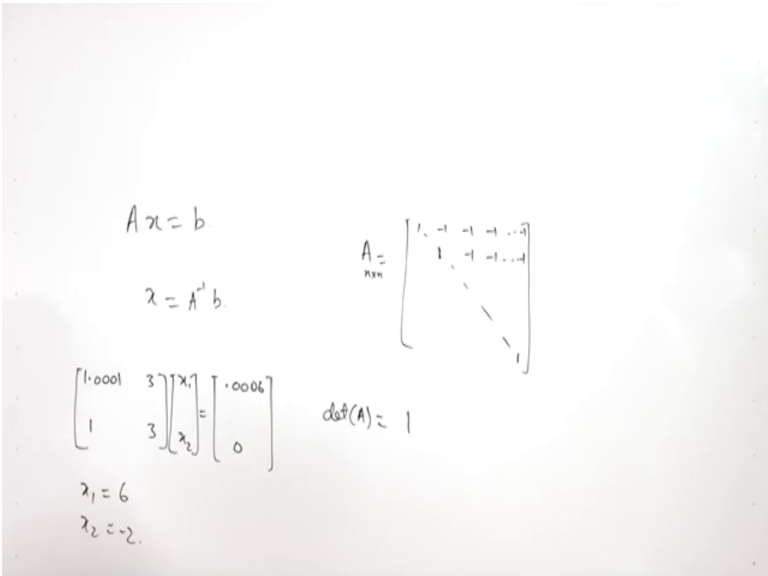
Numerical Linear Algebra
Dr. D. N. Pandey
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 36
Condition number of A matrix: Elementary properties

Hello friends. Welcome to the today's lecture. In this lecture, we will consider an important concept known as a condition number of A matrix. So, how this condition number of a matrix will help in numerical linear algebra. So, if you look at when we try to solve some kind of a linear system A x equal to b.

(Refer Slide Time: 00:47)



Then we when we try to solve this A x equal to b then the first important desirable property of this linear system is that this matrix A if it is say inverses exist or we can say that in a is invertible matrix then our solution can be written as x as A inverse b.

So, that is the basic result which we start with, right? So, if a is invertible we can say that x equal to A inverse b and we can calculate our solution by calculating the inverse effect matrix. Then we may think about that under what condition this matrix A is said to be invertible. Then or we say that the matrix A is non-singular. Then we have seen several result and one of the important result is that if the Eigen values of this matrix A is nonzero or it is other than 0, then we say that it is non singular matrix or we can say that determinant of a is nonzero we say that it is non-singular matrix, but this is may not be a very good criteria to check that a given matrix A is non-singular. The reason is that when you start collecting your data then your system A x equal to b and met entries of A and entries of b are coming through some kind of experiment. And it may happen while doing some experiment or during the experiment or before the collecting the data it may happen that there is some pollution or we can say some perturbation happened. And due to which a may be polluted or you can say that it is possible that the increase of a is not very, very accurate.

So, it may happen that though the matrix A is in current form is non-singular matrix or you can say that determinant of a is nonzero, but it still it may happen that a is very, very near to singular matrix or A is very close to singular matrix. So, how we identify that a given a matrix A is very near to a singular matrix or why we? Because if it is a singular matrix A then of course, this is not possible.

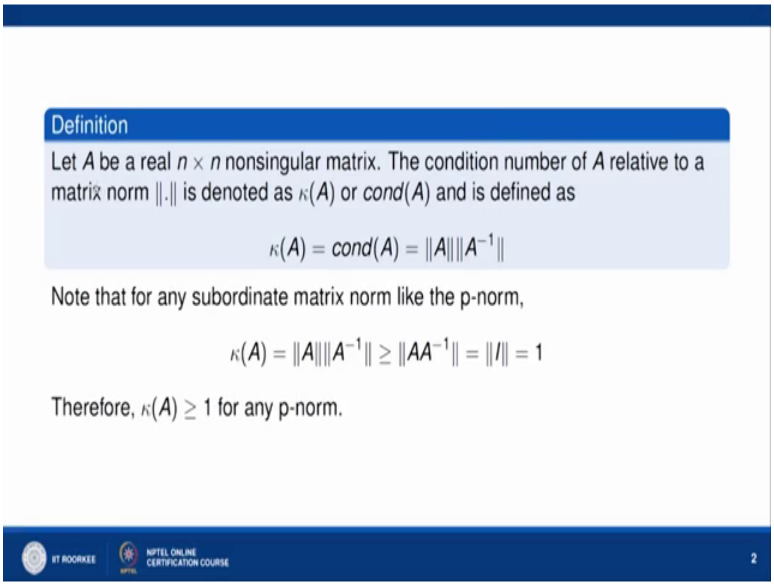
So, it means we want to know that the solution which it is written here x equal to A inverse b is reliable. Reliable in the sense that if a may have some kind of pollution, but still your solution should not worry much with corresponding to this pert perturbation in the matrix A. So, we try to see that how we check that a given matrix A and the system A x equal to b the solution is not sensitive to the perturbation involved in capital A and b. It means that that solution obtained like this is reliable for small perturbation that if even if there is a small perturbation your solution should not vary much.

So, that the concept of that sensitive sensitivity of linear equation or non-singularity of a given matrix is very much related to the particular number which is known as condition number of A matrix. So, we try to see that how this condition number will help you to identify that given a matrix is near to a singular matrix or a given a linear system this system is well behaved with respect to the perturbation. So, that we try to answer with the help of condition number for example, let us take a matrix A like this. So, Aa is a triangular matrix.

