Numerical Linear Algebra
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Lecture - 32
Matrix Norms – II

Hello well welcome to this lecture, in previous lecture we have discuss a some Matrix Norm. So, so far, we have discussed one Matrix Norm 2 infinity Matrix Norm, and in last class we have discuss 2 Matrix Norm and these Matrix Norm are subordinate Matrix Norm. And now we want to discuss one more important kind of Matrix Norm known as Frobenius Matrix Norm and then we try to discuss certain properties, of these norms and we want to discuss why some norms are preferable and some norms are non-preferable.

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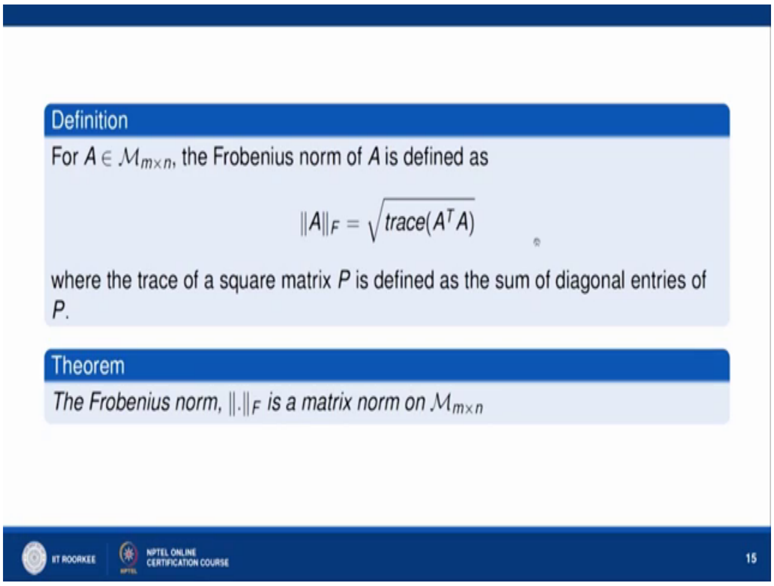

So, let us consider the definition of Frobenius norm. So, let a A be a m cross n matrix. The Frobenius norm of A is defined as under root trace of A transpose A. So, here a trace of a square matrix P is defined as the sum of the diagonal entries of P. So now we first thing we want to prove here, is that this actually define a norm a matrix norm, so we the theorem is that the Frobenius norm is a matrix norm, on a set of all m cross n matrix. So, this actually defines a matrix norm on vector space of m cross n matrices.

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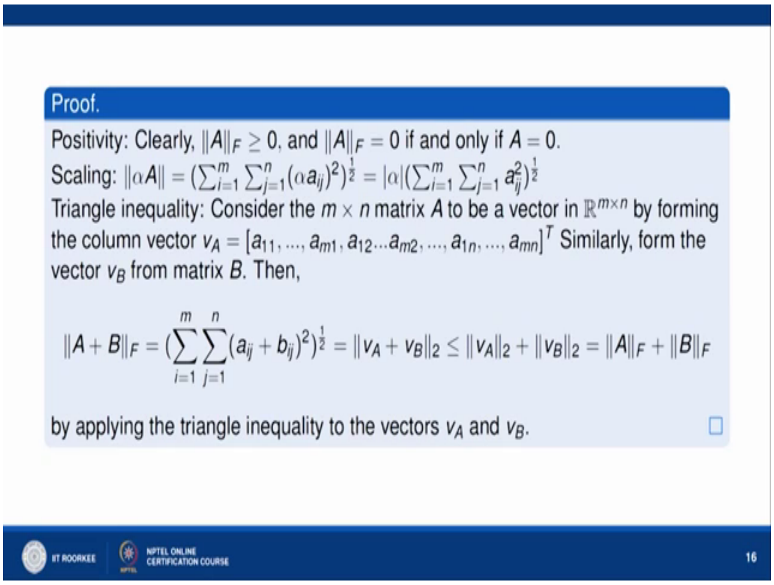

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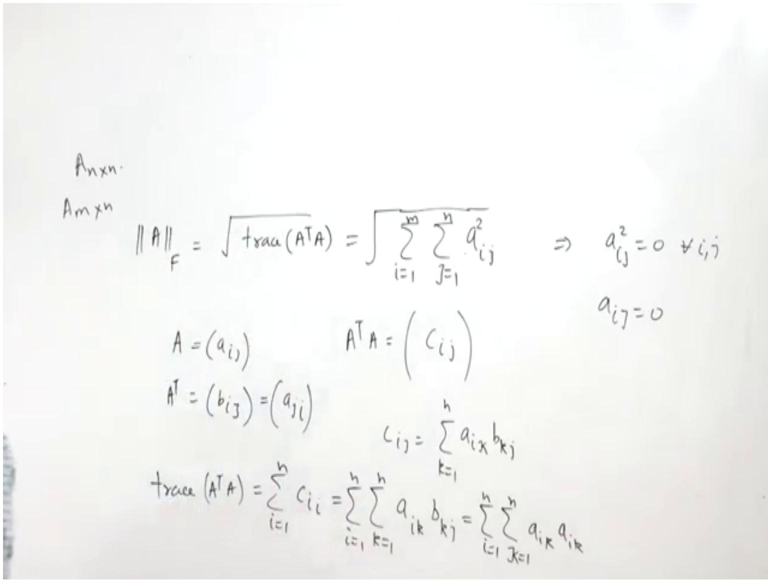

So, we need to prove three conditions first condition is positivity that, F norm of A is nonnegative, so if you look at the F norm of A, F norm of A is actually, what? If you look at F norm of a is basically, under root of trace of A transpose A. Now here, let us look at what is A transpose A so, if A is A i j and A um A transpose will be b i j sorry j bi j and j b i j is nothing, but a j i then to calculate a transpose A, A transpose A is going to be say. Let us say, C i j j what is C i j? C i j is given as summation k is from 1 to n jai j a i k b k j j right?

