Numerical Linear Algebra
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Lecture - 34
Convergent Matrices- II

Hello friends, welcome to the lecture. If you recall in previous lecture we will, we have discussed the concept of convergent matrix the matrix whose if you make a sequence of its powers then it will converge to 0 as the powers are tending to infinity. So, and we have seen a certain example of convergent matrix. So, we have seen that A is set to be a convergent matrix if the sequence A power K, K is from 1 to infinity. This tend to limit k tending to infinity this A k tend to 0 matrix or you can say that limit is a 0 matrix of the same size.

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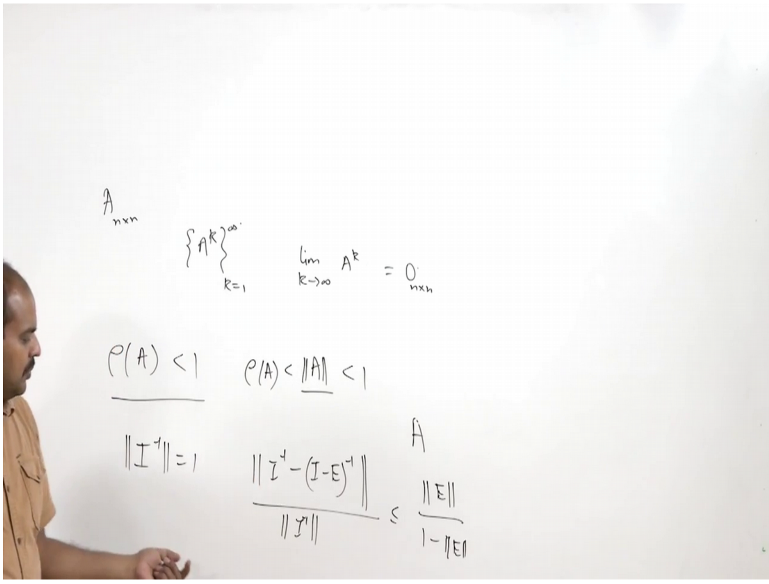

So, if it is n cross n then it is a 0 matrix of n cross n. And we have discussed that A is a convergent matrix if and only if the spectral radius of A is strictly less than 1, spectral radius of A is the maximum of modulus values of eigenvalues of A. So, if this, this is an if and only if condition here. But the only problem is that here to calculate this spectral radius we have to calculate all the eigenvalues which is a little bit difficult or costly problem. So, rather than considering this then we have considered a sufficient condition which say that if norms any wet matrix subordinate matrix norm is less than 1, then this is basically here we have discussed this that the spectral radius is less than matrix norm of A, and if this matrix norm is less than 1 then spectral radius is automatically less than 1. And we say that this is a sufficient condition for convergent metrics at matrix norm of A is less than 1 and we have seen certain example based on this.

Now, in today's lecture we want to see as an application of this convergent matrix. So, idea is this that if we have a say matrix A which is non-singular matrix. So, here we want to discuss that given a non-singular matrix if we perturb our non-singular matrix by small say deviation then we want to find out whether the perturb matrix will remain non-singular or not. So, to begin with we start with identity matrix and see for identity matrix what should be the result.

So, the first result which we want to discuss is this.

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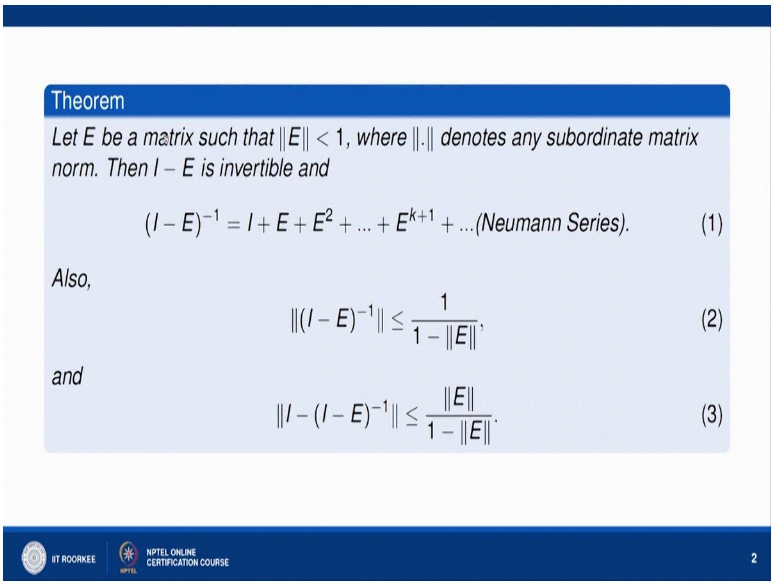

That here let E be a matrix such that norm of E is less than 1, where this norm is any subordinate matrix norm then I minus E is invertible and the formula for I minus inverse of I minus E is given by I plus E plus E square and so on E to power k plus 1 plus this. So, this is an infinite series we can say that and we call this infinite series at Neumann series and for this we can say that norm of inverse of I minus E is bounded by 1 upon 1 minus norm of E and norm of I minus, I minus E inverse is bounded by norm of E and divided by 1 minus norm of E. We will see that why we are calculating this, in fact, this is used this will use to find out related error in finding inverse of E.

