Numerical Linear Algebra
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Lecture - 38
Sensitivity Analysis- II

Hello friends I welcome you to my next lecture on a sensitivity analysis. So now, we are going to assume that there is perturbation only in one data that is either A or b first we shall consider the perturbation only in b.

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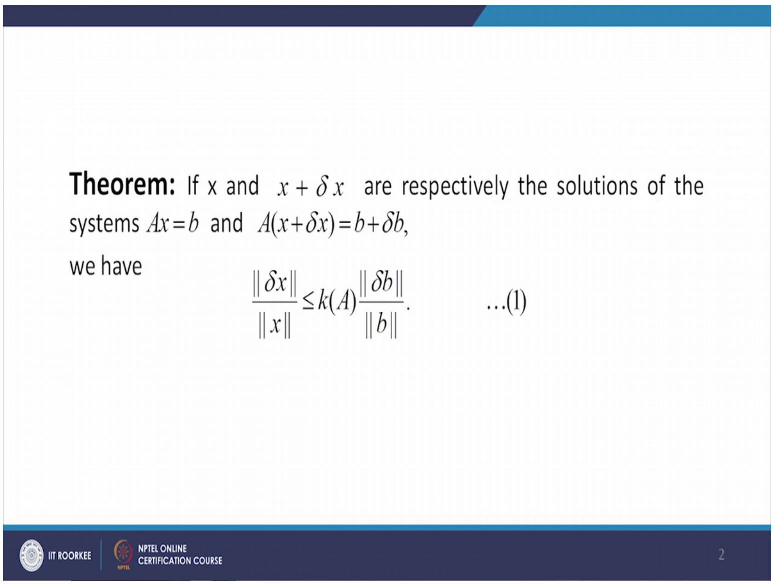

So, suppose x and x plus delta x are respectively the solutions of the systems A x equal to b and A x plus delta x equal to b plus delta b. So, you can see that we are we know exactly there is perturbation only in b which we are taking has delta b. Then we shall see that norm of delta x. So, over norm of x is less than or equal to the condition number of A into norm of delta b over norm of b.

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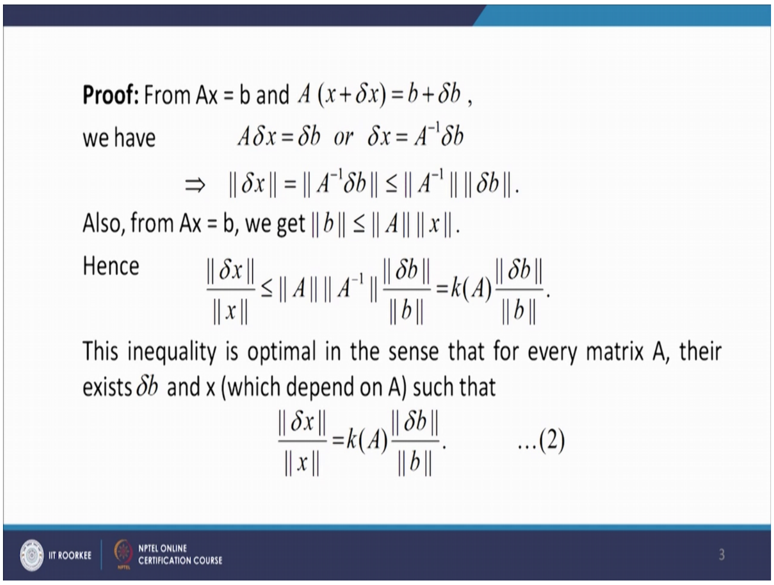

So, let us see how we prove this result we have to prove see we have A x equal to b.