So, a transpose is given by this. Now, trace of A transpose A will be what? A transpose A is basically summation i is from 1 to n C i i so basically it is nothing, but summation I is from 1 to n summation k is from 1 to n, here this is A I k and B k j here B k I here. Now, we already know that B k I is nothing, but A k I so this is given as summation, I is from 1 to n summation j k is from 1 to n and here we have A I k into A I k so this is square of this. So, here this I am writing here for n cross n matrix, so if you want to write m cross n matrix, then the corresponding I n j will change. So, here it is if we take m cross n matrix, then I will run from 1 to n rather than running from 1 to n. So, here this I have given for any square matrix, but by suitable change of notation you can also define F norm of a here.

So, basically F norm of A is given by, summation under root summation I is from 1 to m, j is from one to n, A square I j here. So, it is basically, under root of sum of all the elements square. So, that is your Frobenius norm. So, positivity follows from the definition of F norm and we say that it is equal to zero, it means that summation this this is equal to zero provided that the summation of all the a square term is going to be zero. So, that implies that this happens only when that A i j square is equal to zero for all I and j and this is nothing, but A i j is equal to zero which is nothing, but your matrix A is simply zero matrix.

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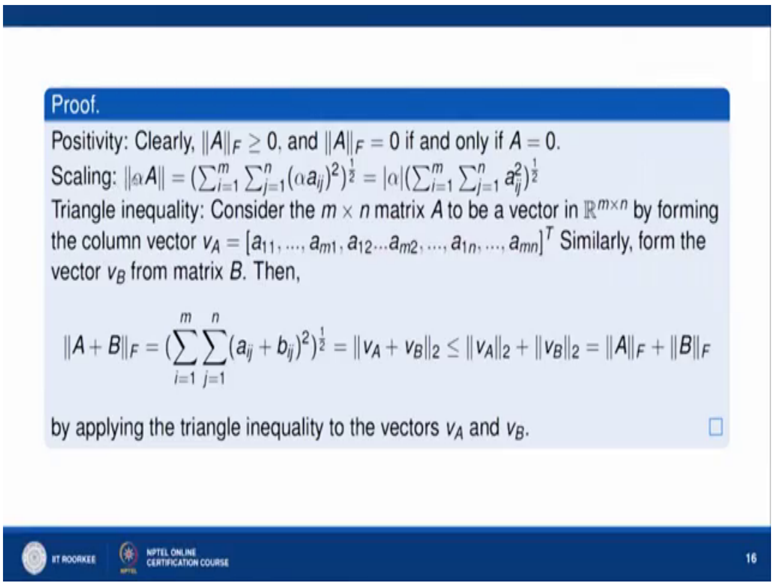

Now, coming out to the second point that is, scaling point calculate the alpha and a F norm of alpha A which is nothing, but summation I equal to 1 to m j equal to 1 to n alpha A i j square to power 1 by 2. Now, here you can take out alpha I square out and when you take out the square root, it is coming out to be modulus alpha summation I equal to 1 to m summation j equal to 1 to n a square I j to power half. Now this is nothing, but F norm of A. So, that satisfy the scaling property that F norm of alpha A is equal to modulus alpha F norm of A here. Now, coming out to be coming out to last property that is triangle inequality.

So, here what we try to do here, we consider this matrix A which is of size m cross n as A vector in R m cross n by forming the column vector V A, as A 1 1 to A m 1 comma A 1 2 to A m 2 comma A 1 n to A m n. So, what we are looking at this matrix as, you take the first column and then second column and third column and write in a column wise right? So, that we are denoting as V of A here.

So, V of A is a vector in R m cross n, similarly we can form the vector V B from matrix B. So, now F norm of A plus B is basically what? So, here if we use the this is nothing, but summation I equal to 1 to m j from 1 to n A i j plus B I j whole square 2 power half. So, you find out A plus B and then you take the square of all the elements and take the square root ah. So, this we can consider at if you consider V A as this and V B as a corresponding.

So, this can be written as 2 norm of V A V A plus V B, now here this is a vector norm. So, we can apply the Cauchy Schwarz triangle inequality for 2 norm and 2 nor[m] - it is given as this, is less than equal 2 norm of V A plus 2 norm of V B. So, 2 norm of V A is nothing, but F norm of A and 2 norm of V B is given as F norm of B. So, it means that F norm of A plus B is less than or equal to F norm of A plus F norm of B. So, here we have looked at this, A as a vector in R m cross n and we have utilized triangle inequality for 2 norm 2 vector norm and we have proved the triangle inequality for, Frobenius Matrix Norm.