So, let us say that size of A is n cross n and your diagonal entries are one and rest all other entries are minus 1 and so on right? So, all other entries are this is one and this is minus 1 minus 1 and so on 1 minus 1 and all these things. And this is a very famous example and we will see that do the determinant of this matrix determinant of a since it is upper triangular matrix. So, determinant is calculated as the product of the diagonal element and it is coming out to be 1. So, it is clear that here this a is A non-singular matrix, but we simply will see that though it is A non-singular matrix, but it is very, very near to a similar matrix and if we perform our perform our ah. If you take this as the matrix involved here in system of linear equation then the solution is a very, very sensitive corresponding to the input data.

So, that we are going to see and we say that this condition number of this matrix A is going to be very, very large. So, let us first define; what is condition number and based on the definition, we try to find out the properties here.

(Refer Slide Time: 06:20)



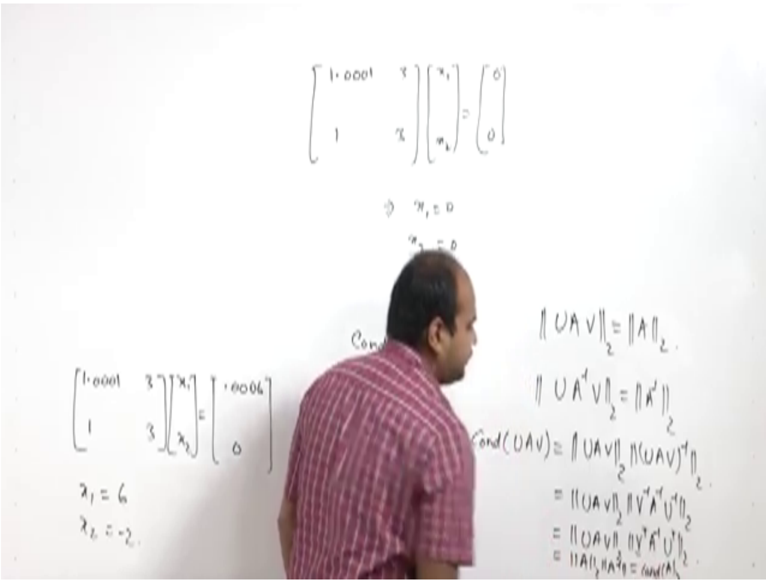
Let A be a real n cross and non-singular matrix. Then the condition number of A related to a given matrix norm this is denoted as kappa A and it is written as condition kind of A and is defined as norm of A into norm of A inverse. And we can check that if we take this norm as any subordinate matrix norm like say p norm then kappa of A which is given as norm of A into norm of A inverse is greater than or equal to norm of A A inverse. Now norm of A A inverse is nothing but I. So, norm of I is given as 1.

So, p norm of p norm of identity matrix is given as 1. So, this implies that in if we define condition number with respect to p norm, then kappa of A or condition number of A is bigger than or equal to for any p norm. Now let us again consider some example once we know what is condition number then consider this example here example is this let us say that we have 1.0001 and we have say 3 here and then it is 1 and this is 3. And we have say x 1 x 2. And here we have say values 3 and 1.

So, it is 0.00 say here say you can take as 6 and it is 0 here and we try to find out the solution of this. If you look at the matrix A and look at the entries of matrix A. So, first row is 1.00013 and second row is 1 and 3. If you look at the it is very, very close to a singular matrix and if you if you want to verify the solution of this metric is what you can say that solution is going to be 6, right and here it is minus 2. So, x 1 is equal to 6 here and x 2 is equal to minus 2. So, if you simply say that it is coming out to be 6 minus this and 6 into this minus 6 is going coming out to be this.

So, it is going to be solution of this linear system is coming out to be x 1 equal to 6 and x 2 is equal to minus 2. Now let us consider the related linear system. So, we have seen that if it is the linear system then solution is given as x 1 equal to 6 and x 2 is equal to minus 2.

(Refer Slide Time: 08:57)

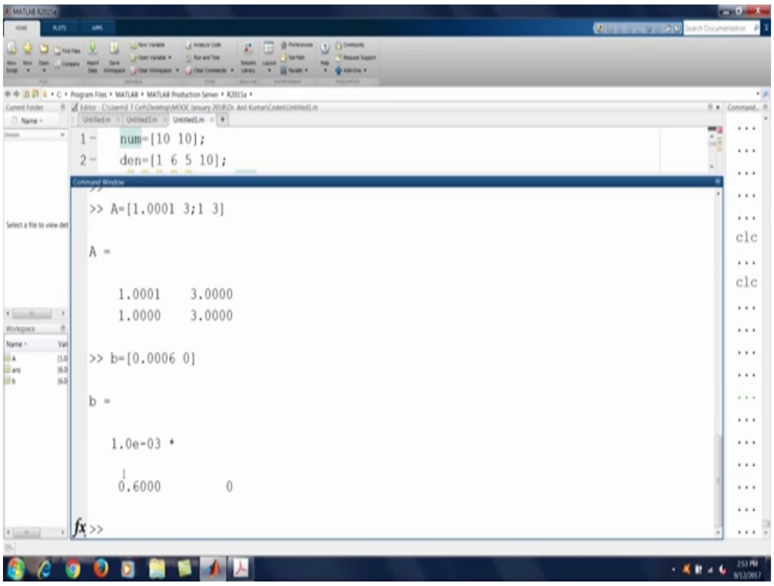


But if we consider this linear system. So, here matrix A is same x 1 x 2 is here the only difference here is that here b matrix is given as 0 0. Here it is 0.0006 and if you look at the solution of this system is given as x 1 equal to 0 and x 2 equal to 0.

So, we can see that with the difference between this system and this system is that the only small deviation in the first component of the matrix b. And if you look at the solution is 6 and minus 2 and solution here is 0 0. So, it means that by giving a small deviation in input data and we have a large change in the solution and ah, but if you look at the matrix and the matrix is non-singular, that we can check determinant is coming out to be nonzero. So, we can say that even if the matrix is non-singular.