So, let us first find out the proof of this theorem.

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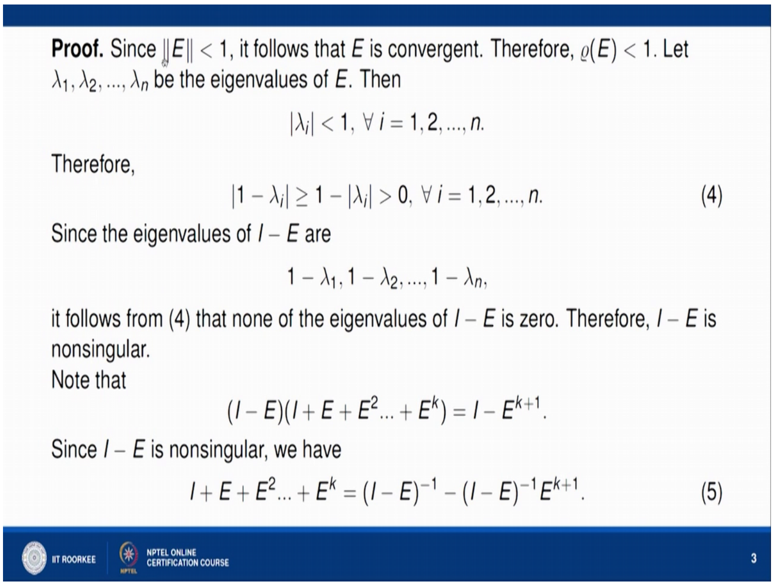

So, here we already know that norm of E is less than 1. So, if norm of E is less than 1 and we already know that a spectral radius of E is less than norm of E, so if norm of E is less than 1, we can say that E is convergent and we can say that spectral radius of E is also less than 1. So, spectral radius of E is less than 1 means the modulus a maximum of modulus of lambda i is where lambda i is are the eigenvalues of a should be less than 1 and this is, so it means that modulus of lambda is less than 1 for every I equal to 1 2 and here and I am assuming that E is a square matrix of size n.

So, here with this we want to we want to show that this I minus E is also non-singular matrix. For that let us use the concept of eigenvalues. So, here we say that modulus of 1 minus lambda i is greater than equal to 1 minus modulus of lambda i this is this follows from the simple triangle inequality of modulus function. So, if since modulus of lambda i is less than 1 for each i, so it means that modulus of 1 minus lambda i is strictly bigger than 0. So, it means that none of the eigenvalues of I minus E are 0 because all the eigenvalues of I minus E which are written as 1 minus lambda 1, 1 minus lambda 2 and 1 minus lambda n they are all positive through equation number 4.

So, it means at this a since no eigenvalues of I minus E are 0. So, we can say that I minus E is non-singular matrix. So, it means that we have concluded with from this fact that modulus of lambda i is less than 1 that I minus E is non-singular. So, this we have proved now we want to find out say a formula for inverse of I minus E. So, for that we simply observe this equality that I minus E if you multiply this I plus E plus E square 2 E, E to power k then it will be what; and if you simplify it will cancel out and it will coming out to be I minus E to power k plus 1.

Now, we already know that I minus E is invertible. So, we can multiply by I minus E. So, if you multiply I minus E, inverse of I minus E then it will be what? I plus E plus E square up to E to power k equal to I minus E inverse multiplied on this. So, here we can simplify that and we can write it I minus E inverse minus I minus E inverse E to power k plus 1, now this is true for every k. So, whatever k you write it is true. So, let us take the limit k tending to infinity.