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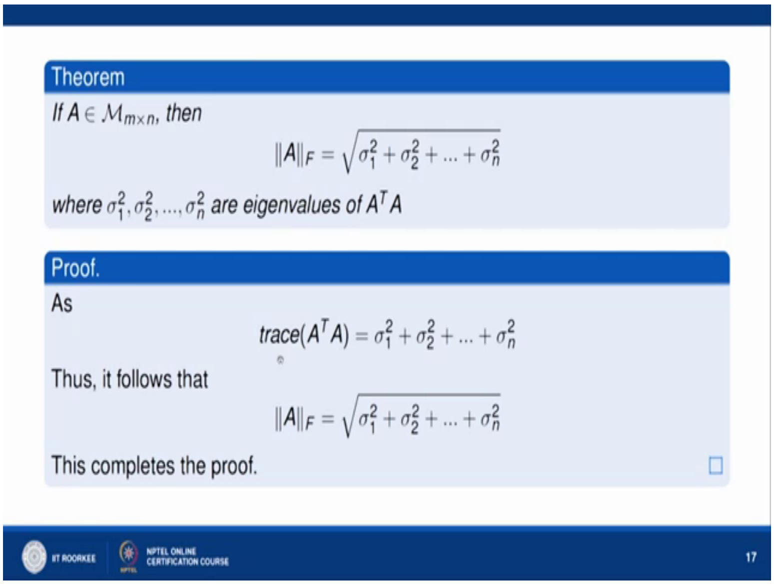

So, now let us consider one more theorem, which simplifies the simplify or we can say that, here Frobenius norm of a is given by under root of sigma 1 square plus sigma 2 square plus sigma n square, where the sigma 1 square sigma I squares are eigenvalues of A transpose A. So, it means that, if we can calculate a the eigenvalues of A transpose A, then F norm of A is given by under root under root of sum of the eigenvalues of A transpose A. So, proving this is not very difficult. So how we can prove let us say, that trace F norm of A is under root of trace of A transpose A. Now, trace of A transpose A is basically sum of the diagonal values. So, F norm of A is given by under root of trace of A transpose A, now trace of A transpose A is sum of the diagonal entries.

Now, we know that some of the diagonal entries, is same as the sum of the eigenvalues of A transpose A. So, it means that trace of A transpose A is same as sum of all the eigenvalues of A transpose A. So, that can be written as sigma 1 square plus sigma 2 square plus sigma n square. So, using this relation we can write F transpose of as F Matrix Norm of a as under root of sigma 1 square plus sigma 2 square plus sigma n square and this complete the proof. So, here this may be used as an alternative way to find out Frobenius norm of a.

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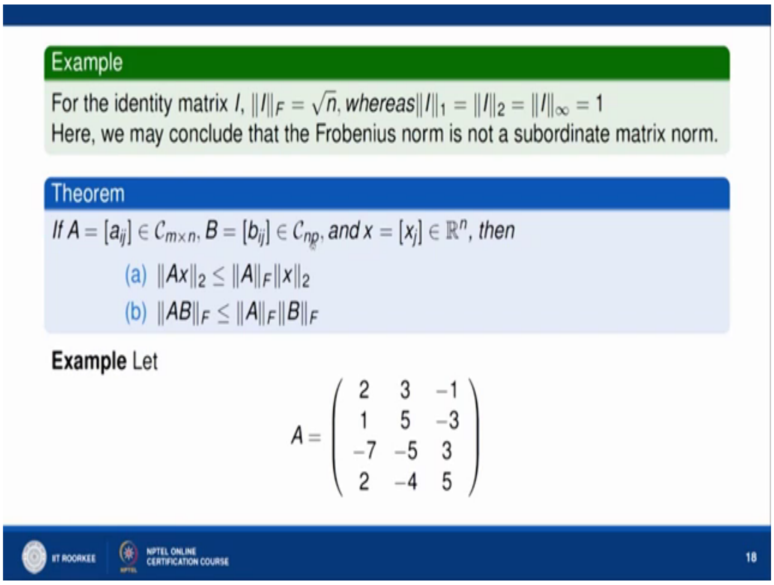

So, let us consider certain example. So, let us consider the identity matrix. So, if you calculate the F norm of A F norm of I it is coming out to be root n. So, here we have utilized the previous result, if you look at identity matrix. For identity matrix 1 1 1 are eigenvalue eigenvalues of A transpose A, so if you look at this sum this is coming out to be n here.

So, F Frobenius norm of I is coming out to be n here. So, for forbenius Frobenius norm of identity matrix is coming out to be root n. But, if you calculate other Matrix Norm, such as 1 norm 2 norm or infinity norm these are coming out to be 1 here. So, here with the help of this, we want to infer that this Frobenius norm is something which is not given as subordinate Matrix Norm. So, how we can prove this let us see here.

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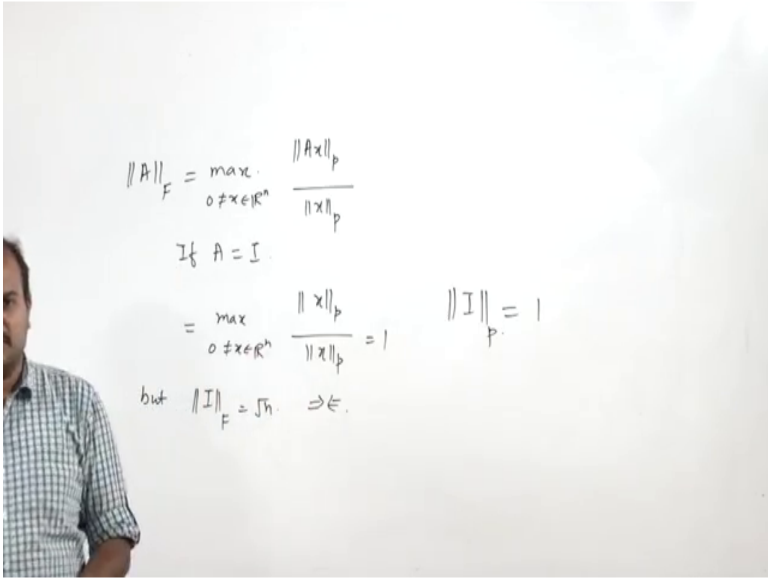