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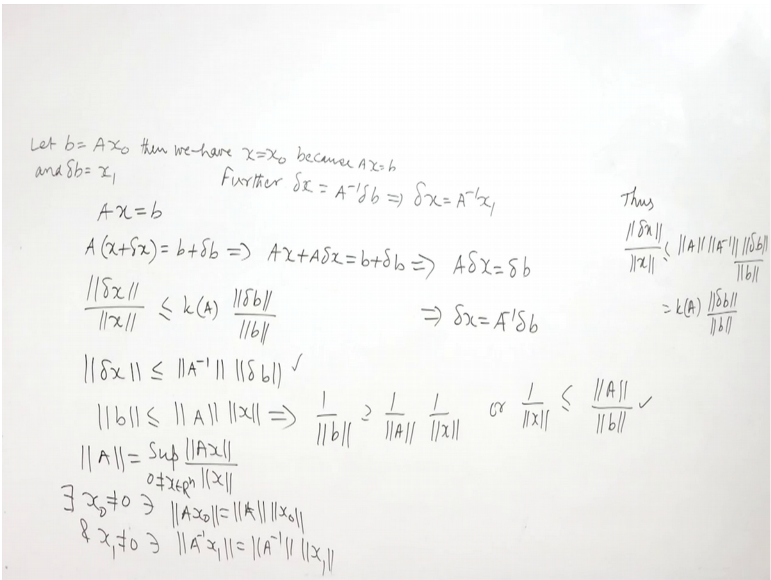

And we have A times x plus delta x equal to b plus delta b we have to show that norm of delta x over norm x is less than are equal to k A times norm of delta b over norm of b. See we have A x equal to b and A x plus delta x b plus delta b and therefore, using A x equal to b we can say that this implies A x plus delta x equal to b plus delta b.

Now, using A x equal to b we have A delta x equal to delta b, or we can say delta x equal to we are assuming A to be real non-singular n by n matrix. So, this implies delta x equal to A inverse delta b and therefore, norm of delta x will be equal to norm of A inverse delta b, but since the norm is subordinate matrix norm. Norm of A inverse delta b will be less than or equal to norm of A inverse into norm of delta b again from A x equal to b norm of b is equal to norm A x which is less than or equal to norm of A into norm of x. So, what we have norm of delta x norm delta x is less than or equal to norm of A inverse into norm of delta b and we have A x equal to b. So, norm of b is less than or equal to norm of A into norm x.

So, combining the 2 equations what we have, a way we have norm of now thus this gives you 1 upon norm b greater than or equal to 1 upon norm of b greater than or equal to 1 upon norm of A into 1 upon norm of x or we can say 1 upon norm of x is less than or equal to norm of A divided by norm of b. So, from this equation and this this equation. Norm of delta x over norm of x is less than or equal to thus norm of delta x over norm of x is less than or equal to norm of A into norm of A inverse into norm of delta b divided by norm b, which is equal to k A times norm of delta b divided by norm b. So, this is the proof.

Now, this inequality is optimal; that means, it cannot be improved further and why it cannot be proved let us. So, that for every matrix the quality can be obtained for every matrix A there exist delta b and x such that norm of delta x over norm of x is equal to k A into norm of b over norm of delta b over norm of b. So, that is why we say that this result is optimal and it is a better estimate then the estimate that be obtained in the previous lecture, the relative error in x be found and it was k over 1 minus k times norms of delta over norm of A into relative error in A plus relative error in b.

So, it is vector estimate than that so let us apply a property of the subordinate matrix norms we know that norm of A, norm of matrix norm of A is equal to supremum of norm of A x over norm of x where 0 is not equal to x belonging to Rn. So, by a property of the subordinate matrix norms there exist x naught not equal to 0.

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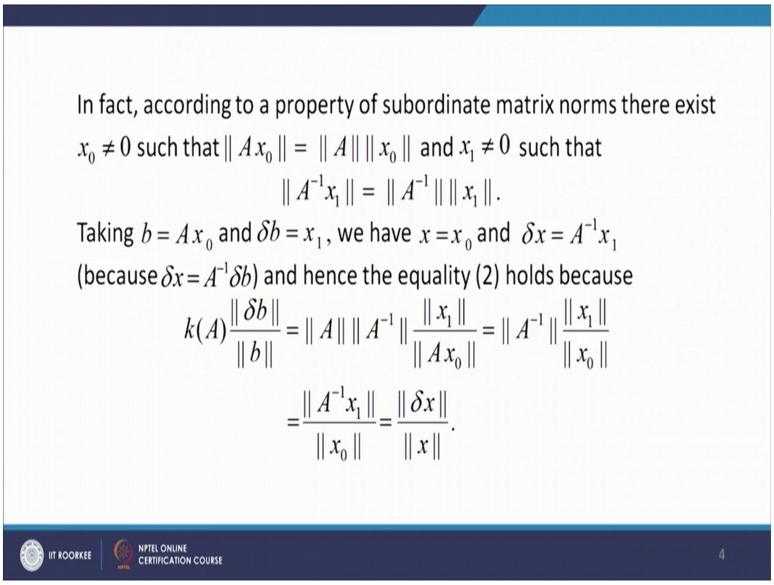