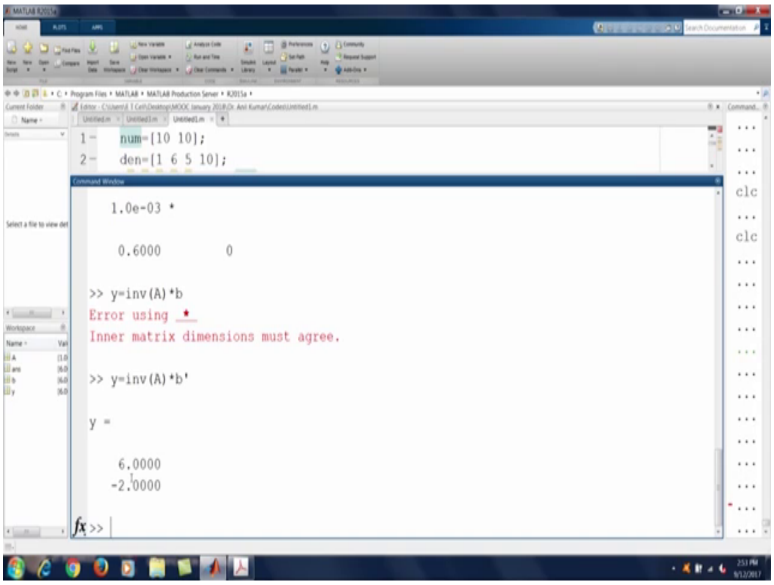
But it is still that is small pertibution in input data gives you a very large change in our large deviation in the solution of the system. So, we try to see that how we can identify that why it is happening, and how this condition number of A is related to this. So, for that let us use the MATLAB here.

(Refer Slide Time: 10:27)



So, here we define matrix A as 1.001, 1.001 I think there is one more 0 0 1 and 3. And in second it is 1 and 3. So, that is how you define your matrix A and then you define b as 0.0006 and 0 here. So, that is your b so now, we say that what is a condition number of a. So, you look at or before that let us find out the solution in first condition your solution is in first condition is let us say that it is say.

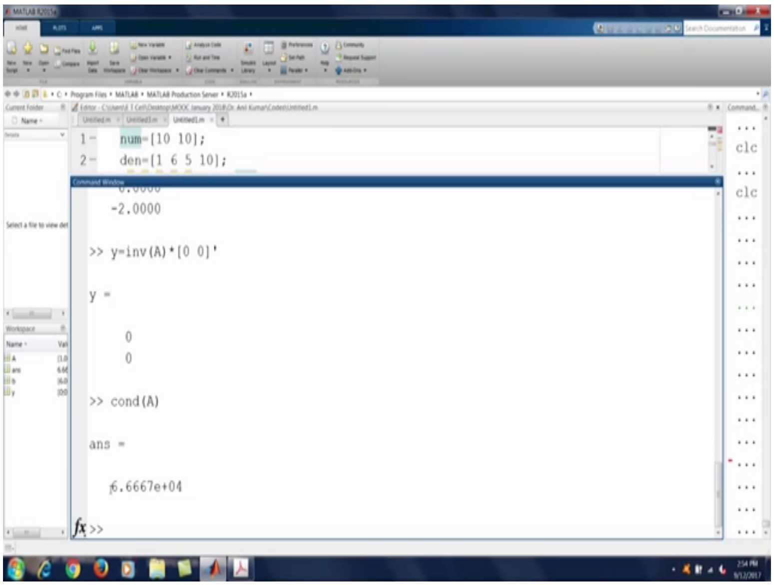
(Refer Slide Time: 11:16)



Um y, y is solution as inverse of A inverse of A into your b and it is coming out to be oh sorry it is b dash here.

So, it is coming out to be 6 and minus 2 which we have just pointed out that solution is of this linear system is a 6 and minus 2.

(Refer Slide Time: 11:37)

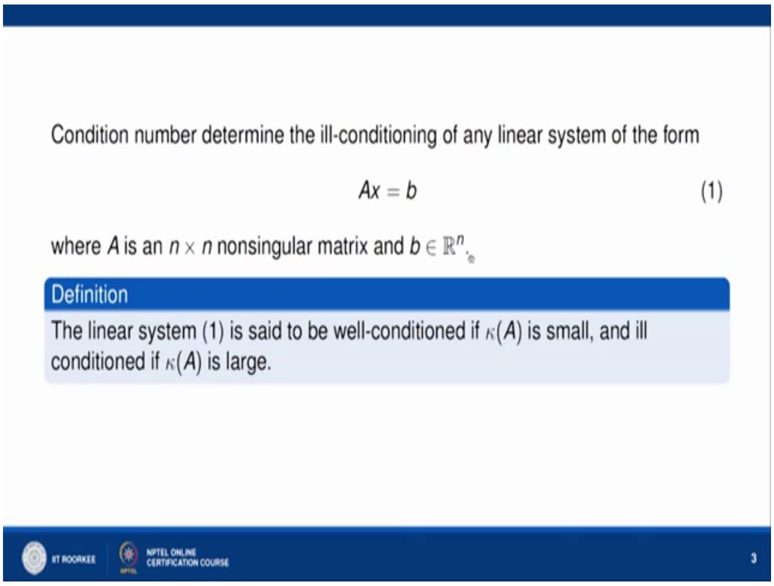


But if we consider in place of b let us consider say values 0 comma 0 and dash. Then solution is coming out to be 0. So, here by a small change in b we have very large deviation in the solution here. So, let us say look at the condition number of the matrix A and if you look at condition number of A it is coming out to be 6.6 6.66 into 10 to power

4, which is quite large. And if you we have seen that condition number of A is greater than or equal to 1, but this is quite large compared to the matrix compared to the value 1. And we will see that condition number of orthogonal matrix is coming out to be one, but if you look at this particular matrix A then it is quite large value.