So, to prove that this Frobenius Matrix Norm is not a subordinate Matrix Norm, as we have considered 1 norm 2 norm or infinity norm. So, to let us say that suppose it is. So, it means that a Frobenius Matrix Norm is a matrix subordinate norm, corresponding to say p norm where p is anything which is bigger than 1. So, if we calculate this. So, if A is equal to I then we can say that this definition reduce to this, that maximum taken over all the nonzero value nonzero vector of R n and now here, A x is nothing, but p norm of x and this is p norm of x and this is nothing, but 1.

So, it means that a maximum of this quantity can be mac maximum. So, it means that if we consider, any subordinate Matrix Norm then, subordinate Matrix Norm of I has to be 1 right? But we have already calculated that Frobenius Matrix Norm of I is nothing, but root of n here. So, it means that here left hand side is not equal to left right hand side is not equal to left hand side, so we can say that this Frobenius metric Frobenius Matrix Norm is not a subordinate Matrix Norm, corresponding to any vector norm p, right? So, now, coming out to next theorem, next theorem says that this 2 norm n Frobenius Matrix Norm is consistent with Matrix Norms.

So, it means that if A is A i j is A matrix of size m cross n, and B is another matrix of size n cross p and x is a vector in R n then 2 norm of A x is less than or equal to F norm of A and 2 norm of x. Similarly, F norm of A B is less than or equal to F norm of A and F norm of B. If you look at the second property second property says, that this Frobenius Matrix Norm is a consistent Matrix Norm. So, let us prove this.

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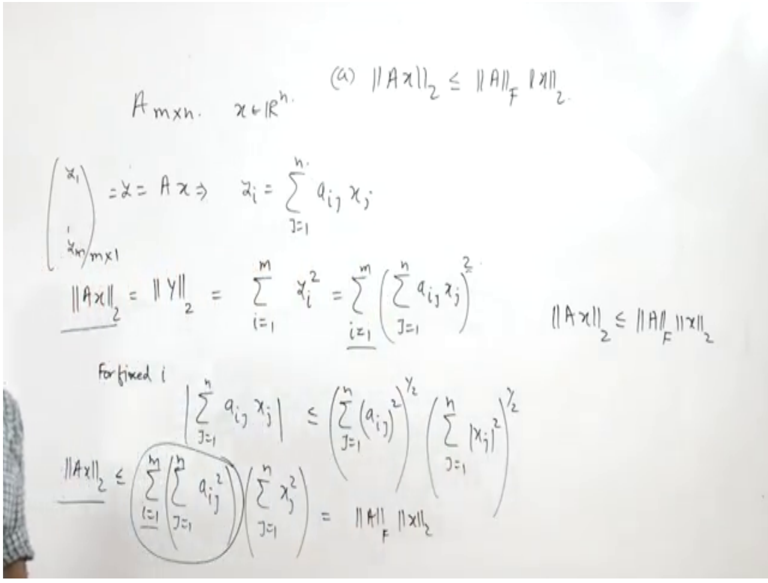

So, to the if to prove the first part, which we want to prove like this that, two norm of A x is less than or equal to F norm of A into 2 norm of x here, this we want to prove. So, to prove this, let us say that since we know that A is m cross n matrix, where x is in R n. So, if we look at y as A x we denote y as a of x here so here y is going to be element in R m. So, here we can say that it is m cross 1 here. So, y I is component of y is given as j equal to 1 to n A i j x j.

So, it means that if we want to calculate a 2 norm of A x which is nothing, but 2 norm of y ah, since we have assume y as a of x. So, 2 norm of y is given as summation I equal to 1 to m y I square. Now y I is given as summation j equal to 1 to n A i j x j, so this can be written as I equal to 1 to m sum summation over j equal to 1 to n A i j x j whole square. Now here to simplify this thing will use the Cauchy Schwarz inequality.

So, here please observe that if you fix I then summation j equal to 1 to n A i j x j. So, here for fix I this A i j represent A vector in R n. So here we can apply the Cauchy Schwarz inequality, which says that this quantity modulus of this quantity is less than or equal to less than or equal to norm 2 norm of this and 2 norm of this. So, we can say that this is equal less than or equal to j equal to 1 to n summation modulus of A i j square 2 power half and into j equal to 1 to n modulus of A i j square 2 power half.

So, using this and applying here. So, this is a power 2 here. So, it means that this can be written as, two norm of A x is less than or equal to summation I equal to 1 to n and the upper bound of this which is given by this. So, which is given as I equal to 1 to m j equal to 1 to n A i j square power half is managed by this, and into j equal to 1 to n x j square. So, here this I equal to 1 to m A, this is independent of I.