So, that norm of A x not is equal to norm of A into norm of x naught and also x 1 not equal to 0. So, that norm of A inverse x 1 equal to norm of A inverse into norm of x 1. So, we have there exist x naught not equal to 0 such that norm of A x naught is equal to norm of A into norm of x naught and x 1 not equal to 0 such that norm of A inverse x 1 equal to norm of A inverse into norm of x 1.

Now, what we do is let us take b to be equal to A x naught. Let us take b to be equal to A x not and delta b the perturbation in b, and delta b equal to x 1. Then we have x equal to x naught because then we have x equal to x naught because A x is equal to b, because we have assumed A x equal to b and delta x is equal to A inverse x 1 because delta x is what delta further delta x by our notations delta x is equal to A inverse delta b. So, this gives you a delta x equal to A inverse x 1 delta b we have assumed to be equal to x 1.

Now, now let us take k A into we have to prove that there is equality here. So, let us take k A into norm of delta v over norm of b.

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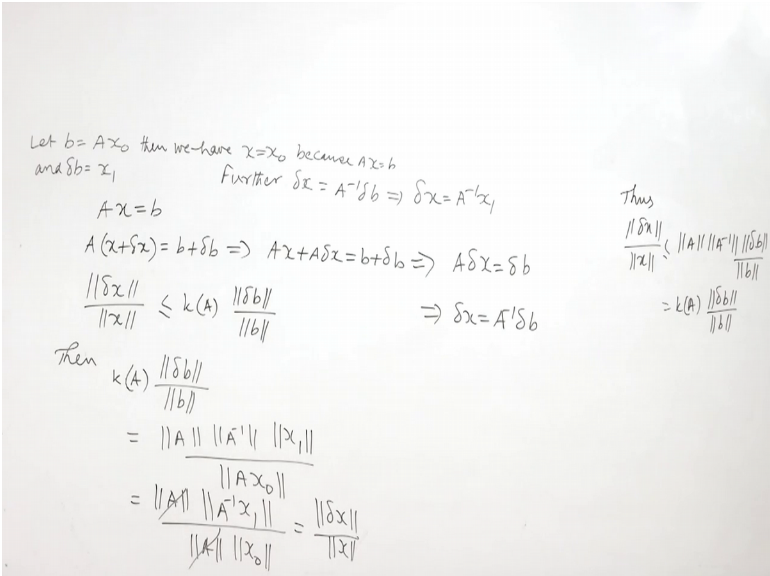

So, then k A into norm of delta b over norm b. K is norm of A into norm of A inverse by definition k A equal to norm of A into norm of A inverse and norm of delta b norm of delta b equal to norm of x 1 and norm of b norm of b is equal to norm of A x naught. Now we have said that there exist x naught set their x naught is norm of A x is not equal to norm of A into norm of A x naught. So, this and this is what norm of A inverse x 1 is equal to norm of A inverse into norm of x 1. So, this is norm of A into norm of A inverse x 1 in the numerator norm of A inverse into norm of x 1 is a norm of A inverse x 1.

But A inverse x 1 is norm of delta x. So, this is norm of delta x and this is norm of A into norm of x naught. So, this cancels with this and this is norm of A inverse x 1 is delta x and norm of x naught is equal to norm of x. So, this is this is how we get the equality here and therefore, we can say that the estimate is optimal.

Now but we know that the upper bound in this a in the quality while optimal is in general very pessimistic there are examples where we see that this optimal is very pessimistic. So, we find another estimate in the other case what we have we a assume that there is perturbation in the matrix a b is known to us exactly. So, if x and x plus delta x are respectively the solutions of the system A x equal to b and A plus delta A into x plus delta x equal to b then we have this estimate.