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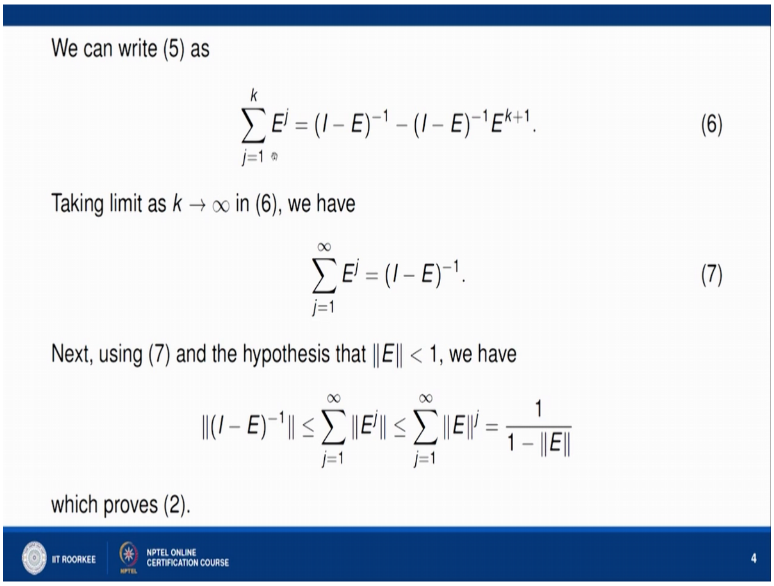

Then we can say that taking limit as k tending to infinity in this equation we have summation j equal to 1 to infinity E to E power j equal to I minus E inverse. Please remember here this is the term where we are applying the limit k tending to infinity. Now, since is convergent then limit k tending to infinity E to the power k plus 1 is tending to 0. So, here we are utilizing the fact that norm of E is less than 1 means the powers of E is standing to 0 as k tending to infinity. So, it means that we have this equation number 7.

So, next using this equation number 7 and hypothesis that norm of E is less than 1, we can write that we can find out the norm of I minus E inverse. So, if you look at norm of I minus E inverse will be what less than or equal to j is from 1 to infinity norm of E to power j. Now, here since this matrix norm is consistent norm. So, we can say that norm of E to power E to power j is less than or equal to norm of E to power j, right. So, here we have utilized j consistency of what vector matrix norm. So, this is less than I equal to j from 1 to infinity norm of E to power j. So, this is geometric series and common ratio here is norm of E and which is less than 1. So, we here we can utilize the sum of a geometric series and it is nothing, but 1 upon 1 minus norm of E; and this proves the result given in equation number 2.

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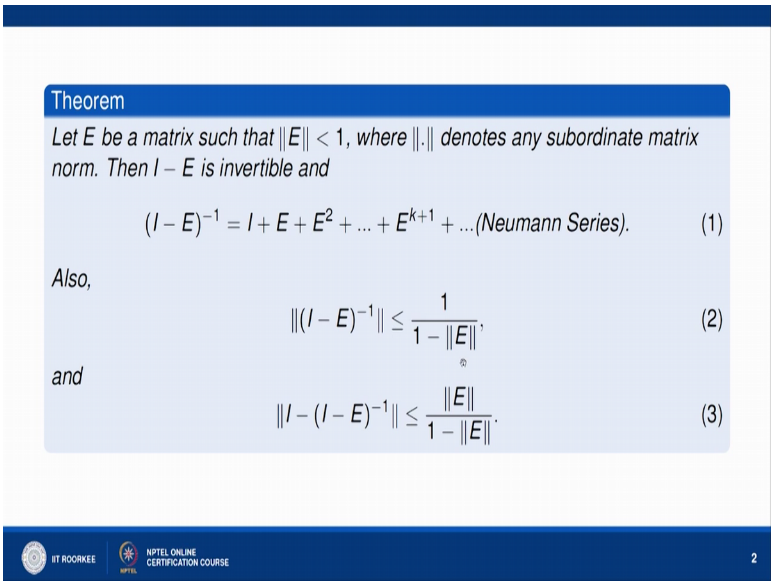

Now, with the help of this we want to prove this result number 3 also. So, to prove result 3 we observe that I minus E into I minus E inverse is identity.

