Numerical Linear Algebra
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Lecture - 35
Stability of Non-linear System

Hello friends, welcome to this lecture. In this lecture we will discuss about stability of a non-linear system. So, we will consider the discrete non-linear system and we try to discuss the stability of these altitude system. So, for that let us first start with linear system, and then we with the help of study we done for linear system, we try to discuss non-linear system as well.

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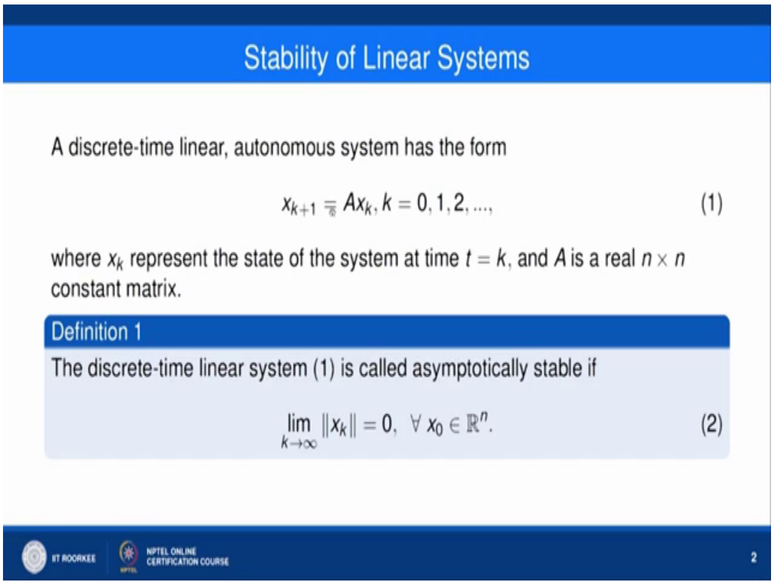

So, a discrete time linear autonomous system has the form X k plus 1 equal to A X k where k is from 0 1 2 and so on. And this X k represent the state of the system at time t equal to k and this matrix A is a real n cross n constant matrix.

So, here this represent the state x at time t equal to k plus 1 is given in terms of A into state of this state and the time t equal to k. Now here we want to see under what condition this solution of this autonomous system is converging to say here reaching to equilibrium position. So, here we define what is known as asymptotically stable. So, the discrete time linear system 1 is called asymptotically stable if limit k tending to infinity norm of X k is equal to 0; For any initial condition x naught belonging to R n. So, if we take any initial condition along with this discrete system, then irrespective of your initial condition solution is turning to 0. In this case we call our system as asymptotically stable solution.

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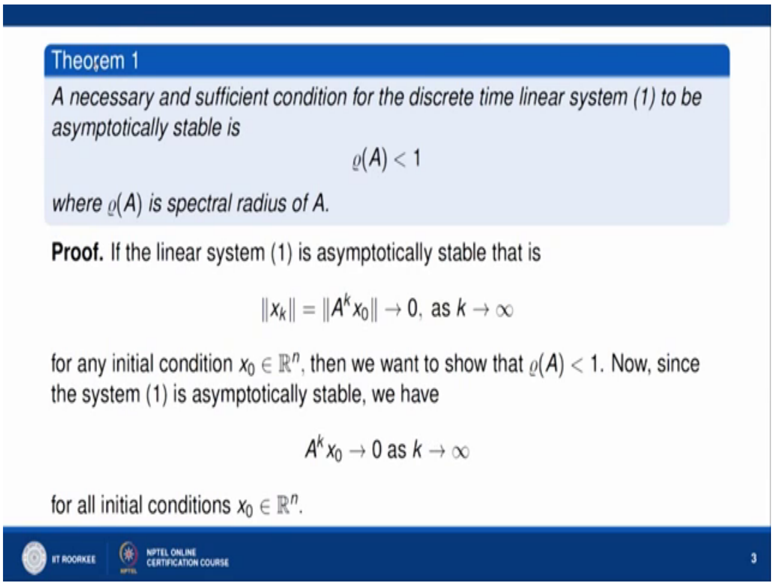

So, here we try to find out say condition if and only condition for discrete time linear system to be asymptotically stable. So, for that we have a theorem 1. So, it says that a necessary and sufficient condition for the discrete time linear system to be asymptotically stable; is that spectral radius of A is less strictly less than 1 and so, let us try to prove it. So, here if you look at if the linear system 1 is asymptotically stable, then we can find out norm of X k in terms of norm of A k x naught. If you look at here that X k plus 1 equal to AX k and if you take the initial condition that x of 0 is equal to a x x naught something, then you can again repeated and you can say that your X k is written as A into X k minus 1.

So, this can be written as X k plus 1 equal to A square X k minus 1 and you can repeat this process and you can write down that X k plus 1 is equal to A to power k plus 1 x naught so, here since I am talking about X k. So, X k is nothing, but A to power k X naught. So, we say that this system is asymptotically stable if norm of X k is tending to 0 or you can say that norm of A k X naught is tending to 0 for any initial condition X naught which is an A vector in R n.

So, here this is given that it is asymptotically stable and you want to show that spectral radius of A is less than 1. Now since it is known that it is a asymptotically stable. So, it means that X k is tending to 0 means A k X naught is tending to 0 as k tending to infinity. Now here we utilize the fact that X naught is any initial condition X naught in R n. So, in particular we can take X naught as the standard basis element.

