matrix, and we took b to be any nonzero vector in r n, x cap we considered as an approximate solution to A x equal to b, where r is equal to b minus A x cap we took A x cap equal to b cap.

So, let us say r is equal to b minus b cap b the residual denote the residual, than norm of x minus x cap be proved that it is less than or equal to norm of A inverse into norm of b minus b cap. And we further showed that one over k A times norm of r that is norm of b minus b cap over norm of b less than or equal to norm of x cap minus x upon norm of x, less than or equal to k A times norm of r into y norm of b cap minus b divided by norm of b where norm is any subordinate matrix norm. So, this is what we have proved is actually the residual theorem. So, now, and this r which is b minus b cap is actually called the residual.

So, now we let us see the condition number of the matrix A determines how much the relative error on the right hand side vector can be amplified.

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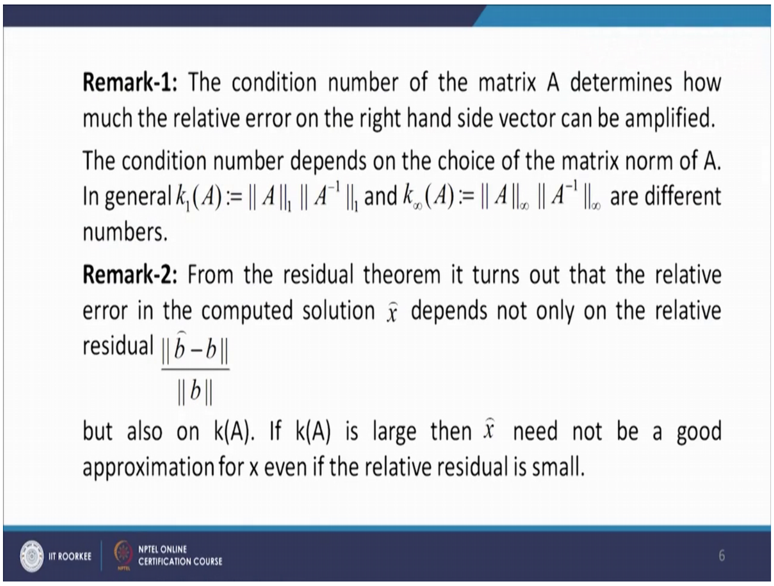

Because right hand side the relative error in b is multiplied by k A. So, this k A decides how much it will be amplified its value actually tells us, how much the right hand side is going to be amplified. The condition number depends on the choice of the matrix norm in general we denote k 1 a 1 norm in a one in one norm k infinity writer for infinity norm, and they are different numbers in. From the residual theorem it turns out that the relative error in the computed solution x depends not only on the residual relative residual norm of b cap minus b divided by norm of b.

But also on the condition number of a that is k, if k is large then x cap need not be a good approximation for x even if the relative residual is a small.

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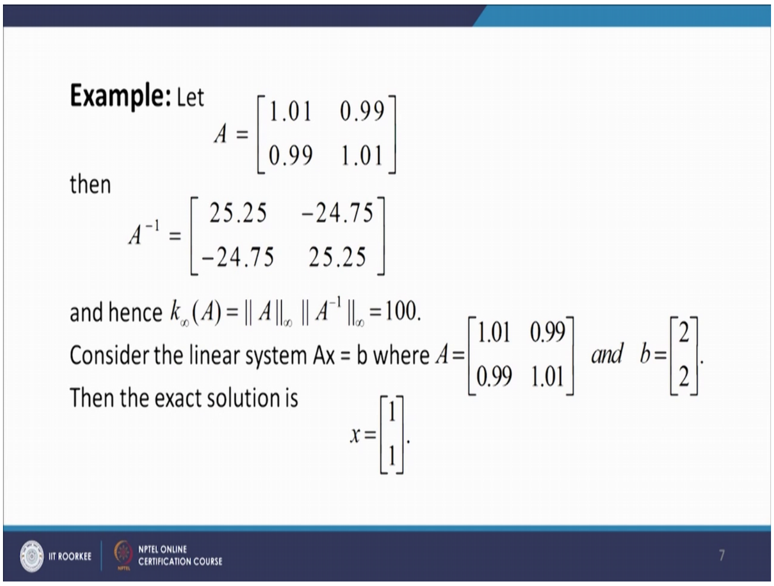