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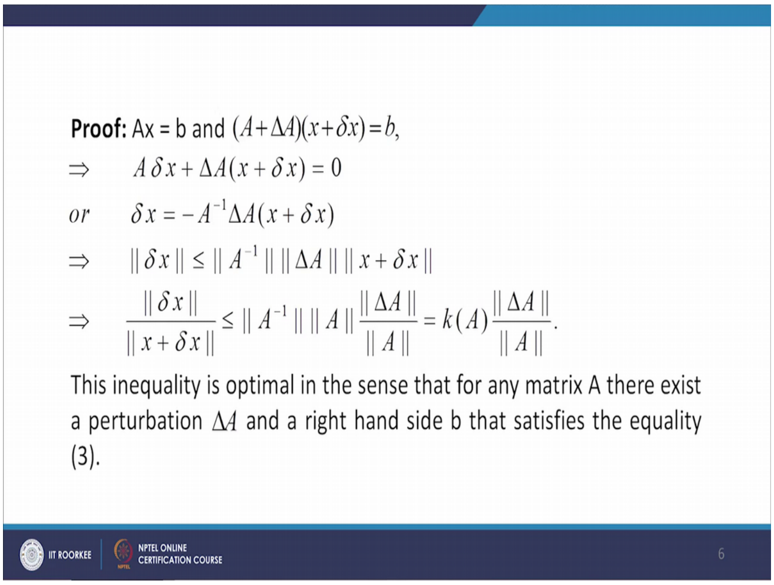

So, norm of delta x over A x is equal to b.

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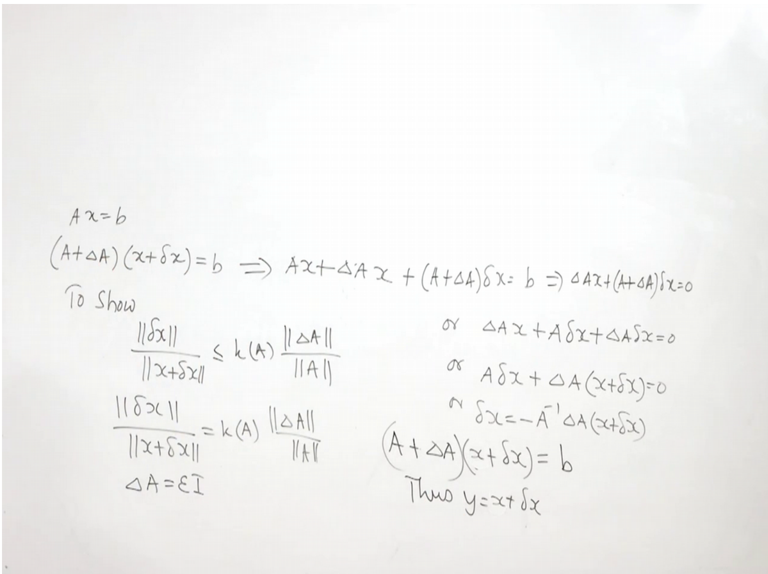

And we have in A plus delta A into x plus delta A x equal to b, b is known to us exactly there is perturbation in the matrix A which is given delta A. Now so this what we are given and we have to show norm of x over norm of x plus delta x, that is the relative error in x the relative error in x is relative to x plus delta x. So, this is less than or equal to k A times norm of delta A over norm of A. Now let us see how we prove this so norm of delta x over norm of x plus delta x is less than or equal to k A into norm of delta A over norm of p.

So, we are getting relative error in x with respect to x plus delta x, now what we have? Let us prove this so A x is equal to b we are given and A plus delta A into x plus delta A x equal to b. So, what we have using A x is equal to b here what we get? We have A plus A x plus delta A into x and then we have plus A plus delta A into delta x equal to b. Now A x equal to b gives you delta A x plus A plus delta A into delta x equal to 0. So, this is what we get and therefore, we have delta x equal to minus A inverse.