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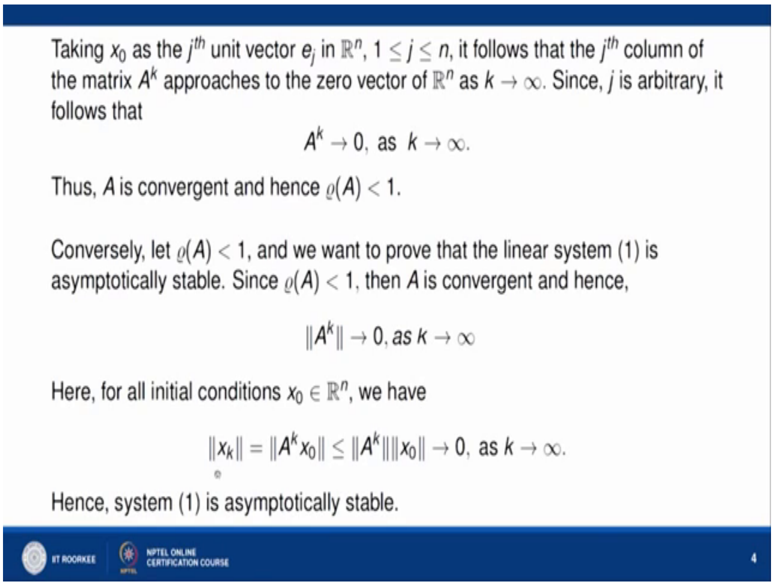

So, let us say that if X naught is e j in R n then it follows that the jth column of the matrix A k approaches to the 0 vector of R n. If you look at this A k X naught is standing to 0 as k tending to infinity for any X naught. So, in particular if we take X naught as e j then A k e j represent the jth column of A k. So, we can say that jth column of matrix A k approaches to 0 vector of R n as k tending to infinity.

Now, here j can be take any value from 1 to n. So, we can say that every column of A 2 power k is standing to 0. So, every column k of A 2 power k approaches to 0 vector means the every term of the matrix A to power k is standing to 0 or we can say that A to power k is standing to 0 as k tending to infinity. Now this is what this implies that A k 8 pow A matrix is convergent. Now A is convergent matrix and we already know that the necessary and sufficient condition for any matrix A to be convergent is that a spectral radius of A is less than 1.

So, this is what we have already we have already discussed in the in the lecture of convergent matrix the first lecture of convergent matrix. So, it means that here if the system 1 is hmm if the system 1 is asymptotically stable, then we have proved that a spectral radius of A is less than 1. Now the prove the other way round. So, other way means it is given that norm of A the spectral radius of A is less than 1, and we want to show that the discrete time linear system 1 is asymptotically stable.

So, we already know that this implies that spectral radius is less than 1 implies that A is convergent. So, A is convergent means that norm of A k is standing to 0, but this norm is any matrix norm. So, a a norm of A k is standing to 0 as k tending to infinity. Now here for all initial condition X naught we already know that norm of X k is nothing, but norm of A k X naught. Now here since this matrix norm and vector norm is consistent to each other. So, we can say that this is less than or equal to norm of A k into norm of X naught and since norm of A k is standing to 0 as k tends to new infinite, then this is a tending to 0 as k tending to infinity. So, it means that norm of X k is standing to 0 as getting into infinity.

So, we can say that our system is asymptotically stable. So, what we have proved here that if we consider this discrete time linear autonomous system, then this matrix A plays a very important role. So, if this matrix A has a spectral radius strictly less than 1, then this system is going to be asymptotically stable system.

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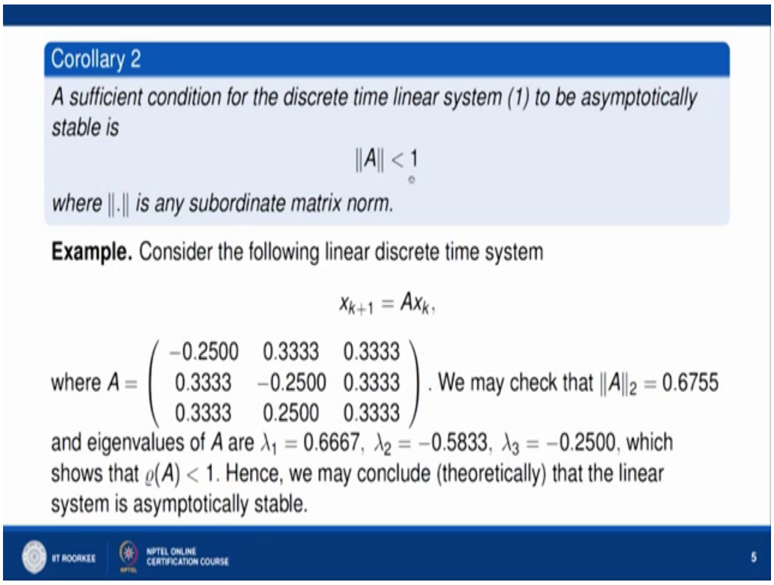

So, now we already know the relation between a spectral radius and matrix norm. So, we can have a sufficient condition for the discrete time linear system on to be asymptotically stable. So, the sufficient condition is that norm of A is less than 1 where this norm is any subordinate matrix norm, and this follows from the relation that spectral radius is less than norm of A and since norm of A less 1 less than 1 then spectral radius is less than 1.