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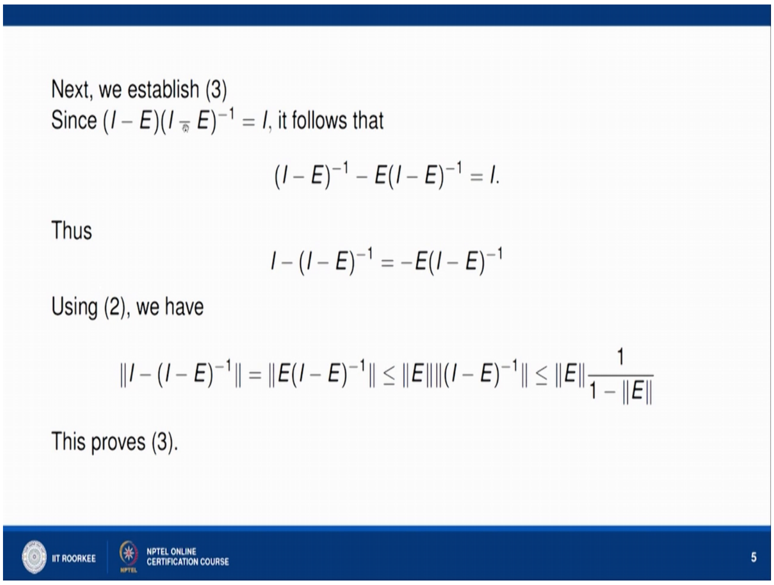

In fact, this is product of 2 matrix whose who which in, so I minus E inverse of I minus E is given here, so product is I. So, it follows that that if you simplify this then I minus E inverse minus E into I minus E inverse is equal to I.

So, now here we can simplify I minus I minus E inverse is equal to minus E 1 minus E inverse. So, here we can write it like this. Now, we want to find out say norm of this. So, norm of I minus I minus E inverse is going to be norm of E into a norm of E into 1 minus E inverse right. Now, here we can use the consistency of matrix norm and it can be further less than equal to norm of E into norm of 1, I minus E inverse and we already know that what is the bound of norm of I minus E inverse which we have just proved that norm of I minus E inverse is less than or equal to 1 upon 1 minus norm of E. So, utilizing this a result we can say that this is bounded by that norm of I minus I minus E inverse is bounded by a norm of E into 1 upon 1 minus norm of E, which proof the required result.

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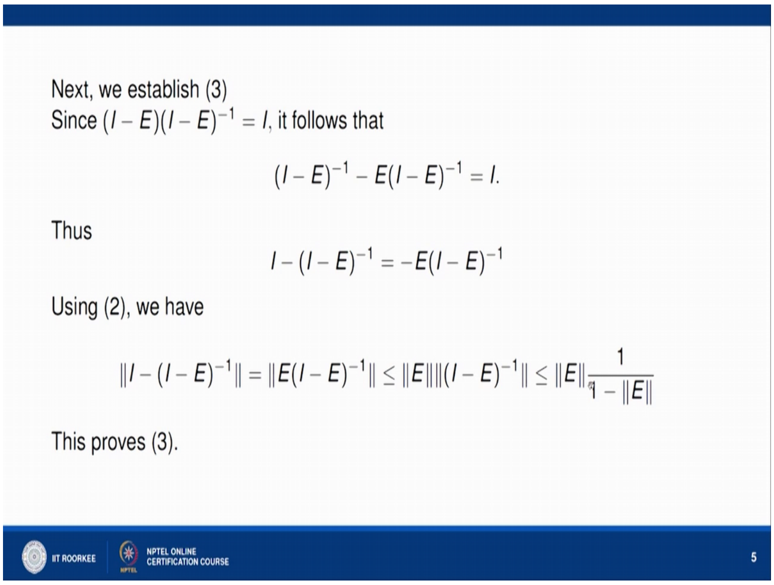

Now if you look at the last result proved is this. Now, with the help of this we can find out same relative either in finding the inverse of I. So, we can say that here look at that that norm of I inverse is basically 1. So, here we can say that norm of I inverse minus I minus E inverse divided by norm of I inverse is a bounded by this E norm of E divided by 1 minus norm of E. So, if you look at this is the formula we say that what is the relative error possible in calculating the I inverse so, here we can say that relative error in calculating I inverse is bounded by norm of E divided by 1 minus norm of E. So, it means that if this matrix E which is known as a perturbation matrix if it has a small norm then relative error is going to be small. So, it means that if perturbation is small then either relative error is also going to small.

So, this is the case when we are considering the identity matrix. Now, let us move to any non-singular square matrix and we try to find out say similar kind of result that whether and we have this kind of result for any non-singular matrix or not.

So, the next theorem which gives the result for any non-singular matrix. So, let A be any n cross n non-singular matrix and let B is equal to A minus E, where this E is a perturbation matrix and here E satisfy the following condition that norm of A inverse E is less than 1.