So, we can write from here we can find from here so this is a I may say or delta A x plus A delta x plus delta A into delta x equal 0. So, we can write it as a. So, I can write it like this or a delta x and then we have delta A times x plus delta x equal to 0. So, or delta A x equal to minus A inverse delta A x plus delta x we can put it like this. So, minus A inverse into delta A x plus delta x and then we can take norm, norm of delta x will be less than or equal to norm of A inverse into norm of delta A into norm of x plus delta x. Now we can divide it by norm of x plus delta x. So, norm of delta x over norm of x plus delta x is less than or equal to norm of A inverse into norm of A and then we multiply by norm of A and divided by norm of A.

So, norm of A inverse norm of A into norm of delta A divided by norm of A which is k A into norm of delta A over norm of A. So, this is a very simple proof and this in equality is optimal in the sense that for any matrix a there exist a perturbation delta A and right-hand side b that satisfies the inequality 3

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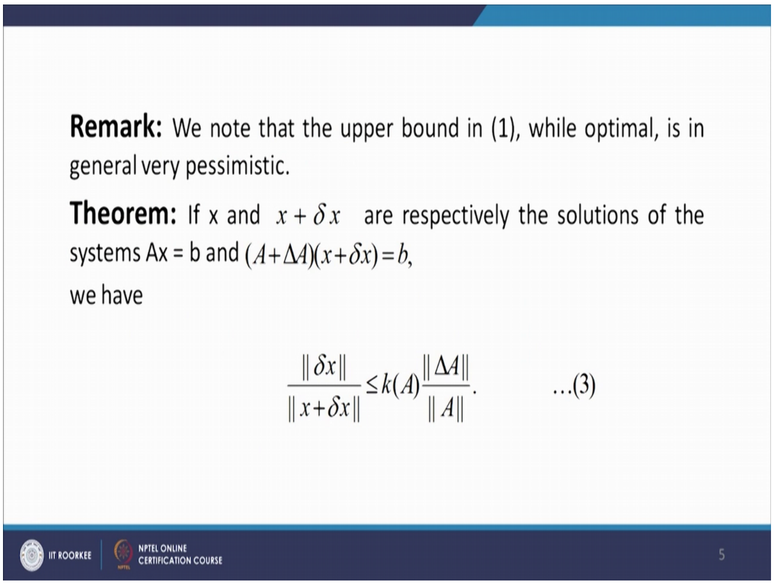

So, equality 3 means here so, this that satisfies. So, we can have this equality. We can take A delta A and b in such a manner that there exist equality. Now let us see how we do this. So, again using a property of subordinate matrix norms.

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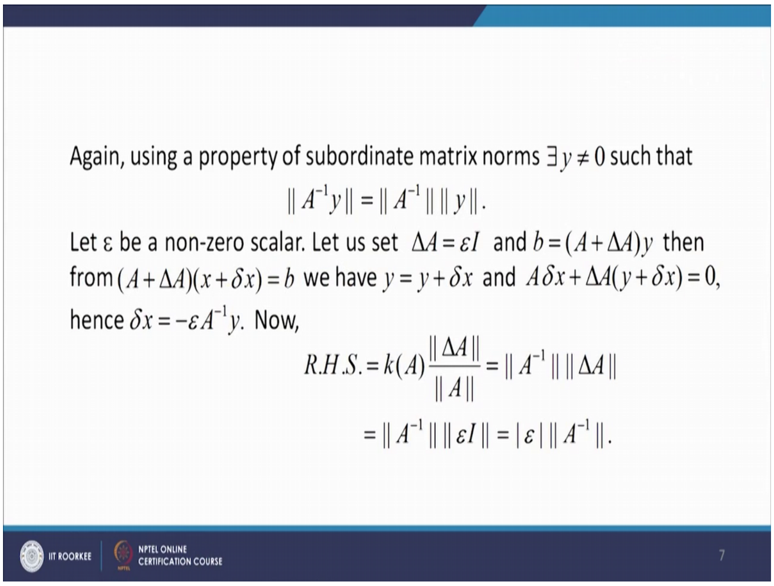

We can say that there exists y not equal to 0. So, that norm of A inverse y is equal to norm of A inverse into norm of y.