Now, this is a sufficient condition, because we have already shown that it may happen that even if norm of A is bigger than 1 is still a spectral radius may be less than 1. So, it is just a sufficient condition that if this is less than 1, then spectral radius is automatically less than 1 and the result which we have proved as previous result we say that a discrete time linear system is asymptotically stable.

So, as an example let us consider the following linear discrete time system X k plus 1 equal to A X k. Where this A matrix is given by minus 0.2500 and so on, here if you look at this is nothing, but minus 1 by 4, 1 by 3, 1 by 3, 1 by 3 minus 1 by 4, 1 by 3, 1 by 3, 1 by 4 and 1 by 3. So, just to show that this is a this system is asymptotically stable I have used matlab and there if we insert this a in format is short I am using short format then we can have this, but if you use rational format or we can say format rational, then you can say that A as minus 1 by 4, 1 by 3 and so on. So, it depend on the format which you are using.

So, here we can check that the 2 norm of A is 0.6755 and by this we can say that sufficient condition is valid. So, it means that if 2 norm of A is less than 1, then your spectral radius is basically less than 1 and this we can verify here since we have a matrix here, we can verify and we can find out the eigen values of A and that for this also we have utilize the matlab and we have seen that the eigen values of A is given by 0.6667 and lambda 2 as minus 0.5833 lambda 3 as minus 0.2500.

But if you look at here we can say the maximum of modulus value of lambda i is strictly less than 1. So, we can say that spectral radius of A is less than 1 which satisfy the condition given an theorem. So, it says that this system is a step asymptotically stable system. So, we may conclude theoretically with the help of our theorem, that the linear system is asymptotically stable.

Now, let us check the same thing, but now with the help of matlab or we can say numerically we want to check that this system is asymptotically stable system.

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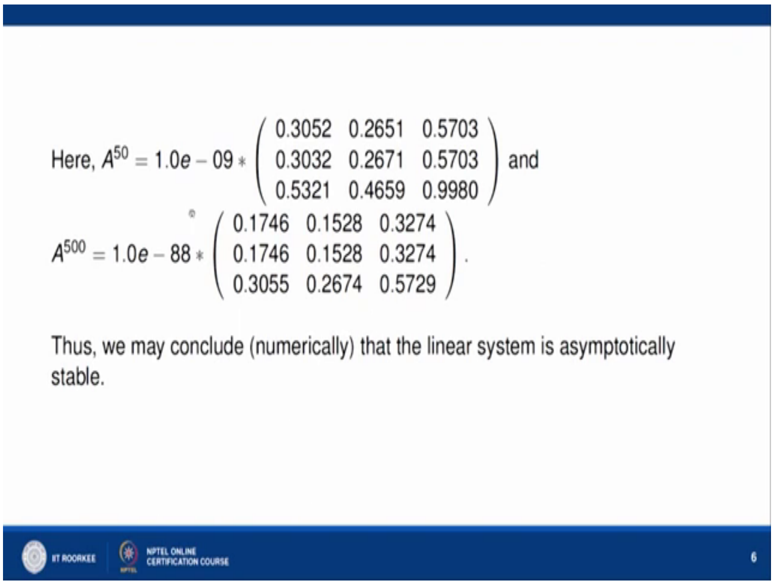

So, for that we can calculate that A to power 50 is given by this quantity. Now here this 1.0 e to power minus 0.9, you can say that this is nothing, but 10 to power minus 9. So, A to power 50 is written as 10 to power minus 9 into this matrix, and if you want to calculate a to power 500 it is given by 10 to power minus 88 into this matrix. So, it means that as your powers are increasing the matrix is standing to a 0 matrix.

So, we say that since A to power k standing to 0 as k tending to infinity, then a to power k into x naught which is nothing, but X k is standing to 0 as k tending to infinity. So, we may conclude now numerically that the linear system is asymptotically stable solution.

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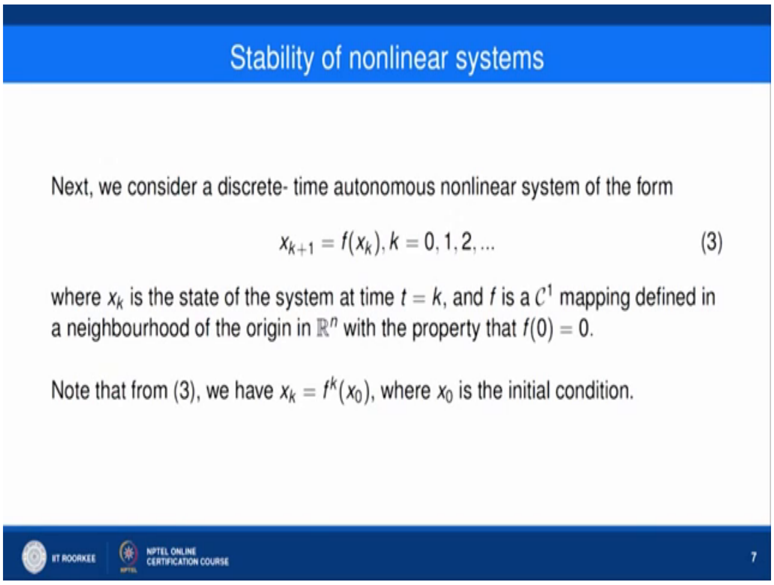

Now, with the help of this, we try to discuss the stability of non-linear system. So, we have discussed only one part that is the asymptotically stable asymptotic stability of linear system. And with the help of that we want to discuss the stability of non-linear system and if you look at we will observe that this is similar to the stability of non-linear system in continuous case.