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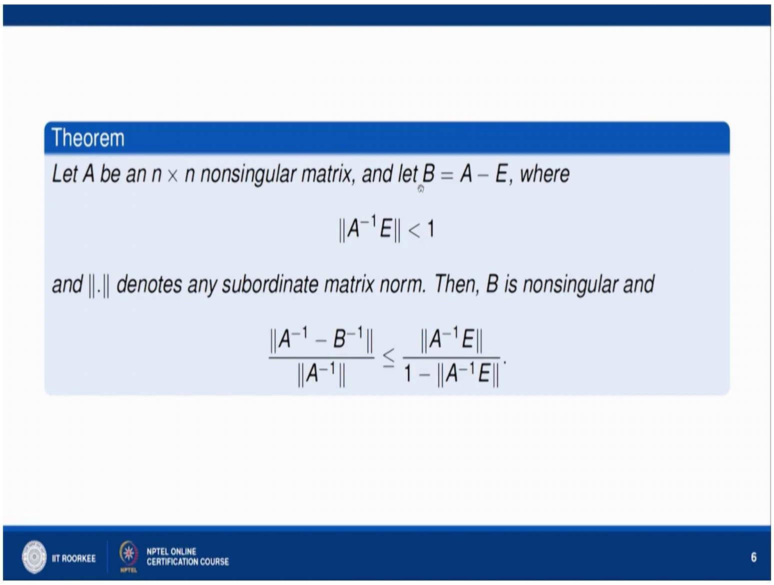

So, if this norm is less than 1 and of course, this norm is any subordinate matrix norm then we claim that B is non-singular and norm of A inverse minus B inverse divided by norm of A inverse less than or equal to norm of A inverse E divided by 1 minus norm of A inverse E and we will see that how we can relate this result in by the relative error in calculating the inverse of B. So, that is what we wanted to know. So, here the let us proof this theorem.

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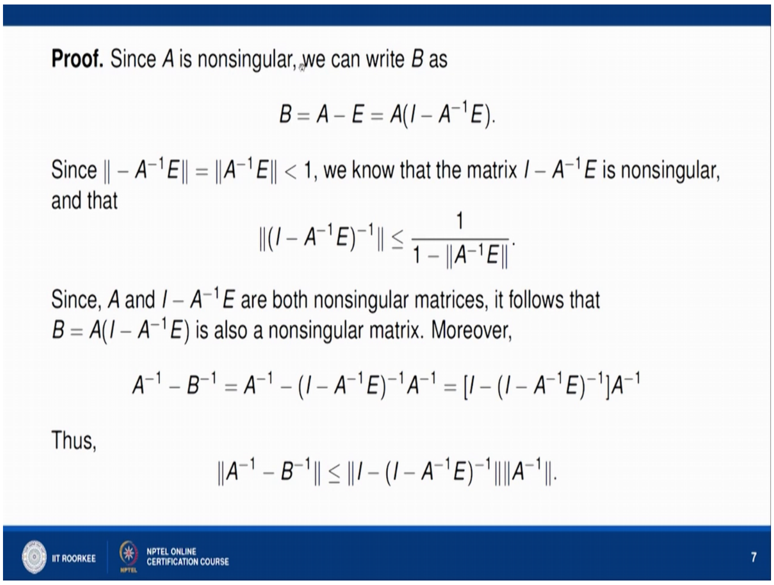

So, here since A is non-singular we can write B as B equal to A minus E and we can take we can write this as A into 1 I minus A inverse E.

Now, here we are trying to utilize the previous result. The previous result says that if you put our identity matrix by another matrix A E, such A norm of E is less than 1 then I minus E remain non-singular. So, here if you look at this is perturbation in identity matrix I minus A inverse E. So, if we consider this A inverse E as new perturb matrix and we already know that norm of minus A inverse E is nothing, but norm of A inverse E which is less than 1, it is already we have assumed here. So, here we have assumed that I is perturbed by this quantity A inverse E. So, it means that this remain non-singular because I is non-singular and we have proved that if perturbation matrix has norm less than 1 then I minus this is going to be non-singular.

So, it means that I minus A inverse E is going to be non-singular. So, this is non-singular is non-singular. So, we can say that B is non-singular. So, that proves the first part that B is non-singular now we want to find out say this bound. So, for that let us observe that norm of I minus A inverse E whole inverse is less than I equal to 1 upon 1 minus norm of A inverse E. So, that is nothing, but this equality inequality to. So, we are utilizing this now here in place of E we have A inverse E. So, norm of I minus A inverse E inverse is less than equal to 1 upon 1 minus norm of A inverse E.

So, utilizing this we are coming here. So, here we have this bound and we have already proved that B is non-singular matrix now let us calculate this A inverse minus B inverse. So, A inverse is as it as B is A, A into I minus A inverse E. So, we can write on B inverse as I minus A inverse E whole inverse into A inverse. So, here we can take out A inverse outside and in this I minus I minus A inverse equal inverse. So, here we can take out A inverse outside and it is I minus I minus A inverse E whole inverse. So, here taking the norm on both the sides we have norm of A inverse minus norm of B, norm of A inverse minus B inverse less than or equal to norm of I minus I minus A inverse E whole inverse into norm of A inverse.