Let us look at an example on this suppose we consider the matrix A equal to 1.01, 0.99, 0.99, 1.01 which is 2 by 2 matrix, then the inverse of this matrix is 25.25 minus 24.75 minus 24.75, 25.25.Here we are finding the condition number in the infinity norm. So, condition number in the infinity norm is norm of A infinity into norm of A inverse infinity because finding condition number in the 2 norm is difficult. So, here we have considered infinity norm. So, norm of A infinity into norm of A inverse infinity; and norm of A infinity is equal to maximum absolute row sum in the matrix A.

If you look at the matrix a the absolutely row sum in the first row 25.25 plus 24.75 which is equal to 50 and in the next row the absolute row sum is 24.75 plus 25.25. So, absolute row sum is equal to 50. So, in both rows the absolute row sum is 50 and therefore, the maximum is 50.

So, this is equal to 50 and this is equal to a norm of A inverse, because this is what I was talking about the inverse matrix a inverse. So, norm of A inverse infinity A is equal to 50 of the matrix A inverse in the case of A in the case of a similarly the maximum absolute row sum is 2, because 1.11 plus 0.99 is 2 and in the second row also 0.99 plus 0.01 is 2. So, norm of A infinity is equal to 2 and so, the condition number in the infinity norm of the matrix A is 50 into 2. So, we get 100 ok.

So, we have we got the condition number in the infinity norm, let us consider the system norm A x equal to b where a is the 2 by 2 matrix which we have considered, and b and let us take to b 2 2. Then A is 1.09 and 0.99 and here we have 991.01 this is our this is our x A matrix . So, A x equal to b, where b is b is 2 it means that x is equal to 11 we can easily see that x is equal 11 here which is the exact solution.

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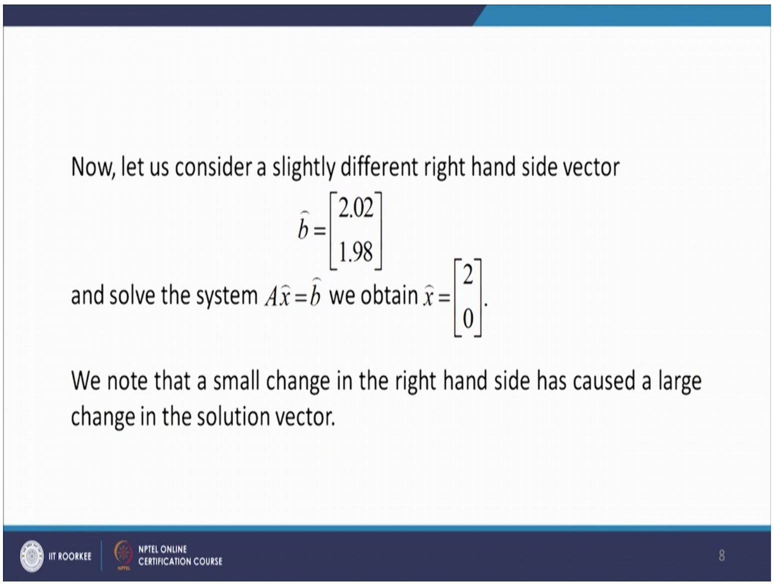

Slightly different right hand side vector, let us take a slight perturbation matrix b cap equal to 2.02 and 1.98 should be kept we have taking to be equal to now let us consider again the A x cap equal to b cap ok.

So, we have 1.09, 0.999 and here we have 0.99 and 1.01 into x cap is equal to 2.02 and 1.98. So, here we get x cap to be equal to 1.01 So, you can see that here x cap is equal to 2 0 vector. So, when we make a slight change in the right hand side vector b, there is a drastic change in the vector x earlier vector x was 11.