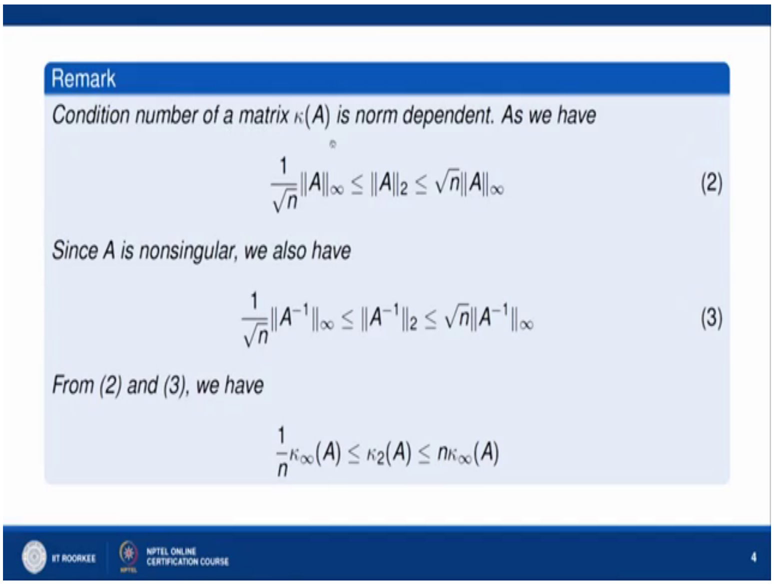
So, here this may justify that if we have a matrix A with large condition number, then we may have say we can say that our linear system is very, very sensitive to the small perturbation in the data here. So now, coming back to our definitions. So, the condition numb number of a matrix is defined like this.

(Refer Slide Time: 12:59)



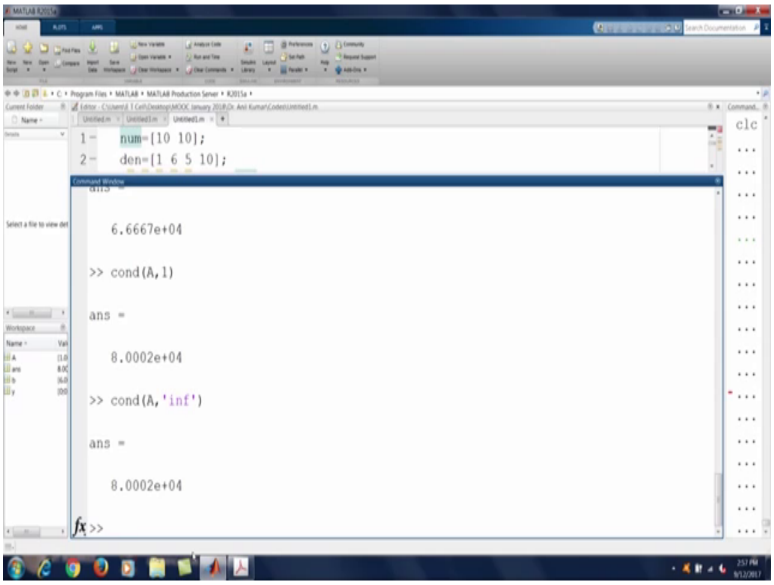
Now, let us we define A new definition which says that linear system a A x equal to b, where A is n cross n sin[singular] non-singular matrix and b is a vector in R n. And we say that this system A x equal to b is said to be well conditioned if kappa of A is small and ill condition f kappa of A is very large. So, in this notation we say that the considered system this is ill condition because here condition number of A is quite large.

(Refer Slide Time: 13:38)



So now we also observe that condition number of A matrix kappa A is nondependent, right? Because we already know this result that 1 upon root and infinity norm of A is less than or equal to 2 norm of A is less than or equal to infinity norm of A into root of n. And since we have already assumed that a is non-singular. So, for A inverse also we can have the similar result and we say that 1 upon root n infinity norm of A inverse into a is less than or equal to 2 norm of A inverse. And it is less than or equal to infinity norm of A inverse into root of n. If we combine these 2 then we can say that 1 upon n condition number of A with respect to infinity norm is less than or equal to condition number of A with respect to 2 norm is less than or equal to n times infinity norm of A infinity condition number of A in terms of infinity norm is related by this.

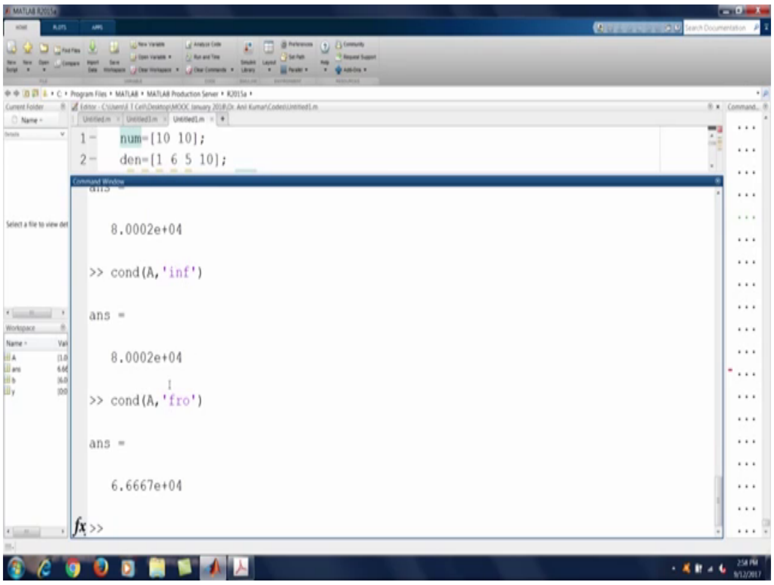
(Refer Slide Time: 14:44)



So, condition number is also norm dependent, that we can here see when we write only condition number of A. So, that will give you condition number of A with respect to 2 norm. So, here p norm is 2 norm.