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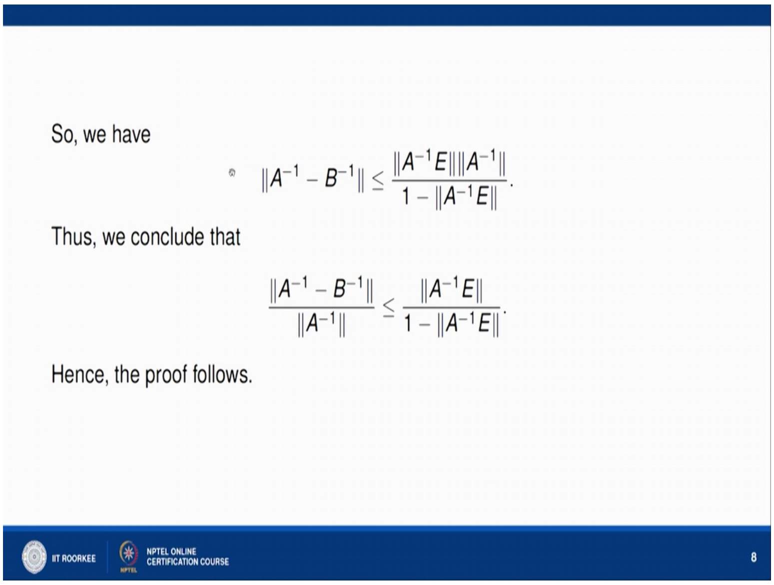

So, here we have that norm of A inverse minus B inverse and here we can utilize the bound of this now bound of this is basically what. Now, here this is something which we have already calculated in previous result, here A inverse, A is a perturbation matrix, so this is bounded by the norm of a perturbation matrix divided by 1 minus norm of the perturbation matrix. So, here perturbation matrix is inverse A, so we can say that this is bounded by norm of A inverse E divided by 1 minus norm of A inverse E into norm of A inverse which is already there here as norm of A inverse. So, we can say that norm of A inverse minus B inverse is bounded by this quantity. So, here if we divide by norm of A inverse. So, we can say that we can write it like this. Now this is what we wanted to prove here.

Now, if you look at what is our B here, B is your A minus E. So, we can say that A inverse minus A minus E whole inverse divided by norm of A inverse is bounded by this. So, if you look at this represent the relative error in finding A inverse. So, we can say that that this relative error in calculating A inverse is bounded by A inverse norm of A inverse E divided by 1 minus norm of A inverse E. Now, again here this quantity norm of A inverse E is bounded by norm of A inverse into norm of E.

So, it means that it is quite related to norm of E. So, if norm of E is less than means if norm of E is small then this quantity is going to be small and hence we can say that relative error is going to be small. And with this we can say that proof follows and we say that that if perturbation matrix has a smaller norm, then identity matrix will remain non-singular similarly any non-singular matrix will also remain non-singular provided that norm of A inverse is less than 1 or you can say that if norm of E is sufficiently small then we can say that perturbation of a non-singular matrix will remain non-singular. So, now with the help of this and the concept of norm we can say that a set of all matrices forms a matrix space and we can define a metric on set of all met matrices say of size n.

Now, we have one more result in that direction we say that if is denote the set of all n cross n non-singular matrices in set of all n cross n matrices then S is an open set in this set, right.

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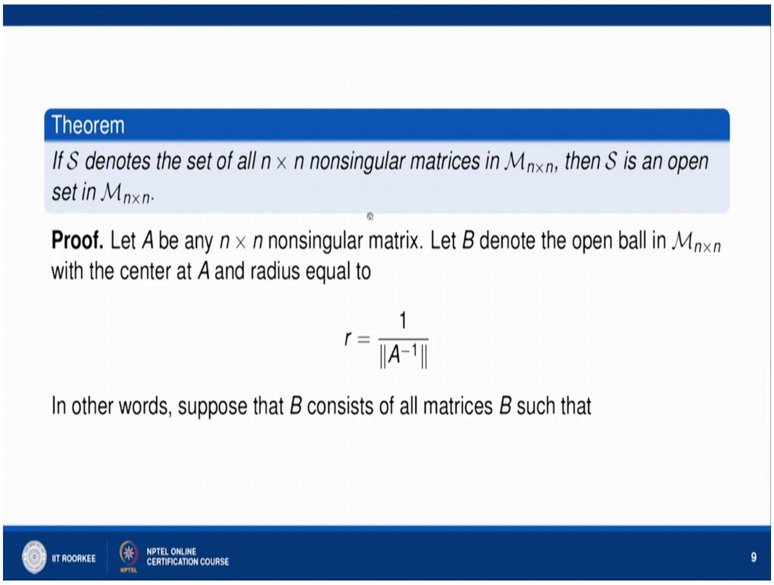