So, this we can take it out and what is left here, if you look at this is nothing, but Frobenius norm of matrix A. So, this can be written as less than this is this quantity is less than or equal this quantity is nothing, but Frobenius norm of A and this is nothing, but 2 norm of x. So, this implies that that, 2 norm of A x is less than or equal to Frobenius norm of A into 2 norm of x here, so which proves the first part of the theorem.

Now, to prove the second part of the theorem that, this Frobenius Matrix Norm is a consistent Matrix Norm, let us look at here ok. So, to prove the consi that Frobenius norm is a consistent Matrix Norm, we have to show that Frobenius norm of A B is less than equal to Frobenius norm of A and Frobenius norm of B here. So, to prove this, we consider that A is m cross n matrix and B is n cross p matrix. So, let us denote A by A i j and B as B I j. So, it if you form the product here A B, A B is given as say, C i j where C i j is can be obtained by k equal to 1 to n A I k B k j. So here I is running from 1 to m and j is running from 1 to p.

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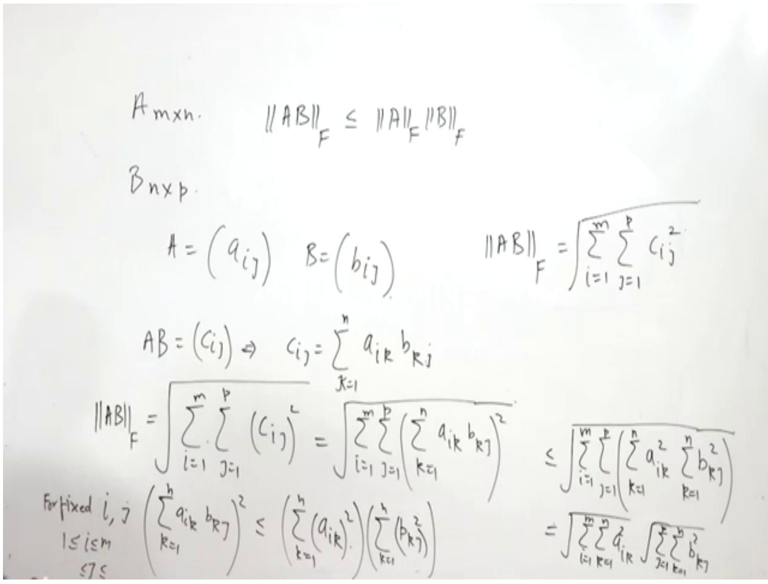

So, it means that, if you look at Frobenius norm of A B will be what Frobenius norm of A B is given as summation , I is from 1 to m and and j is running from 1 to p and C i j square here. So here a using this F norm of A B is given by under root I have to apply under root here. So, it means that under root I equal to 1 to m j is from 1 to p C i j square. Now use a value of C i j which is given here k equal to 1 to n A I k B k j.

So, here utilize the value of C i j which is nothing, but k equal to 1 to n A I k B k j square. Now to handle this we again use the Cauchy Schwarz inequality now what is Cauchy Schwarz inequality. So, if we fix this I so let us say, that for fix I where I is running from 1 to m ah, we can see that A I k is A member of R n and, similarly B k j is a member in R n here right? So, for fix I n j, you can say j also, because of fix fix I and j.

So, j is running between say, your j is running between 1 to p here. So, we can say that here, this is given as k equal to this is less than or equal to k equal to 1 to n A I k square into k equal to 1 to n B k j square. So here we have utilize the Cauchy Schwarz inequality. Now utilizing this bound here, this can be written as less than or equal to summation I equal to 1 to m j equal to 1 to p, which is as it is only thing we have applied the upper bound of this upper bound we have taken this. So, here we can write it k equal to 1 to n k equal to 1 to n A square I k into k equal to 1 B k j square. So, this we have written here.

So, now if you look at here we do not have j and here we do not have I. So, here this expression can be simplified further and here, we can write as product of 2 vector, we can write it this as I equal to 1 to m k equal to 1 to n A square I k into under root of quantity j equal to 1 to p k equal to 1 to n B square k j. Now, if you look at what is this quantity this quantity is nothing, but Frobenius Matrix Norm of a, similarly this quantity is what this is Frobenius Matrix Norm of B here.

So, this we can write it here that, Frobenius norm of A B is less than or equal to Frobenius norm of matrix A and Frobenius norm of matrix B which proves the claim ah, which we have stated here, that Frobenius norm of A B is less than or equal to Frobenius norm of A and forbenius norm of B here. So that prove the theorem here, now let us consider an example for which, we consider a as 2 1 minus 7 2 and 3 5 minus 5 minus 4 minus one minus 3 3 and 5 and here. We want to calculate the Frobenius norm of A here for that we calculate a transpose A. So, A transpose A, we have calculated like this, but here rather than consider -. So, we need to consider the trace of A transpose A.