So, in continuous case things are going in similar manner. So, here we want to consider a discrete time autonomous non-linear system of the form X k plus 1 equal to f of X k. Please remember here autonomous means what? That that this f does not involved any time in explicit manner. So, though this X k is again x of t k, but f naught; f is not involving any time in explicit manner. So, in time is involved in implicit manner, but not the explicit manner. So, these kind of systems are known as autonomous system.

Now, this is f is non-linear. So, we are calling this as discrete time autonomous non-linear system. And again X k is the state of the system at time t equal to k and here this function we are assuming as a C 1 function. C 1 function means whose derivative is a continuous function. So, here because we are going to utilize the Taylor series of this function f. So, for that at least first order at least f should be that C 1 function, it may if you want to consider more term like we may consider f belongs to C 2 C 3 function C 2 or C 3 class.

Now, here f satisfied one more condition that f of 0 is equal to 0 ok. So, and this f is a C 1 function defined in a neighbourhood of the origin in R n. Now from this again we can say that X k plus 1 can be considered as f k X naught. So, how we are writing this? We can say that X k plus 1 is what f of X k. Now again I can use this X k I can write as f of X k minus 1. So, it means that f operating on f of X k minus 1. So, it is can be written as f square X k minus 1. So, if you keep on doing this. So, we can say that X k can be written as f k X naught where f k represent composition of f k times right. So, X k is equal to f of k X naught, where X naught is initial condition which is provided with this non-linear system.

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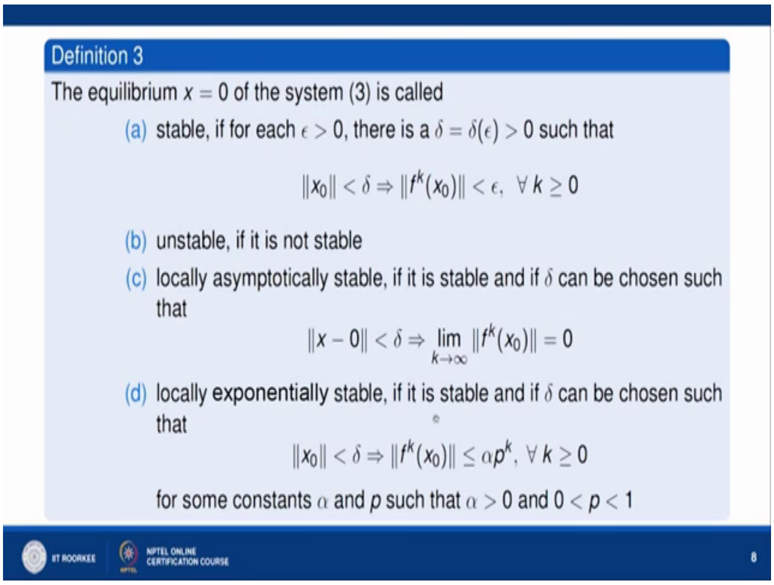

So, here first let us define what do you mean by stability unstability of solution on this discrete autonomous non-linear system. So, here we say that the equilibrium solution x equal to 0 of the system 3 is called stable. If for every epsilon greater than 0 they exist a delta, which depend on this epsilon and greater than 0 such that norm of x naught is less than delta implies that norm of f k x naught is less than epsilon please remember this f k x naught is your X k. So, it means that if initial condition is small, then the final means X k is norm of X k is less than epsilon and norm of X k is nothing, but norm of f k x naught.

So, this is less than delta implies that norm of f k x naught is less than epsilon for every k greater than or equal to 0. And it is x equal to 0 is said to be unstable, if it is not stable means and though your initial condition is small, but norm of X k is not small.

So, in that case we say that solution x equal to 0 equilibrium solution is unstable solution. And then locally asymptotically stable if it is stable and if delta can be chosen such that norm of X minus 0 is less than delta implies that limit k getting into infinity f k x naught is equal to 0 or you can say asymptotically stable means in limiting sense it is going to the solution x equal to 0. So, here your X k is in the neighbourhood of say 0, but this asymptotically stable means as k getting into infinity your solution is standing to 0 solution. So, that is what we know as asymptotically stable solution.

Now, locally exponentially stable. If it is stable and if delta can be chosen such that norm of x naught is less than delta implies that norm of f k x naught is less than or equal to alpha p 2 power k, for every k greater than or equal to 0 for some constant alpha and p says that alpha is positive, and p is lying between 0 to 1. And in that case we say that our discrete non-linear system is yes exponentially stable solution.

So, here we may observe that this exponentially stability implies the asymptotically stability. So, here if you look at this alpha is positive and p is something which is lying between 0 and 1. So, if you take limit k tending to infinity, then this right hand side is tending to 0 and we can say that that limit k tending to infinity norm of f k x naught is standing to 0. So, it means that exponentially stability implies asymptotically stability.