So, to show that it means that if we have a non-singular matrix then in a neighbourhood of that matrix we may have we have several other non-singular matrix. So, here let us proof our, prove our result. So, let A be any n cross n non-singular matrix. So, let capital N denote the open ball in set of all n cross n matrices with the centre at A and radius equal to r, where r is given by 1 upon norm of A inverse or in word in other word we can say that this capital N consists of all matrices B such that norm of A minus B is less than r.

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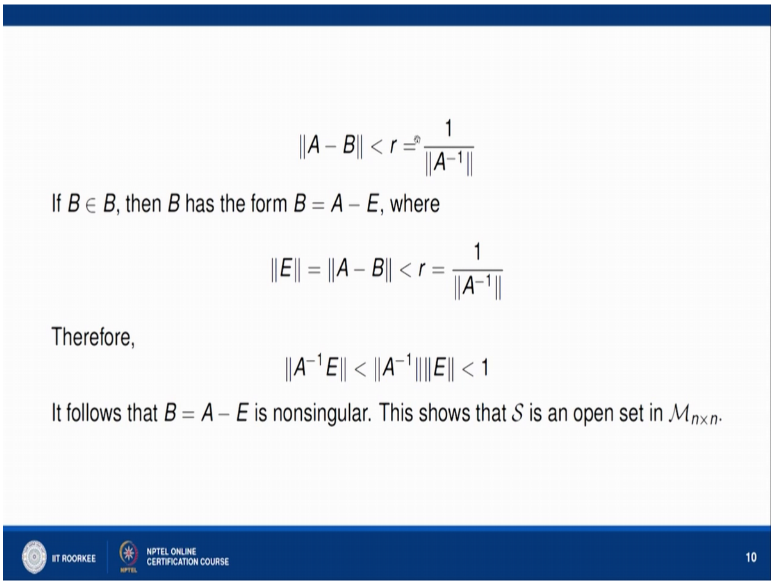

And here r is 1 upon norm of A inverse, it means that we say that the distance of these matrices from A is this our maximum this quantity. So, here if B belongs to B then B has a form B equal to A minus E, where E is the difference between A minus B. So, we can say that norm of E is going to be norm of A minus B and it is less than r. So, here norm of E is less than 1 upon norm of A inverse.

So, if you simplify we can say that norm of A inverse in which is bounded above by norm of A inverse into norm of E. So, we say that this norm of E is strictly less than this then norm of A inverse E is strictly less than 1. So, here by a previous result we can say that if norm of A inverse E is strictly less than 1, then B is equal to A minus E is non-singular. So, it means that what is B? B is an any arbitrary matrix in the neighbourhood of A with the radius r. So, it means at this S is an open set, where S represent the set of all n cross n non-singular matrices.

So, it means that if you have a non-singular matrix then if you look at in the neighbourhood of that non-singular matrix then we can find out another non-singular matrix. So, here we say that if A k be a sequence of real n cross n matrices which converges to a non-singular matrix A, then we can say that then A k is non-singular for large values of k and a limit k tending to infinity A k inverse is given by A inverse.

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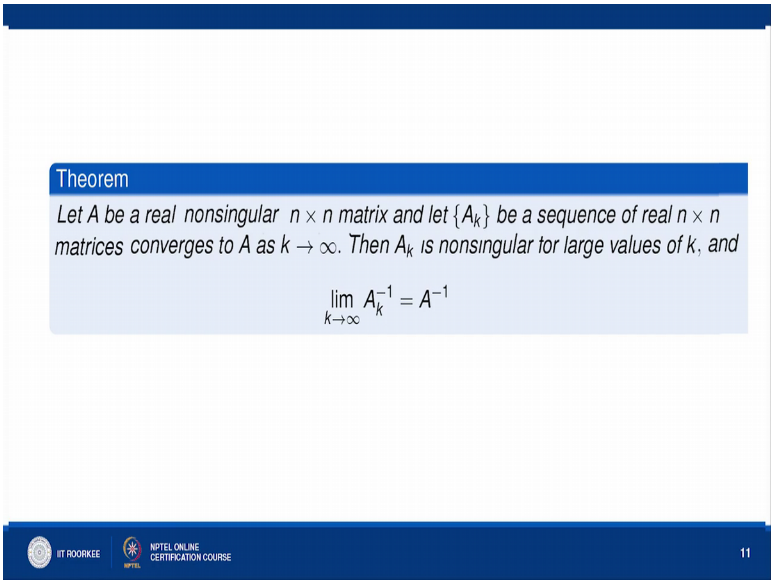