So, that we can verify here we can write it 2 and it is coming out to be same, but we can also find out condition number of A in terms of say 1 norm. So, that we can look at here and 1 norm it is different here you can say that condition number in A, in 2 norm is coming out to be this, but condition number of A with respect to 1 norm is a 8.002 A into 10 to power 4. And we can also define condition number of A in terms of infinity norm and it is given by this command. So, condition number of A come A and with respect to infinity norm and it is given by the same as this.

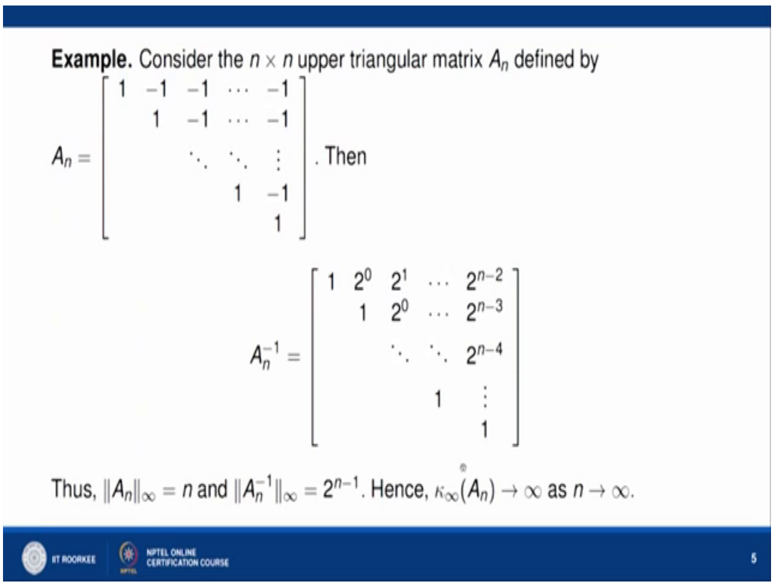
(Refer Slide Time: 15:36)



Now here this is [vocalized-noise h given here, now we can also define condition number in terms of frobenius norm, right. So, and it is coming out to be 6.667 into 10 to power 4, which is similar to your a condition number of A in terms of 2 norm.

So, we can say that here condition norm a condition number is depending on the norm you are using. So, if you are using 2 norm 2 matrix norm then we have a this result if we are using 1 norm or infinity norm for calculating the condition number then it is some other values, but it is related by this relation that 1 by n kappa infinity of a is less than or equal to kappa 2 a is less than equal to n times kappa infinity of A.

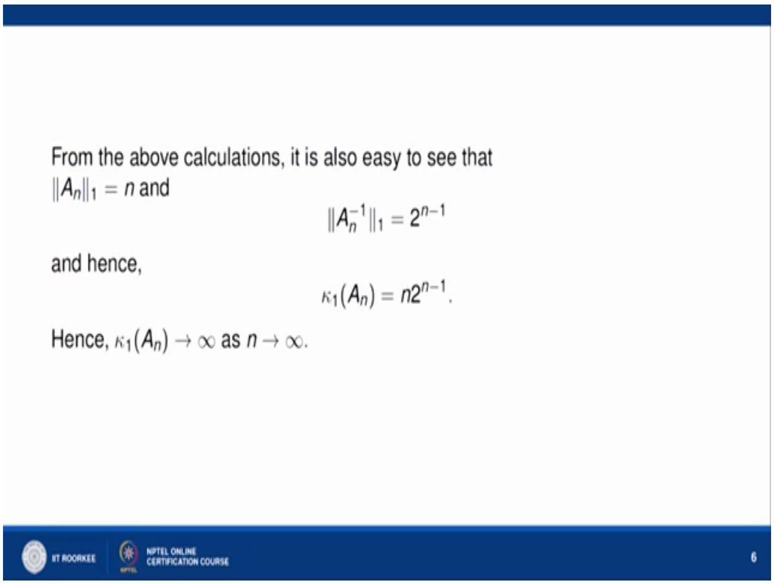
(Refer Slide Time: 16:20)



Now, let us consider one example which we have started with that, let A and b the this n cross n upper triangular matrix A n which is in diagonal it is one and rest all this minus 1. Then we can calculate the inverse of A and it is given by 1 1 diagonal entries are 1 and this entry is 2 to power 0, 2 to power 1 and so on. You can easily verify with the help of mathematical induction that inverse of the matrix A, n is given by A A n inverse and it is given by this upper triangular matrix having this particular form. And if you find out say infinity norm of A n which is given by. So, A n is given here. So, infinity norm of A n you can calculate and is the absolute value of this column thing and it is coming out to be n and similarly you can calculate the infinity norm of A n inverse which is coming out to be 2 to power n minus 1.

So, you can say that condition number of A n with respect to infinity norm is given by n into 2 to power n minus 1. And you can say that as n t[tending] n is tending to infinity then condition number of A n is also tending to infinity with respect to infinity norm. So, it means that as you are taking large dimension a matrix A and of this kind, then condition number is going to be very, very large. And if condition number of this matrix is going to be very large then if we consider any mat[matrix] any linear system like A n x equal to b then it is very, very sensitive to the data input data.