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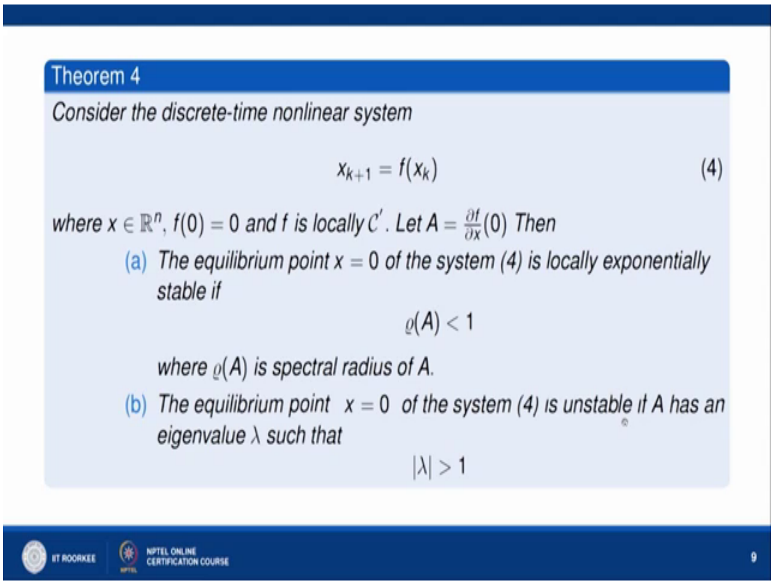

So, now consider the next theorem in which we consider the condition for exponentially stability and unstability. So, here we consider the discrete time non-linear system X k plus 1 equal to f of X k, where x belongs to R n here X k belongs to R n, f 0 is equal to 0. So, x vanishes at 0 and f is locally C 1 function and here A denote the derivative of f with respect to x at 0. Now here if it is a scalar system then it is simple partial derivative, but if it is a vector which is the case here then this a this symbol dou f by dou x evaluated at 0 represent the Jacobean matrix of f with respect to the vector x. So, this is just a symbol which represent the Jacobean of f with respect to the vector x.

Then the first part which says that the equilibrium point x equal to 0, of the system 4 is locally exponentially stable if spectral radius of A is less than 1 and B that the equilibrium x equilibrium solution x equal to 0 of the system 4 is unstable. If A has an eigenvalue lambda whose modulus value is bigger than 1; So, it means that if all the eigenvalues have modulus value less than 1 strictly less than 1 then it is locally exponentially stability locally exponentially stable, but if at least 1 lambda whose modulus value is bigger than 1 then we have a unstable solution we are not providing any proof for this, but it is going in a similar way as we have as we can prove for continuous non-linear system.

Again we are using the same pattern, we are using the Taylor series expansion for this function f in the neighbourhood of 0 and we can we can neglect the higher order term because X k we are considering the solution in the neighbourhood of origin. So, it means that we can ignore the higher order term, and we can consider only say linearized version of this non-linear system and we can say that this non-linear system and linear system behaves in a in a similar manner.

So, here we are not giving any proof of this, but we say that the proof is simply analogous to the proof given in a continuous case. And for continuous case if you want to consider the proof, we refer the book of brown martin that is for an introduction to ordinary differential equation, there the proof of continuous version continuous non-linear system is a given.

So, let us take this as a granted, and let us consider another corollary of this.

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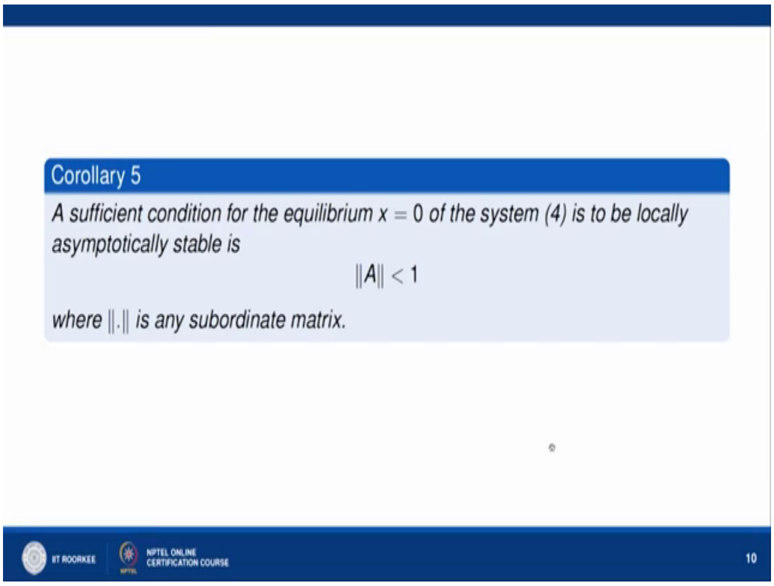

So, here corollary says that a sufficient condition for the equilibrium solution x equal to 0 of the system 4 is to be locally asymptotically stable is that norm of A is less than 1, where this norm is any subordinate matrix norm and this the proof follows immediately from the theorem this and the relation that spectral radius of A is less than norm of A here. So, this proof of this corollary 5 follow from the proof of the theorem.

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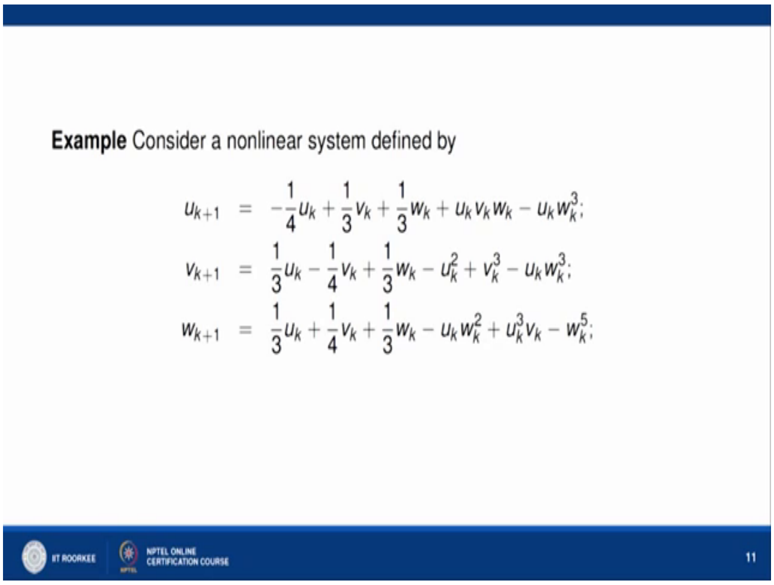