Now, let us take epsilon to b any nonzero scalar and let us said delta A equal to epsilon into I, I is the identity matrix. So, let us take the perturbation in A to b epsilon into I and let us choose b to be equal to A plus delta A into y. So, we are taking delta A and b in such a manner that we will have equality. So, delta A b to take equal to epsilon into I and b we take to be equal to A plus delta A into to y. Then from A plus delta A into x plus delta x equal to b. What do we get we get? Y equal to y plus delta x. And so, a from A plus delta A into y plus delta x.

So, this is A plus delta A into x plus delta x equal to b. So, what we get A plus delta A into x plus delta x equal to b and so we have a y equal to thus we have y equal to we are taking y equal to x plus delta x the y equal to x plus delta x. So, again using a property of subordinate matrix norms there exist y not equal to 0. So, that norm of A inverse y equal to norm of A inverse into norm of y. Now let us take epsilon to be a nonzero scalar and let us said delta A equal to epsilon into I.

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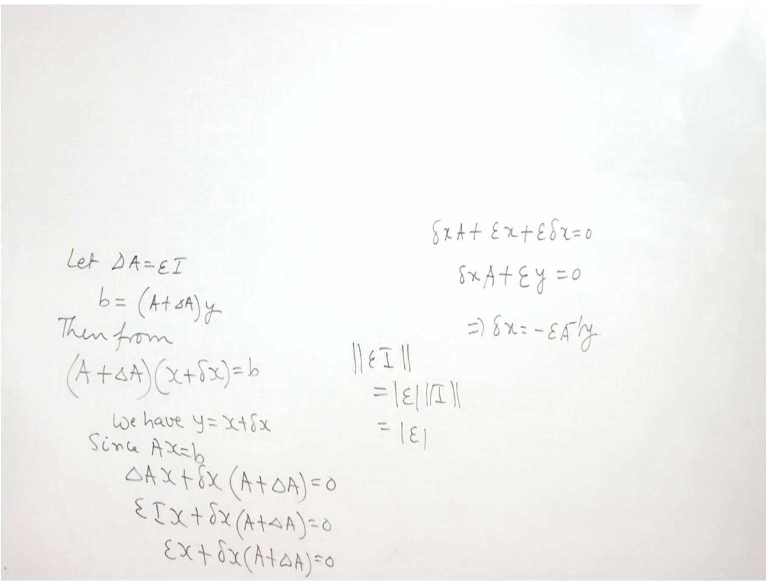

Also let us take b to be equal to A plus delta A into y, then from A plus delta A into x plus delta x equal to b, what do we notice is that if you compare these 2 equations we have y equal to x plus delta x b equal to A b we have chosen b to be A plus delta A into y, but we have A plus delta A into x plus delta x equal to b equation. So, we have y equal to x plus delta x and not using A x equal to b, since A x equal to b we get delta A into x plus delta x times A plus A x plus A delta x A plus delta A into delta x. A plus delta A into delta x is here A plus delta A into x gives A x plus delta A into x, but A x gets cancelled with b. So, we have this is equal to 0 and then what we have is we can get delta x the value of delta x to be equal to minus epsilon A inverse y. See delta is equal to epsilon into I into x plus delta x times A plus delta A equal to 0.

So, epsilon x we have y equal to x plus delta x. Y equal to x plus delta x plus delta x in A delta x A delta x plus delta A times delta A. So, this I can also write it as A delta x plus delta A times delta A is here also now. So, also, I can write it as delta x into A and we have epsilon x and then we have epsilon x into delta A, delta A is equal to what delta b we have taken to be epsilon into I. So, what do have front I. So, dealt x into a delta x into epsilon into I. Actually, I have written already delta x into delta x into A plus epsilon x plus delta x into the epsilon the delta A. So, we have delta A equal to epsilon into I.

So, epsilon delta x from here epsilon I delta x delta x into this. So, what do we get now delta x (Refer Time: 20:41) A plus epsilon we are getting x times minus A epsilon A inverse y equal to epsilon delta x. So, delta x into A plus epsilon times x plus delta x is equal to y. So, y is equal to 0 and therefore, I get delta x is equal to minus epsilon into A inverse y. So, what we get here is that A delta Actually we are replacing by epsilon into y and when we replace delta x equal to epsilon into y we arrived at this and then be combined the term this term and this term.