Now, the vector new vector x is 2 0. So, there is a small change when there is small change in the right hand side, it has caused a large change in the solution vector and this is because the condition number is large the condition number is 1000. So, it has played a significant role in in the in amplifying the error in the solution vector thus a small residual.

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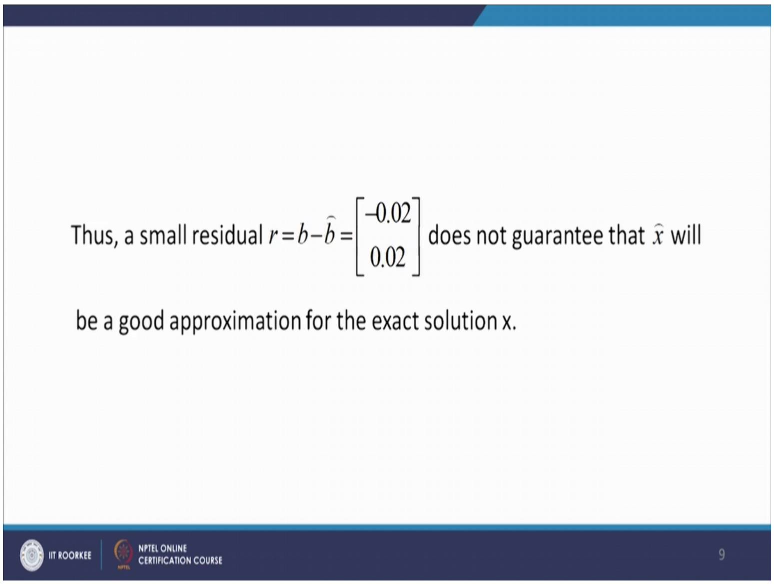

R equal to b minus b cap, which is minus 0.02 and 0.02does not guarantee that x cap will be a good approximation for the exact solution.

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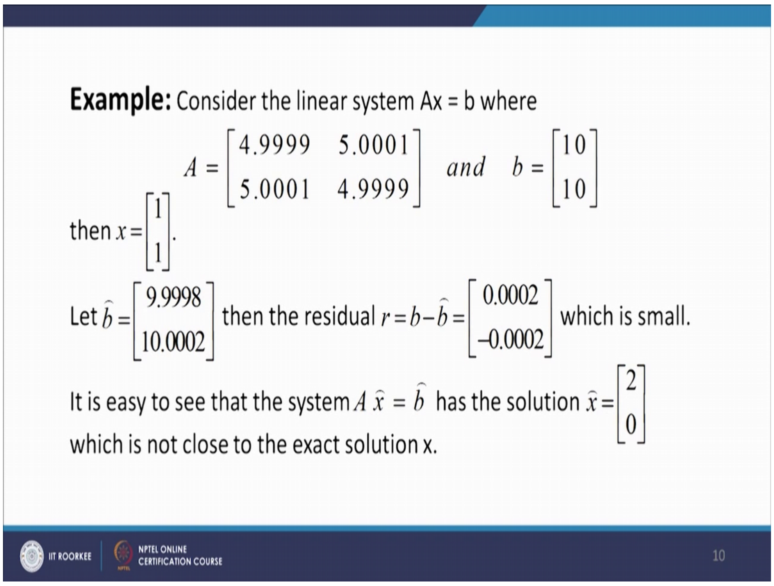

Now let us consider next linear system another example these are where A is again 2 by 2 matrix, 4.99,99 5.001, 5.0001 4.9999 and b is equal to 10 by 10 ,then you can see here that x is equal to one, one when we take A x equal to 1.