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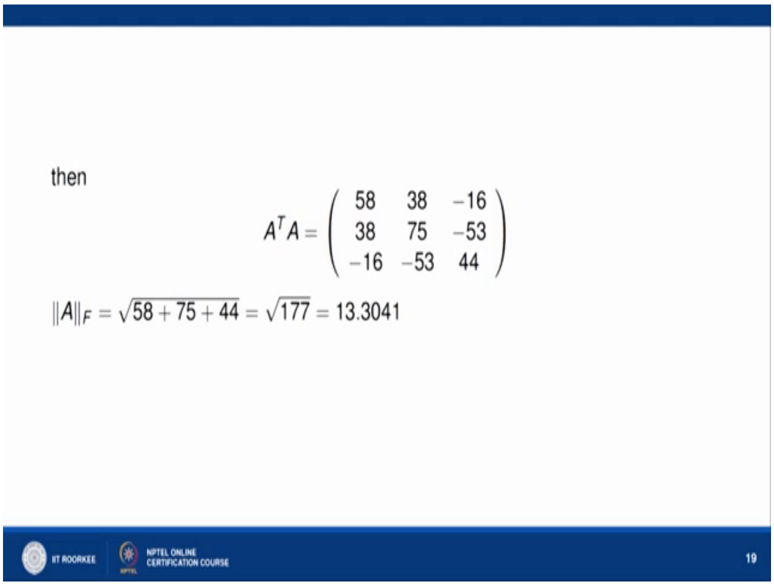

So, we simply consider trace of a which is coming out to be under root once trace is 171. So, Frobenius norm of A is given as under root of trace of a transpose A, which is given as 13.3041. So, Frobenius norm of A where A is given here, is given by under root of trace of A transpose A, which is nothing, but 13.3041.

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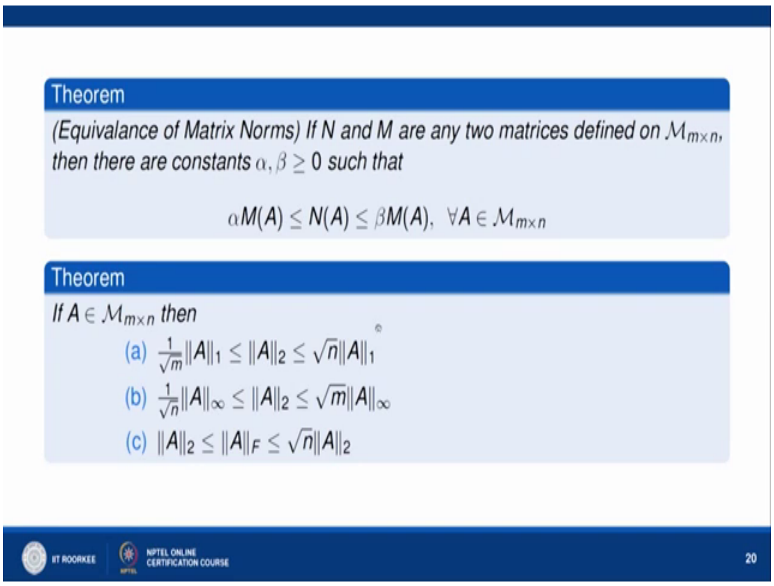

Now, as we have discuss the equivalence of vector norm the, similar concept we can do for equivalence of Matrix Norm. So, two norms say n n m are said to be equivalence, if they exist constants alpha and beta nonnegative, such that alpha m is less than n A is less than beta m A for every A in vector space of m cross n matrices.

So, if such an alpha beta exists, we say that m n n are 2 equivalence metric equal equivalent Matrix Norm. So, using this we can prove the following theorem, we say that 1 and 2 norm are related by this. So, it means that 1 and 2 norm are say m equivalent and following this condition, that A 1 norm of A divided by root of m is less than equal 2 norm of A and less than or equal to root of n 1 norm of A and so on. So, this I am not going to prove ah, but I can give you a small hint, I you just use the definition of 2 norm of A ah, since it is a metric subordinate norm and then, this can be written as maximum of 2 norm of A x divided by 2 norm of x and then use the use the, similar of similar relation which we have proved for vector norms and this we can easily be proved.

So, this I am skipping it. So, one last thing regarding this part c that 2 norm of A is less than or equal to F norm of A is less than or equal to root n 2 norm of A, this we can easily prove, because this we have not discussed earlier.

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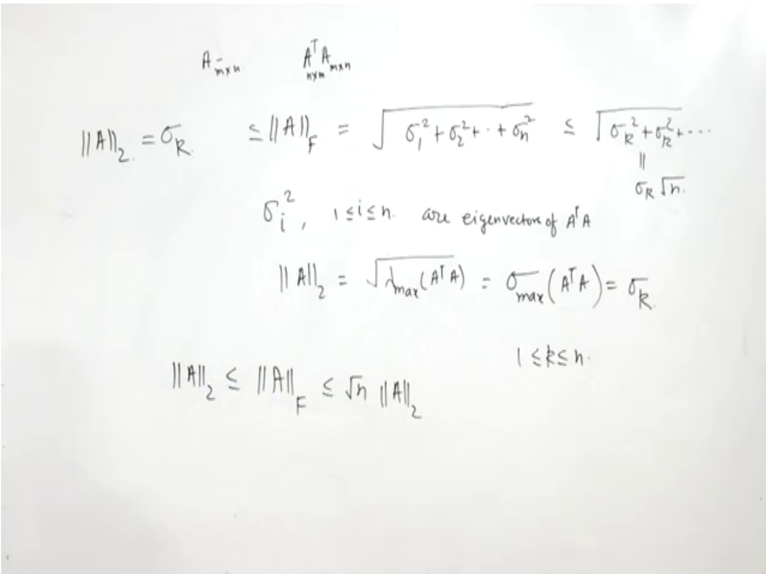

So, this let us consider so to show that Frobenius norm and 2 norms are equivalent to each other. So, it let us observe this fact that f norm of A is given by under root of sigma 1 square plus sigma 2 square plus sigma n square, this we have already proved here and here, sigma I squares are eigenvalues of A transpose A and I is running from 1 to n and 2 norm of A. If we recall this is nothing, but under root of lambda max of A transpose A and, which is known as sigma max of A transpose A ah. Let us suppose that this k is from 1 to n A is a is a value for which we achieve the maximum.