So, x plus delta x y. So, we get delta x equal to minus epsilon A inverse y and then what we have right hand side is k into norm of delta A over norm of A k is norm of A inverse into norm of A. So, norm of A gets cancelled and we get norm of A inverse into norm of delta A and delta A is epsilon into I. So, norm of A inverse into norm of epsilon I. Now norm of epsilon I is equal to mod of epsilon into norm of I and we know that norm of I is always equal to 1. So, this is mod of epsilon we have. So, mod of epsilon into norm of A inverse we have.

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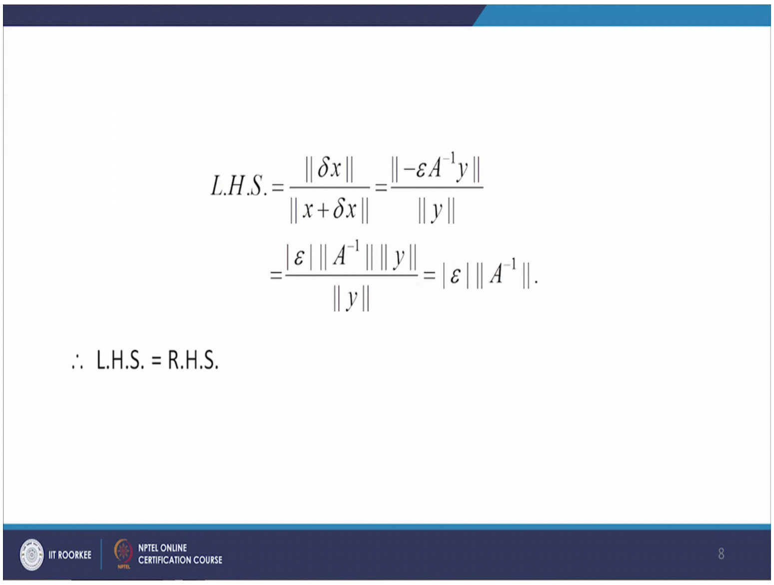

And then the lab 10 5 it is norm of delta it is over norm of x plus delta x and delta x we have seen is equal to minus epsilon into A inverse y. So, norm of minus in epsilon into A inverse y and x plus delta x is y x plus delta Axis y plus y x plus delta x is y. So, norm of in the numerator we will have mod of epsilon into norm of A inverse into norm of y and the denominator is norm of y.

So, we get mod of epsilon into norm A inverse and therefore, left hand side and right-hand side are equal. So, this is how we prove that the (Refer Time: 22:57) quality is or the up upper bound is that estimate is optimal now the f theorem shows that the error in x.

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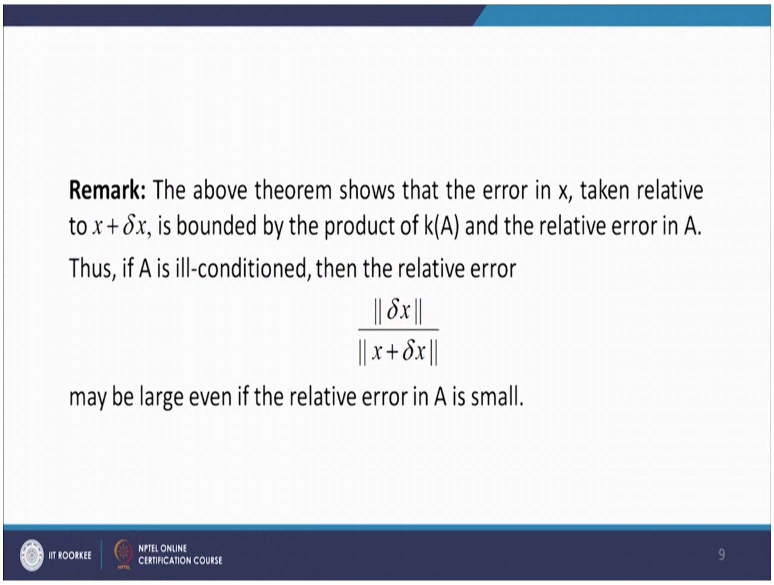

Taken relative to x plus delta x is bounded by the product of the condition number and the relative error in the A, thus if a is ill conditioned the relative error may be large even if the relative error in a is small.