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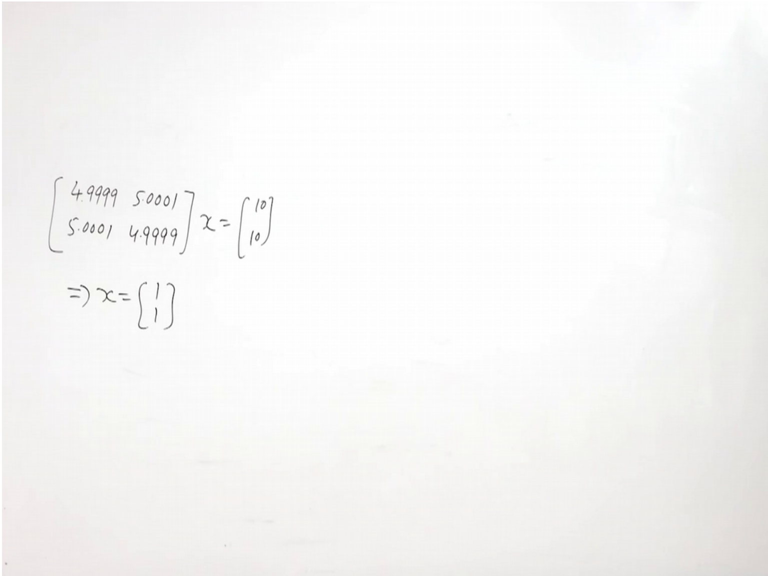

So, A is 4.999, 5.0001 and then we have here 5. 00010, 4.999 and here we have x and here we have 10 10; some of these 2 is 10, and sum of these 2 is also 10. So, we can see that x is equal to 11. So, we can easily find the solution vector here, now let us b cap we take to be 9.998 and 10.0002, then the residual r equal to b minus b cap which is 0.0002 minus 0.0002, which is again a small and we can see that if you take this instead of 10 10 you take now 9.9998

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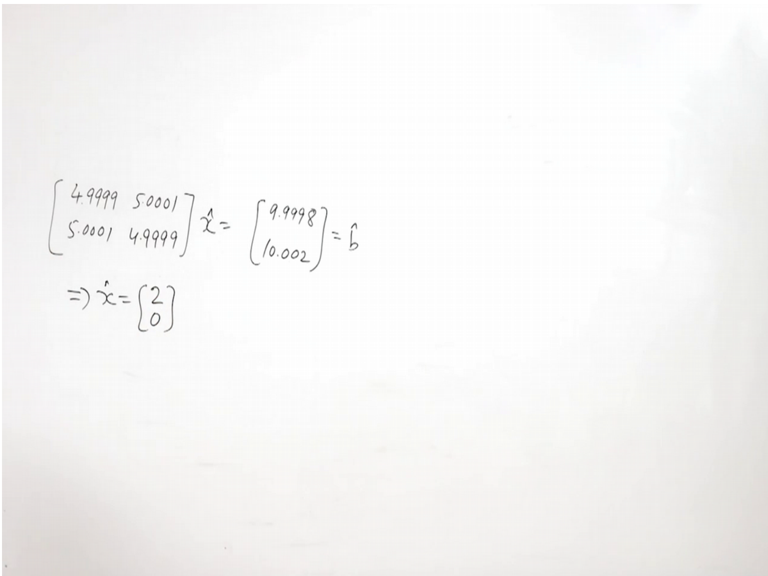

10.002 set off this we take this x then x cap let me write here x cap. So, this is now a b cap than x cap is equal to 2 0 So, we can see that when there is a small change in the right hand side vector b, there is a drastic change in the solution vector x earlier it was 11.

Now, it is 2 0 and which is not close to the exact solution x.