So, it means that sigma max A transpose A is given as sigma k. So, it means that 2 norm of A is given by sigma k. Now, with the help of this we want to show that F norm of A and um 2 norm of a are equivalent to each other. So, if you look at this, quantity is what if we look at this quantity is bigger than or equal to sigma k square. So, it means that this quantity is bigger than under root of sigma k square and we can write it sigma k.

Now sigma k we can write as 2 norm of A similarly, we can show that, if we replace sigma I square by sigma k square then, sigma 1 square is less than sigma k square sigma 2 square is less than sigma k square and so, on. So, every sigma a square is less than or equal to sigma k square. So, we can write this as under root sigma k square. So, these are n number of values. So, we can write it sigma k under root n. So, which is this proves that 2 norm of A is less than equal to F norm of A is less than or equal to under root n 2 norm of A which is the content of the result.

So, this we have proved here, so this we have already proved. So, we have proved the last part of this theorem and let us move to next result and this theorem is important theorem.

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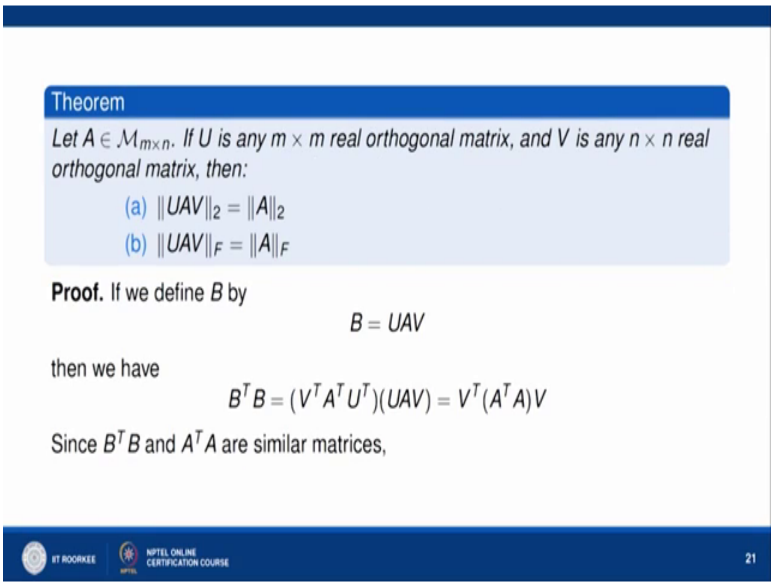

Let us understand the importance of this, let A be a matrix of size m cross n and U is any m cross m real orthogonal matrix and V is any n cross n real orthogonal matrix, then we you want we want to prove that 2 norms of U A V is equal to 2 norm of A, and F norm of U A V is equal to F norm of A. So, if you understand what is mean what the meaning of this is. So, it means that that 2 norm and F norm are invariant Matrix Norm under the multiplication of orthogonal matrices.

So, it means that if we multiply by orthogonal matrices A, still there is no change in 2 norm and F norm. So, that is the say usefulness of this theorem. So, will see how useful is this. So, to first let us prove this proof is not very difficult. So, let us define this U A V as a new matrix say B. So, B is U U A V. So, we can calculate B transpose B as V transpose A transpose U transpose into U A V, now U is orthogonal matrix. So, U transpose U is nothing, but identity, it is nothing, but V transpose A transpose A V. Now it means that B transpose V can be written as V transpose A transpose A V. So, this shows that B transpose B and A transpose A are similar matrices.

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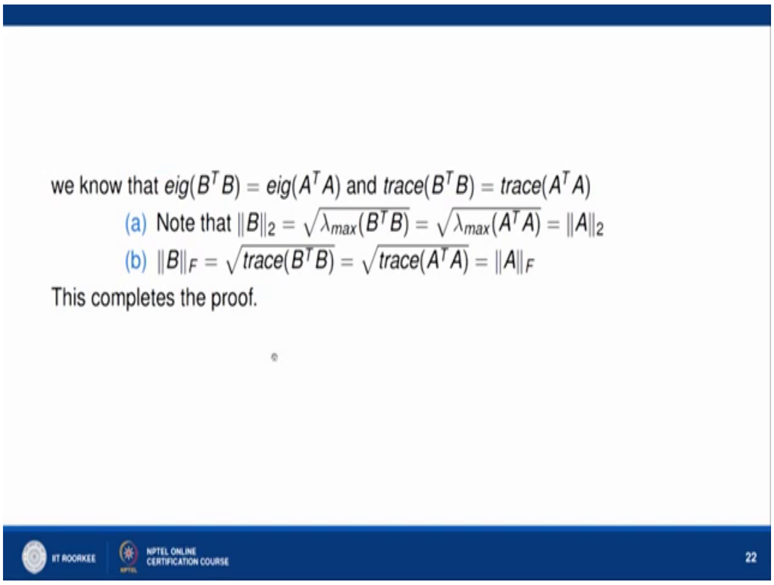

So, we already know that similar matrices are having a lot of good properties the important property is that; they share the same set of eigenvector. So, it means that eigenvectors of B transpose V B is same as eigenvectors of eigenvalues of A transpose A. So, they have the same eigenvalues and is. So, we can also prove that trace of B transpose B which is nothing, but sum of all the eigenvalues of B transpose B is same is equal to trace of A transpose A. Now, with the help of this we can prove that 2 norms of B which is given as under root of lambda max B transpose B and since, we already know that eigenvalues of B transpose B as same as eigenvalues of A transpose A.