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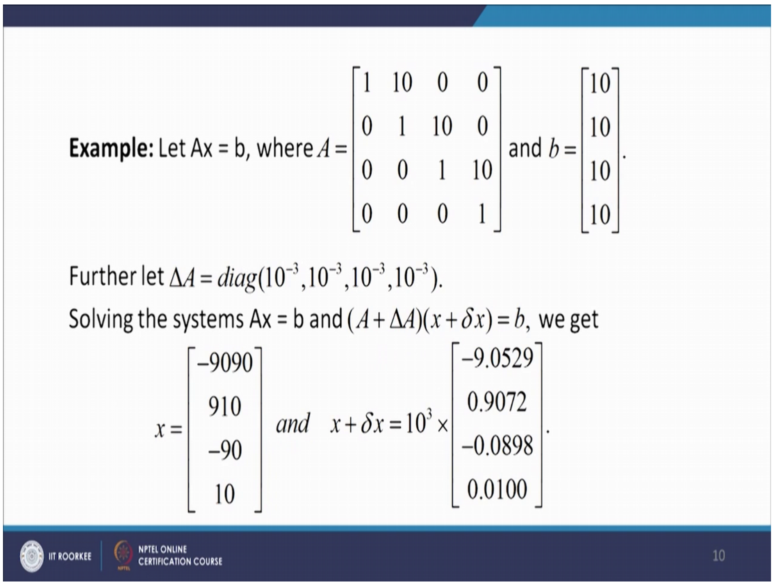

Now, we have taken an example on this theorem. So, let us take A x equal to b where is this 4 by 4 matrix and b is this column matrix and let us take the perturbation in a given by delta A to be diagonal matrix 10 to the power minus 3 all the diagonal elements are equal 10 to the power minus 3 each.

So, there is this the perturbation matrix, now we can solve the system A x equal to b and A plus delta A into x plus delta x is equal to b the values of x and x plus delta x turn out to be this x is this column matrix and x plus delta x is 10 to the power 3 into this column matrix.

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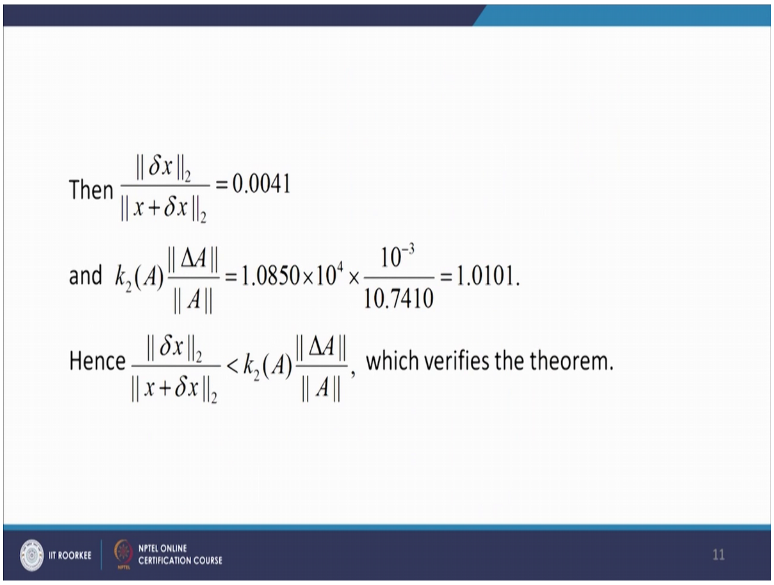

Now when we calculate the relative error in x related to the x plus delta x that into norm, norm of delta x to over norm of x plus delta x to it turns out be 0.0041 and when we consider the find the condition number in 2 norm that is k 2 A what we get is 1.0850 and the relative error in A, k 2 A is 1.0850 into 10 to the power 4 and norm of delta A upon norm of A is 10 to the power minus 3 divided by 10 plus 10.7410 which is actually 1.0101. And so, we can see that norm of delta x 2 over norm of x plus delta x 2 is strictly less than k 2 A into norm of delta A over norm of A, which verifies the theorem. With this I would like to conclude my lecture.

Thank you very much for your attention.