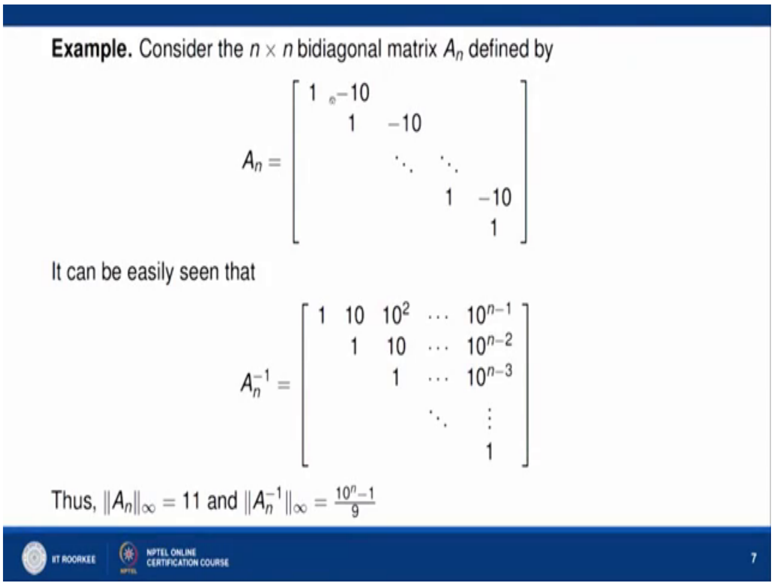
(Refer Slide Time: 18:07)



Now, let us look at the same matrix, but now with respect to 1 norm. So, from the other calculation we can say that 1 norm of A n is 1 n and similarly 1 norm of A inverse is given by 2 to power n minus 1. So, we can calculate that condition number of A n with respect to 1 norm is also given as n into 2 to power n minus 1 and it is also tending to infinity as n tending to infinity. So, here we have seen that for this particular matrix which is upper triangular matrix and given in this form then here your condition number of this matrix is tending to infinity as n tending to infinity. So, this matrix is very, very sensitive to the input matrix are it is very, very say ill condition.

Now, consider another example of a matrix who which is very which is ill conditions or whose condition number is quite high.

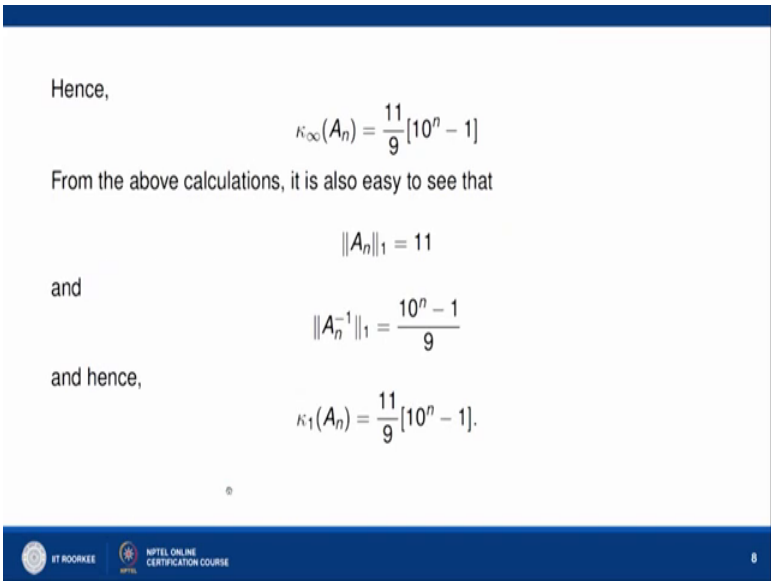
(Refer Slide Time: 19:04)



And here it is this A n is equal to indignant is 1 and the upper diagonal it is minus 10, you can take any number and rest are all 0. So, it is a bidiagonal matrix A n. now we can verify that the inverse of the matrix A n is given by this, that a you can verify with the help of mathematical induction. And it is seen as that an inverse is given by diagonal entries it is 1 the above diagonal entries is 10 they and then above diagonal entries next above diagonal entries is 10 to power 2 and so on and is coming out to be 10 to power n minus 1. And we can calculate the infinity norm of A n it is coming out to be 10 plus 1 it is 11 here. And similarly, we can calculate the infinity norm of A inverse that is 1 plus 10 plus 10 is square up to 10 to power n minus 1.

So, if you sum them, it is coming out to be 10 to the power n minus 1 divided by n this is a simple geometric series. So, you can find out infinity norm of A n inverse is given by 10 to power n minus 1. So, we have infinity norm of A n and infinity norm of A inverse.

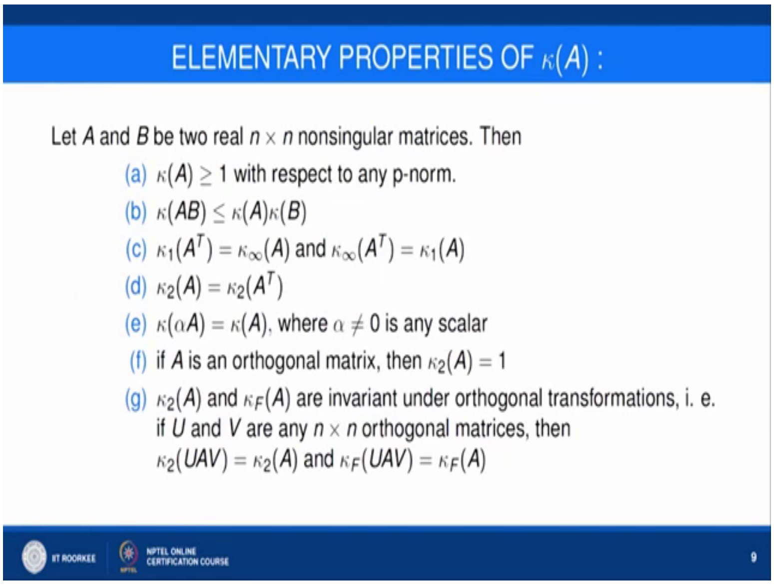
(Refer Slide Time: 20:18)



So, we can say that condition number of A n with respect to infinity norm it is coming out to be 11 by 9 10 to power n minus 1. And you can see that it is tending to infinity as n tending to infinity. So, this is also one example of ill conditioned matrix.