So, this quantity is equal to lambda max of A transpose A which is nothing, but 2 norm of A. So, it means that 2 norm of B is same as 2 norm of A. Similarly, we can say that consider the F norm of B, F norm of B is nothing, but trace how B transpose B, now trace of B transpose B is nothing, but trace of A transpose A. So, this is given as under root trace of A transpose A. So, this is nothing, but F transpose of A. So, it means that F transpose F norm of B is same as F norm of A and A 2 norms of B is same as 2 norm of A and, what is B here B is a some matrix which is obtained from A by multiplying the orthogonal matrices.

So, it means that this 2 norm and F norm is invariant under the multiplication of orthogonal matrices. So how why this result is so important? Let us, try to understand.

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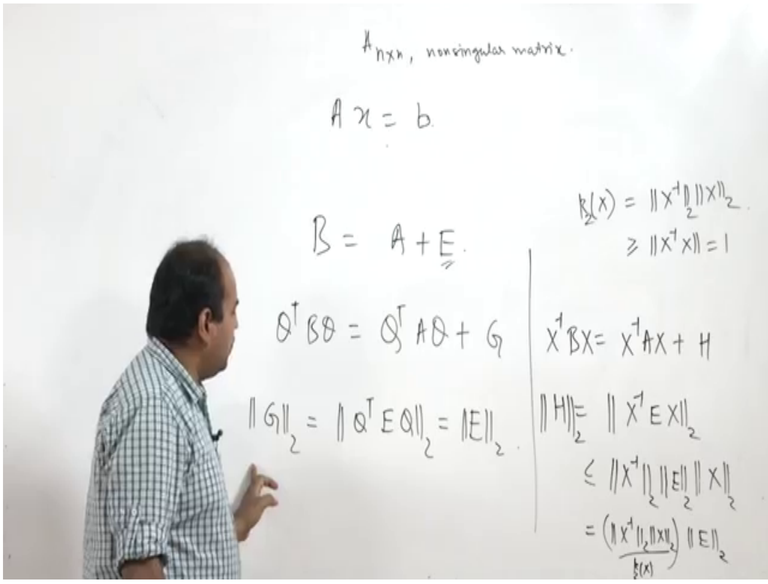

So, here we try to understand why this 2 norm and Frobenius norm is useful? Because if we look at in practical, if you want to solve a simple matrix equation say A x equal to B, here A is n cross n non-singular matrix. Then to solve this matrix equation, we try to use a say our canonical form and then we try to solve this equation, but our idea is at suppose it may happen that, while calculating these matrices A and B we may have certain error.

So, let us say that let B is a matrix which can be written as A plus E, where this E represent the error in calculating the coefficient matrix A and then, we try to see that, if we use A orthogonal matrix to solve this equation and what happen, if we use any other non-singular matrix to simplify this. So, if we use an orthogonal matrix to handle this situation, then we can say that Q transpose B Q is equal to Q transpose A Q plus G where G is what G is Q transpose E Q. So, G is written as Q transpose E Q and then if we take the 2 norm or say F norm then, we can say that 2 norm of G is written as 2 norm of Q transpose E Q and this we have already proved that, since Q is an orthogonal matrix, then this is nothing, but 2 norm of E.

So, it means that when we perform any operation with the help of orthogonal matrix, then in this step we do not have any say increase in error term. So, earlier in this term we have error as the E and if in next step also when we performed operation by with the help of orthogonal matrix, there also the term G has the error whose magnitude is not changed. In fact, we have not we have seen that magnitude remains the same so it means that error is not propagating.

But A the same thing we can show for F norm of G here. So, because F norm of G will also hold the same situation, but in case if we use any other matrix other than orthogonal matrix. So, let us say that x is A any non-singular matrix, and we are using that to solve this A x equal to B then x inverse B x is equal to x inverse A x plus H, where H is given as x inverse E x. So, if we use 2 norm of here, then since x is not orthogonal matrix. So, we cannot say anything here. So, let us say that 2 norm of H is written as 2 norm of x inverse E x. So, which can be written as less than or equal to 2 norm of x in x inverse 2 norm of E and 2 norm of x here.

So, which we can say that some constant multiple of norm of 2 norm of E now this we can this quantity 2 norm of x inverse into 2 norm of x, we call this as kappa 2 x and kappa 2 x ah, we can write it like this, and we can easily prove that this quantity is greater than or equal to 1 here, and equality is achieved when we have x as orthogonal orthogonal matrix, but in general your this quantity is greater than or equal to 1. So, it means that in place of orthogonal matrix, if we use any other non-singular matrix and use the 2 norm then, here your error may propagate.

But if we use orthogonal matrix operation then, and use the 2 norm or F norm then, your error is not going to propagate and this is the reason, why? This 2 norm and Frobenius norms are more popular and the use of orthogonal matrix is more popular in numerical linear algebra.

So, here we stop with this point, and in next lecture we will discuss some more points of or some more concept of numerical linear algebra.

Thank you for listening us. Thank you.