Now, let us take certain examples to validate our theorem or we can say that as an application of previous theorem, we consider the following 2 examples. So, first example says that we have a non-linear system where you U k plus 1 is given as minus 1 by 4 U k plus 1 by 3 V k plus 1 by 3 W k plus U k V k W k minus this. So, here your non-linear system similarly V k plus 1 and W k plus 1 is also defined, and we can say that this is a non-linear system because non-linear terms are available here, and then we want to see whether this non-linear system is a stable solution a stable system or unstable system.

So, for that we need to consider the Jacobean now if you look at Jacobean of this here this present your f part. So, here we are considering this non-linear system and then we try to apply the result of previous theorem.

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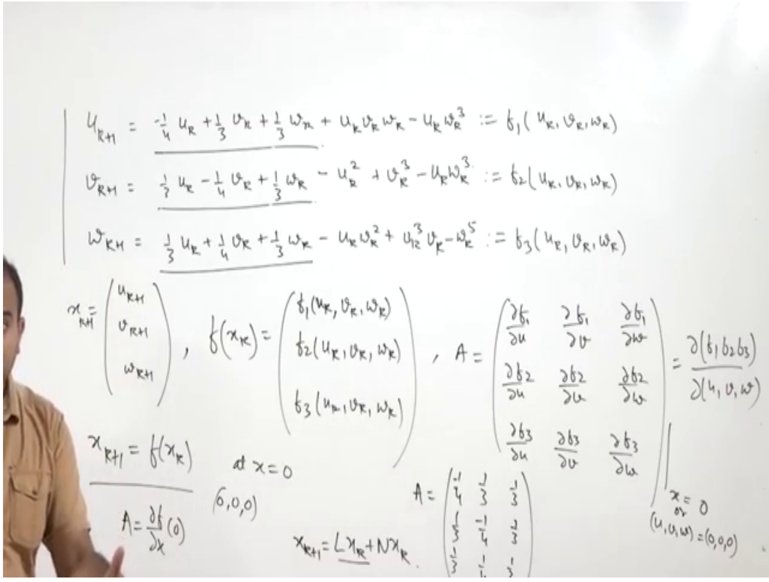

So, let us look at here we have U k plus 1 is defined as this v k plus 1 w k plus 1. So, if I denote this first equation as the function f 1 of u k, v k and w k. Similarly, the second right hand side of the second equation as f 2, u k, v k, w k on right end of third equation as f 3, u k, v k, w k and if we denote the vector x k plus 1 as u k plus 1, v k plus 1, w k plus 1 then we can say that this vector is an r 3 and we can defined f of x k like this, f of x here as f 1, f 2, f 3 then we can write this as x k plus 1 equal to f of x k. So, it means that this system is given by this.

Now, we know that the stability of this non-linear system at x equal to 0 or you can say at 0 0 0 here, here we are considering in r 3 is related to the matrix A which is given as dou f by dou x at 0. So, this is the Jacobean of f with respect to this factor. So, here to find out this here you look at the matrix A which is the Jacobean of f with respect to x here, here f is having f 1, f 2, f 3 component and x is having component u, v, w.

So, this is given by dou f by dou u, dou f dou f 1 by dou u dou f 1 by dou v dou f 1 by w and similarly we can define. And this we are calculating at x equal to 0 or you can say that u comma v comma w equal to 0 comma 0 comma 0. So, if you consider this as function of u v w, this as another function of u v w, and so it is f 1 u v w, f 2 u v w and f 3 u v w and you can calculate the Jacobean a and if you look at at 0 comma 0 comma 0.

If you look at this a is coming out to be the coefficient matrix of this part. So, here if you look at A is coming out to be the matrix A as. So, here we have minus 1 by 4, 1 by 3 and

1 by 3 here, 1 by 3 minus 1 by 4 1 by 3 and here it is 1 by 3 1 by 4 and 1 by 3 or. So, here if you look at this is a linear part of this non-linear equation or you can say that this X k plus 1 equal to f of X k, we can write this as some X k plus 1 equal to L of x k plus n of x k.

So, here L of x k is a linear part in terms of x k and n of X k is in non-linear part in terms of X k and the coefficient of linear part is your matrix A. So, here you can look at that linear part of this non-linear system is basically this. So, your L matrix is you can easily calculate it and you can call this matrix A as this linear matrix is it and this you can verify by finding the Jacobean. So, either you do it do like this or you simply find out say linear part of your non-linear system. So, if linear part is available, you can directly find out your A, if linear part is not available then you can use this is it ok.