Now, let us do the same calculation, but with the help of 1 matrix norm. So, again in one matrix norm your A n is coming out to be 11 and 1 norm of A n inverse is again same 10 to power n minus 1 divided by n. So, condition number of A n with respect to 1 is also coming out to be 11 by 9 10 to power n minus 1. So, with this we have seen that we have seen 2 example of ill conditioned matrices.

(Refer Slide Time: 21:09)



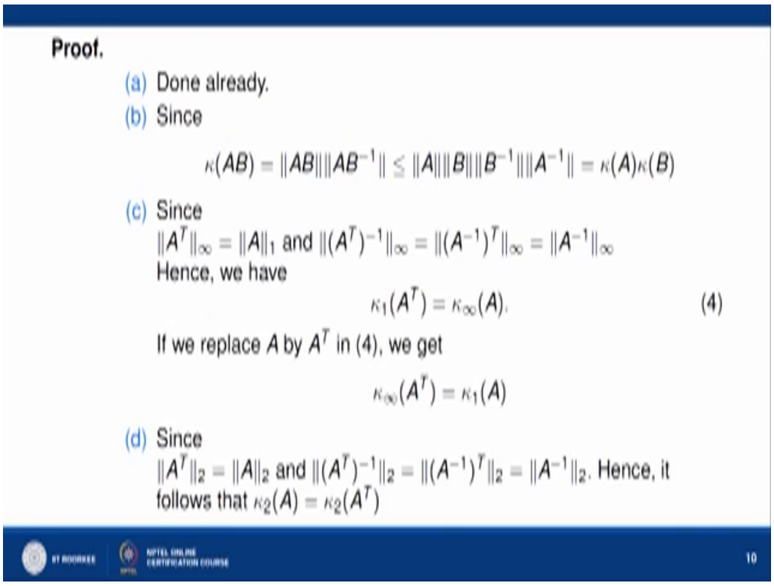
Now, let us consider some elementary properties of condition number of a given matrix A.

So, there are some properties so, let A and B be 2 real n cross and non-similar matrices then we have following properties. So, first property which is evident that a condition number of Any matrix is greater than or equal to 1 with respect to any p norm. Now second is condition number of A B is less than or equal to condition number of A into condition number of B. third is that condition number of A transpose with respect to 1 norm is equal to condition number of A with respect to infinity norm and vice versa vice versa means condition number of A transpose with respect to infinity norm is given as condition number of A with respect to 1 norm. And condition number of A and condition number of A transpose with respect to 2 norm is same. And condition number of Alpha A is same as condition number of A where alpha is any nonzero A scalar value. And if a is an orthogonal matrix then condition number of this orthogonal matrix A is going to be 1.

So, this may be and this is the reason why we prefer all the time orthogonal matrices. So now, we also have one more important properties of orthogonal matrices, that condition number of A with respect to 2 norm and forbenius with respect to frobenius norm are invariant under orthogonal transformation. It means that if U and V are any in cross n orthogonal matrices. Then a condition number of UAV is same as condition number of A. So, it means that by multiplying your matrix A with respect to with the orthogonal matrices will not change the condition number of this matrix A. So, it is in invariant under product of orthogonal matrices. And you can say that it is invariant under orthogonal transformation. Similarly, condition number of UAV with the respect to frobenius norm is also unchanged.

So, it means that condition number of UAV with respect to frobenius norm is condition number of a with respect to frobenius norm. So, these are some properties which we wanted to have, this is very useful properties and we will use and very often in our calculation or in numerical in course of numerical linear algebra.

(Refer Slide Time: 23:52)



So, regarding the proof of A this is the one first a observation point here we can say that kappa of A which is given as norm of A into norm of A inverse and which is greater than or equal to norm of A into A inverse A norm of A into A inverse is basically identity matrices. We say that condition number norm of A identity matrices is given as 1. So, if we have condition number of a then it is going to be greater than or equal to 1. and regarding the proof of second that is condition number of A B is less than equal to condition number of A and condition number of p observe that condition number of A b is given as norm of A B into norm of A B inverse.

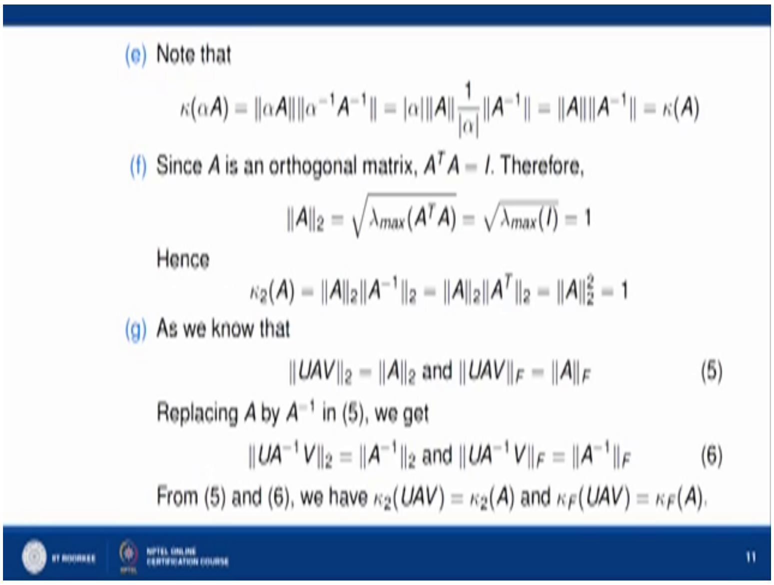
So, when you write norm of A B is nothing but norm of A into norm of B. And condition norm of A B inverse is less than norm of B inverse into norm of A inverse. And if you take this term norm of A into norm of A inverse then this will give you condition number of A matrix A. And if you look at this condition norm of B into norm of B inverse this will give you the condition number of B. So, it means that this is written as condition number of A to condition number of B. So, which says that condition number of A B is less than or equal to condition number of A into condition number of B here. Now regarding the third that is condition number of A transpose with respect to 1 norm is same as condition number of A with respect to infinity norm.