So, what is the meaning of this theorem? It says that if we have a sequence which converges to non-singular matrix then after say for large terms your sequence matrices are also non- singular. And we can also say about the inverse of those non-singular matrices that those sequence of non-singular matrices will also converge to A inverse that is what is the content of this theorem. Let us prove this.

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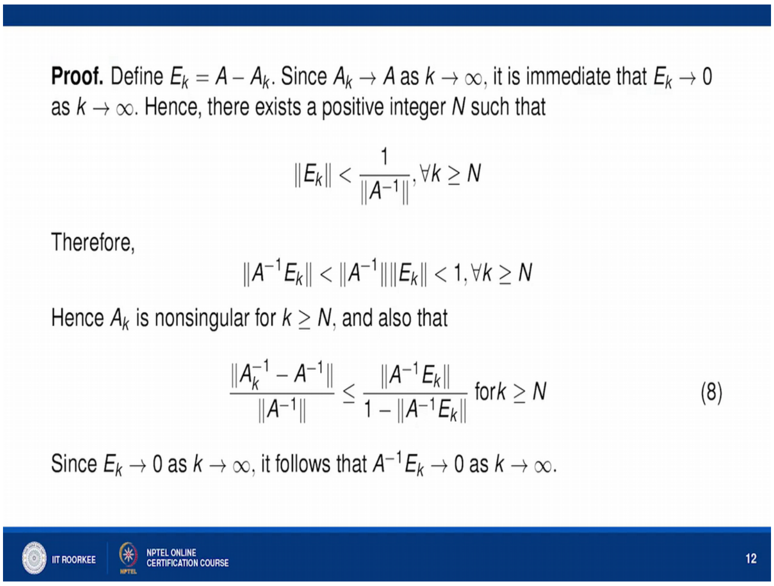

So, here define A k as a minus A k N. We already know that A k is converging to A, it means an E k is tending to 0 as k tending to infinity. So, we can say that E k is a convergent, E k is a matrix a; is a matrix which converges to 0. And hence we can find out a positive integer N such that norm of E k is less than 1 upon norm of A inverse. So, here we know that this E k standing to 0 means after say large k the norm of E k is less than some epsilon. Now, here epsilon I am taking as 1 upon norm of A inverse.

So, now with this if you look at this norm of A inverse E k which is bounded above by norm of A inverse into norm of E k then it is going to be strictly less than 1, for every k greater than or equal to this number N. So, this implies that this A k which is perturbation of A by this perturb matrix E k is non-singular for k greater than or equal to N. And also by a result which we have proved we can say that this norm of E k inverse minus norm of A inverse A k inverse minus A inverse norm divided by norm of A inverse is bounded by norm of A inverse E k divided by 1 minus norm of A inverse E k.

Now, if you look at the numerator, numerator is norm of A inverse E k and we know that as E k standing to 0 we can say that norm of A inverse E k which is bounded by norm of A inverse into norm of E k will also tend to 0. So, it means that since E k is standing to 0 as k tending to infinity it follows that A inverse E k is also tending to 0 as k tending to infinity. It means that this numerator that this norm of A k inverse minus A inverse is also tending to 0 as k tending to infinity. So, we can say that limit k tending to infinity A inverse E k is equal to 0, taking limit as k tending into infinity we have this limit skating into infinity A k inverse minus A inverse norm of this is equal to 0 and this is true it means at limit k tending to infinity A k inverse is your a inverse and which complete the proof of this.

And here also we conclude our lecture. So, in this lecture what we have seen that with as an application of convergent matrix we have seen that if we perturb our non-singular matrix by a small perturbation matrix whose norm is strictly less than 1, then this perturbation matrix A minus E, is going to be non-singular. And we have seen certain result in terms of relative error in calculating the A inverse, inverse of the non-singular matrix.

And as an application we have seen that if we have a sequence of matrices converging to a non-singular matrix then this sequence will consist after large values of course, consists non-singular matrices and inverse of these non-singular matrix will also converge to the inverse of non-singular matrix. So, that is what we have discussed in this lecture. So, here we conclude our lecture. The next lecture we will continue our study, thank you very much.

Thank you.