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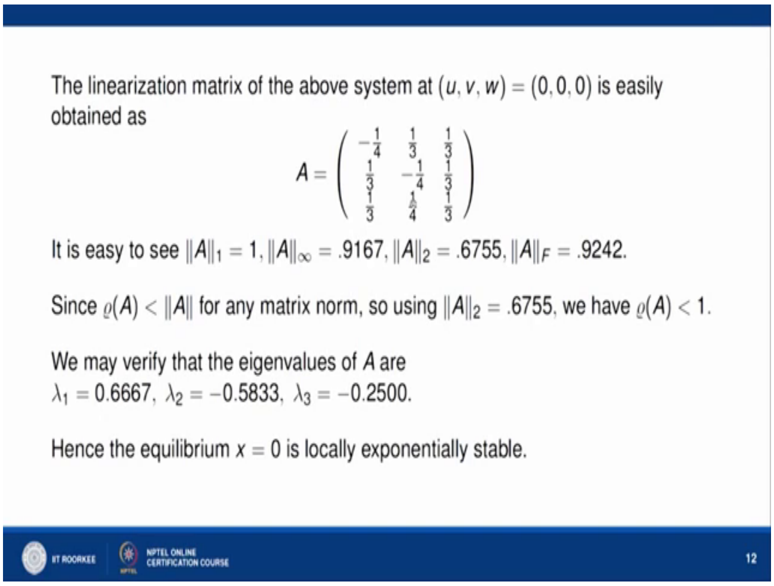

So, now we can find out say the linearization matrix of the above system at u comma v comma w as 0 comma 0 comma 0 is easily obtained as this. Now here once the matrix is given to us let us find out say matrix norm. So, matrix norm of A 1 matrix norm of A is 1 infinity norm of A is 0.9167, 2 norm of A is 6.6755 and frobenius norm of A is 0.9242 and we know that this spectral radius is less than any matrix norm. So, you can take any of these 3 norms and you can say that let us say that 2 norm of A is 0.6755. So, we can say that a spectral radius is going to be less than less than 1. So, please remember here that this is any matrix norm. So, I can take any of this and spectral radius is less than all matrix norm all matrix norm.

So, here we can say the spectral radius is less than 1, you can easily verify by calculating the eigenvalues of A. So, here if we calculate the eigenvalues of A, it is coming out to be lambda 1 equal to 0.6667, lambda 2 as minus 0.583, lambda 3 as minus 0.2500. So, here we say that it is easily proved that spectral radius of A is less than 1 which we have obtained earlier by theoretical result. So, either you do it like this or this, but this is the easier way, but the only problem is that it may happen that none of these norms is going to be less than 1, in that case you have to find out the eigen values. So, but if these norms to find these norms is a little bit easy in terms of if you are doing the calculation in matlab.

So,. So, it means that ultimately you have to find out say the spectral radius. So, either you do it with the help of matrix norm or you can find out the eigenvalues of the matrix. So, here we have seen that a spectral radius is less than 1. So, equilibrium solution x equal to 0 is locally asymptotically stable. So, theoretically we have proved that the solution 0 0 0 of this non-linear system is asymptotically stable solution. So, it means that any solution standing to 0 0 0 as k tending to infinity. So,. So, that is that we have proved theoretically with the help of previous theorem, now let us use matlab to verify numerically also.

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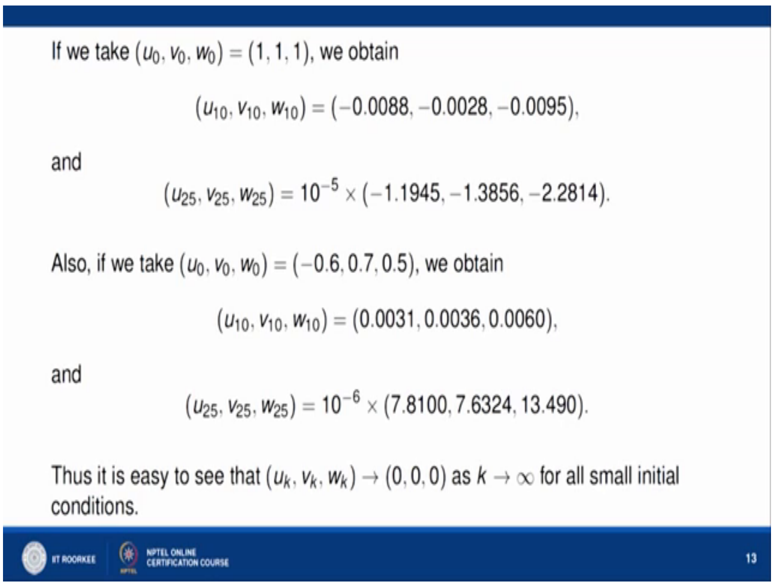

So, let us take initial condition 1 1 1, and then we can calculate the U 10, V 10 and W 10. So, it means that k equal to 10 I am considering here and if you use matlab it is coming out to be minus 0.0088 minus 0.0028 minus 0.0095. So, it is quite a small, but let us consider one more iteration let us say that in place of 10 let us consider the 25th iteration and it is coming out to be 10 to power minus 5 into this vector.

Now here you can say that as iteration we keep on considering the high iteration, and then this matrix is this vector is tending to 0 vector. So, here we say that as iteration means as k tending to infinity, your solution is tending to 0 oh. So, it means that. So, here we are taking this initial condition if we change the our initial condition. So, let us say if we change this initial condition as minus 0.6, 0.7 into and 0.5, we can obtain that that 10th iteration is this, and the 25th iteration is given by this.

So, we can say observe that whatever be the initial condition your U k, V k, W k is tending to 0 comma 0 comma 0 as k tending to infinity. So, since due to space constant I am considering only these 2 initial condition, you may verify it for any any initial condition and using matlab. So, this we can do it no problem. So, here we say that if the coefficient matrix has a spectral radius less than 1, then your solution is standing to the equilibrium solution here equilibrium solution we are considering a 0 comma 0 comma 0. So, in this case we can say that your system is non-linear system is a asymptotically stable solution asymptotically stable system.