So, here you observe that we have already seen this um equality that condition A norm of a transpose with respect to infinity norm is same as norm of A with respect to 1 norm. So, using this relation we can say that infinity norm of A transpose inverse is same as infinity norm of A inverse transpose which is nothing but infinity norm of A inverse. So, using these 2 equality we can say that condition number of A transpose with respect to 1 norm is nothing but condition number of A respect to infinity norm. And this in this if you replace a transpose by a a transpose or you can say when you replace this metric a by a transpose, then this can be written as kappa infinity of a transpose is nothing but kappa 1 A. So, this is this can be easily proved by this relation that infinity norm of A transpose is same as a 1 norm of A.

So, using this you can easily prove that. Now regarding the 2 norm we say we already knew that 2 norm of A transpose is same as 2 norm of A. And using this we can say 2 norm of A transpose inverse is equal to 2 norm of A inverse transpose and which is nothing but 2 norm of A inverse. So, using this we can say that 2 norm of condition number of A with respect to 2 norm same as condition number of A transfers with respect to 2 norm. So, by this we are we are done here that condition number of matrix A with respect to 2 norm a same as condition number of A transpose respect to 2 norm is same. And here one word about this part a that here we have a shown that kappa of a is greater than equal to 1 with respect to any p norm.

So, here I am not considering any other norm we are considering only say subordinate matrix norm p, but if it is a if it is say even frobenius norm then also this result is true. So, that you can easily check.

(Refer Slide Time: 27:53)



So now, regarding the A next a this that condition number of alpha A is equal to by definition it is norm of alpha A into norm of alpha A whole inverse which is given as alpha inverse A inverse. Now norm so norm of alpha a is given as modulus of alpha into norm of A norm of alpha inverse A inverse is 1 upon modulus of alpha norm of A inverse. So, 1 upon modulus of alpha and 1 upon into norm of modulus of alpha this will cancel out and this will give you A norm of A into norm of A inverse; and which is nothing but a kappa of A it means that condition number of alpha A and condition number of A is same when n alpha is A nonzero a scalar value.

So, here that condition number is not going to change by same multiplication of any nonzero constant that is evident from this proof. Now moving on the next and that is if a is an orthogonal matrix then condition number of A with respect to 2 norm is coming out to be 1. So, for that we already know that A transpose A equal to y because A is an orthogonal matrix. And to norm of A which is given as under root of lambda max of A transpose A. So, a transpose A is I lambda max of I. So, maximum Eigenvalues of identity matrix is nothing but 1. So, 2 norm of A is coming out to be one similarly we know that I I inverse of identity matrix is identity itself. So, inverse 2 norm of A inverse is also coming out to be 1. And hence, a condition number of A with respect to 2 which is nothing but 2 norm of A and 2 norm of A inverse, which is nothing but coming out to be 2 norm of A whole inverse which is coming out to be 1.

So, here so it means that if A is an orthogonal matrix then condition number of A with respect to 2 norm is equal to 1. Now we try to prove the important property that this condition number is invariant under orthogonal transformation. So, for that we start with a 2 norm. So, 2 norm of UAE V, we already know that it is nothing but 2 norm of A and F norm of UAV is given by F norm of A. That we have already proved when we discussed the matrix norm.

So, we have seen that matrix norm is invariant under orthogonal transformation. So now, here in this relation if we replace a by A inverse then also the similar result told. So, it means that U of 2 norm of u A inverse v is same as 2 norm of A inverse. And f norm of U A inverse V is same as F norm of A inverse and if you multiply 2 norm of A into 2 norm of A inverse you will get 2 norm of UAV into 2 norm of U A inverse v and. So, it means that we can so that 2 norm of A means condition number of A and condition number of UAV is same. Similarly, frobenius norm of U A condition number of UAV with respect to forbinius norm is same as condition number of A with respect to frobenius norm. V here how we are getting this let us consider here, we already know that 2 norm of UAV is equal to 2 norm of A and 2 norm of U A inverse V is given as 2 norm of inverse.

So, if I look at the condition number of UAV then condition number of UAV is condition number of UAV, transpose into UAV inverse 2 norm of this. And it is nothing but UAV into here we write V inverse A inverse U inverse. Now here V inverse is nothing but v transpose. So, it is UAV we transpose A inverse U transpose 2. And V again if V is orthogonal then V transpose is again orthogonal. And if U U is orthogonal. So, U transpose is again orthogonal. So, this is nothing but 2 norm of A and this is what a 2 norm of A inverse and which is nothing but condition number of A respect to 2 norm right.

So, similarly, we can do for frobenius norm of forbenius condition number with respect to frobenius norm and we can. So, we can see that it is equal to this right. So, here we stop our discussion and we see that in this lecture we have discussed one important concept known as condition number of A given matrix and some elementary properties of condition number of A. And we have just try to say realize that if condition number of A matrix is high then the corresponding linear system is very, very sensitive to input data. And the ill condition matrix ill condition matrix means the condition number of A matrix is quite large and ill condition matrix is very near to a singular matrix.

So, here will end our lecture will continue our discussion in next class thank you very much.

Thank you.