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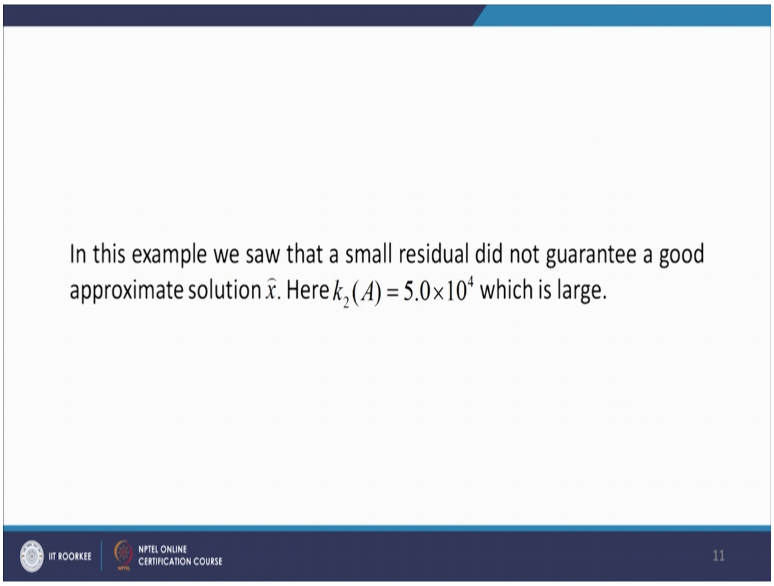

So, in this example we have seen that small residual did not guarantee a good approximate solution x cap. Here if we find condition number in 2 norm k 2 a then k 2 a turns out to be 5 into 10 to the power 4 which is large. So, again the condition number has played a role in the relative error in oxidative error in x is not a small, even if the relative error in the solution when the right hand side vector b is small.

So, the fact that the matrices now here we notice one more thing the matrices that we have considered in the examples 1 and 2 both of them have large condition numbers; in the first case be found in condition number in the infinity norm it was 100, in the second case be found the condition number in condition number in the case of and 2 norm it came out 5 into 10 to the power 4. So, which are which is again large.

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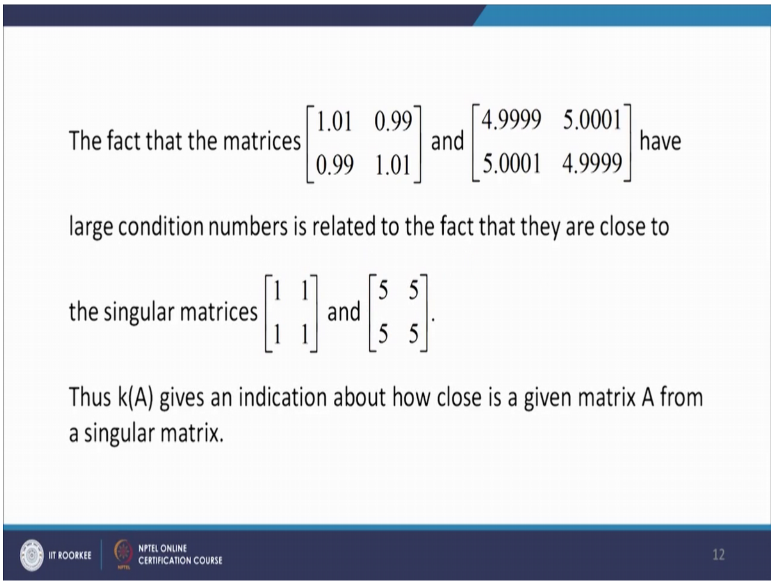

So, this large condition numbers is related to the fact that they are close to the singular matrices actually condition number; actually is very important to see whether a given non-singular matrix is close to a singular matrix.

So, if condition number is less than the larger the condition number, the closer is the non-singular matrix to the singular matrix. This we shall be seeing in our next lecture when we discuss nearness to singularity. So, the condition number in the 2 matrices which we consider is large, is related to the fact that they are close to the singular matrices 11 11 and 5 5 5 5 thus condition number gives an indication actually how close is a given matrix non singular matrix A from a singular matrix.

To this, we are going to see when we discuss the next in the next lecture the nearness to similarity. The condition number plays a significant role in deciding whether a given non singular matrix is close to a singular matrix. So, with that I would like to conclude my lecture.

Thank you very much for your attention.