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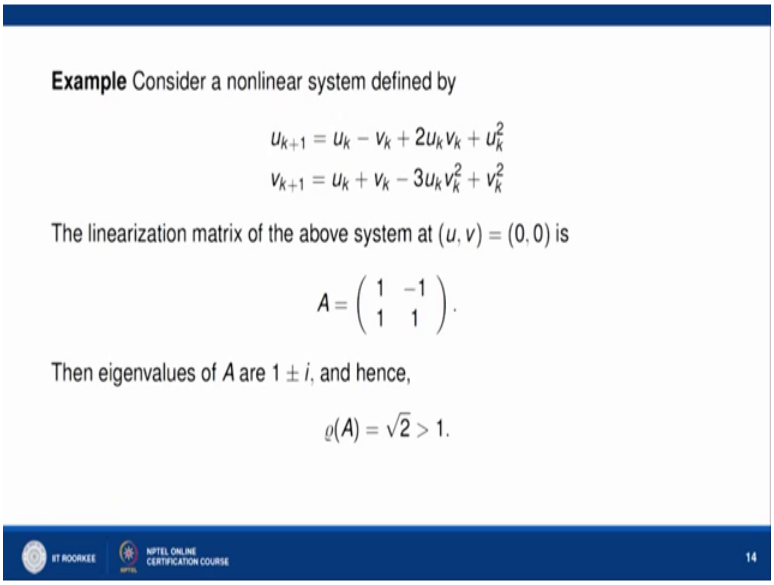

Now, consider another non-linear system here U k plus 1 is equal to U k minus V k plus 2 U k V k plus U k square and V k plus 1 equal to U k plus V k minus 3 U k V k square plus V k square. Now here if we want to check the stability of this non-linear system at 0 comma 0. So, if you look at we have a linear part available. So, your a matrix is given by the coefficient of linear matrix linear part that is 1 minus 1 1 and one. So, A is given by this. So, whether you do it with the help of Jacobean evaluated at 0 0 or look by just by looking at the linear part.

So, A is coming out to be this you can verify that these two are coming out to be the same one. So, find out the eigenvalues of A. So, eigenvalues of A is given as 1 plus minus i and we can say that the spectral radius is coming out to be under root 2 which is greater than 1. So, here we say that that at least a has 1 of the eigenvalue say 1 plus i, whose modulus value is bigger than 1. So, we can say that by theorem this system is unstable solution unstable system.

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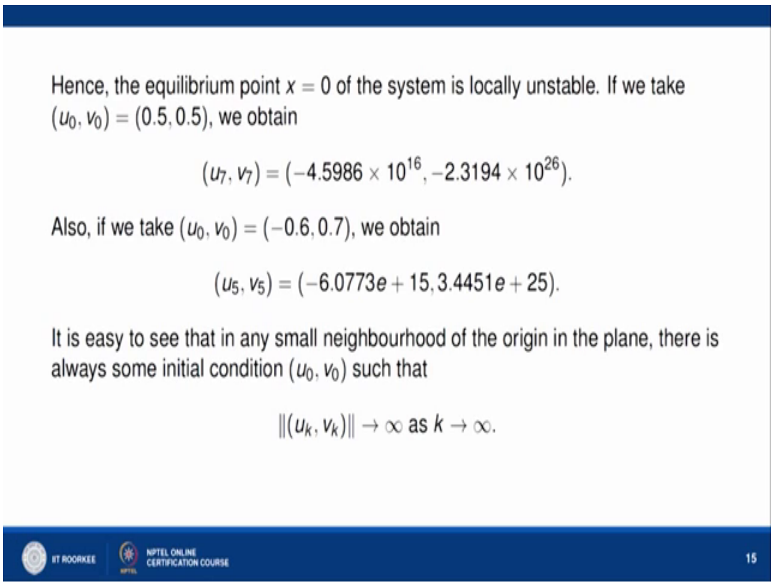

So, let us verify using matlab. So, here we want to show that equilibrium point x equal to 0 of the system is locally unstable. So, for that let us take initial condition as 0.5 and 0.5 and we obtain the 7th iteration U 7, V 7 as minus 4.5986 into 10 to power 16 and minus 2.3194 into 10 to power 26. If you look at these are quite large even if your initial condition is quite small and it is near to 0 and if we take different initial condition that is minus 0.6 in an 0.7, we obtain that the fifth iteration itself is minus 6.077 into 10 to power 15 and 3.44 into 10 to power 25. And please remember here in matlab you generally have the output in this way e plus e plus 15. So, that is nothing, but 10 to power 15 and it is 10 to power 25.

So, here we say that even if your initial conditions are quite small, but still fifth iteration or you can say seventh iteration is going to be very very large compared to your compared to your 0 0. So, you can say that your U k V k is tending to infinity as k tending to infinity. So, it means that your the system non-linear system is locally unstable system.

So, here we conclude our lecture. So, if you see what we have discussed in today's lecture. In today's lecture we discuss the stability and unstability of the non-linear system for that we have considered for the linearized part and we discussed the stability unstability result for linear part and with the help of linear part we discuss result for non-linear system and we have seen say two example to say validate our theoretical result.

So, with this we stop here thank you very very much for listening us.

Thank you.