Numerical Linear Algebra
Dr. D. N. Pandey
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 31
Matrix Norms- 1

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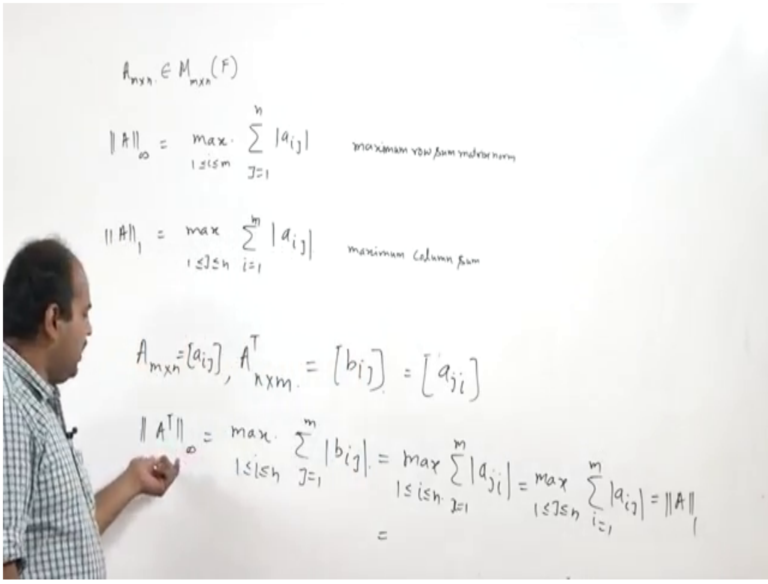

Hello friends, welcome to this lecture. In this lecture, we will continue our study of matrix norm. So, if you recall, in previous lecture, we will discussed matrix norm subordinated by vector norm. So, in previous class, we have discussed infinity matrix norm, one matrix norm. So, infinity matrix norm is given by maximum over i from 1 to m summation j equal to 1 to n modulus of a i j, where this metric A is of size m cross n.

And it is an element of set of all vector space, vector space of matrices of size m cross n. And this infinity norm is known as maximum row sum matrix norm. And we have proved in previous class that this actually defines a norm that it satisfy all the three properties. First properties are non-negativity, and then property of scaling, and then triangle inequality.

Similarly, we define one norm which is given by maximum j from 1 to n summation i equal to 1 to m some modulus of a i j, and this is known as maximum column sum matrix norm. So, this we have proved in previous class. In today’s class, we want to prove a corollary of previous theorem.

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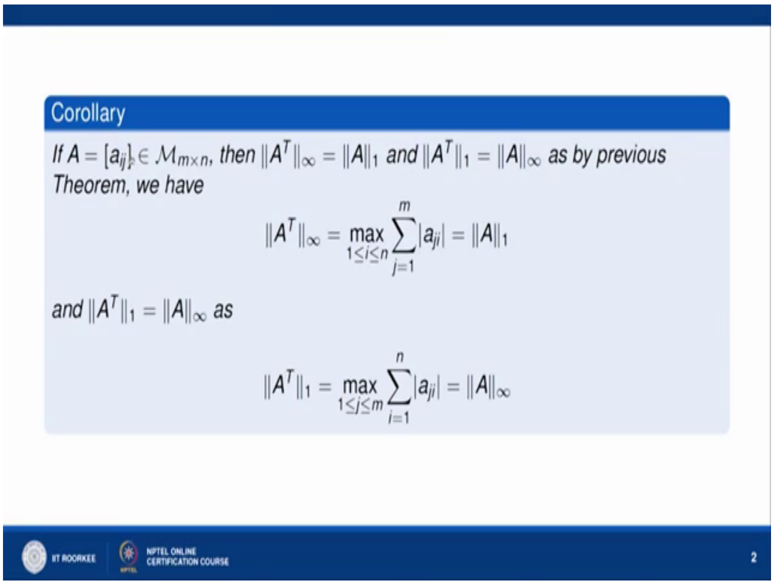

It says that if A which is given as a i j is a matrix of size m cross n then infinity norm of A transpose is same as one norm of A; and one norm of A transpose is equal to infinity norm of A. So, the statement is quite obvious. And if you look at the proof is also not very difficult. If you look at here we have A whose size is m cross n, then A transpose as of size n cross m. Then if you look at if you represent A transpose as b i j, then according to the definition here infinity norm of A transpose is equal to maximum. Now, here this i is coming from rows, so here rows are from n. So, here we can say that i is from 1 to n and summation j is from 1 to m modulus of b i j, so b i j here.

Now if you look at the relation between b i j and a i j. So, this is given as a i j. So, here b i j is nothing but you can write this as a j i. So, this I can write it a j i. So, here we can write this as equal to maximum 1 less than i less than n summation j is equal to 1 to m modulus here I can write it a j i here. And if you look at this is what this is nothing but here maximum where j from 1 to n summation i equal to 1 to m from a i j. So, if you just change the notation here we just interchange the notation i and j.

So, here we can say that maximum over taken over j from 1 to n and summation here we n place of j, I am writing i, i equal to 1 to m modulus of a i j. And this is nothing but one norm of A. So, this is nothing but one norm of A. So, it means that infinity norm of A transpose is same as one norm of A. So, similarly you can prove the other claim that is one norm of A transpose equal to infinity norm of A.

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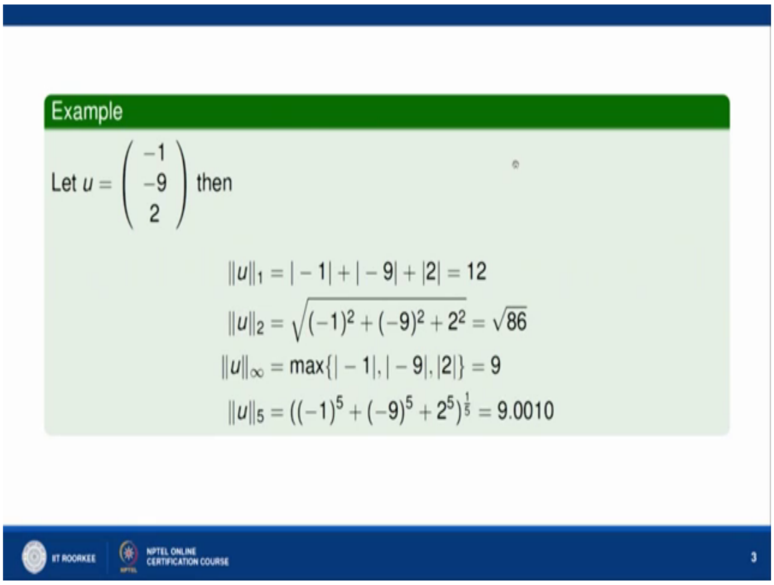

So, with this let us move to one example. So, here we start with the example having say vector norm. So, u be a vector of R 3. So, u is given as minus 1, minus 9 and 2. Then one norm is basically sum of modulus value of components of u. So, here components are minus 1, minus 9 and 2. So, one norm is given by modulus sum of all the modulus value. So, it is coming out to be modulus of minus 1 plus ma modulus of minus 9 plus modulus of 2 and is given by 2 - 12. And similarly, if you look at the two norm here, so two norm is the under root of sum of all the elements square, all the components square. So, it is minus 1 whole square plus minus 9 whole square plus 2 square. So, it is coming out to be under root 86, 86.

Now, infinity norm of u is given as maximum of modulus values of these components. So, it is maximum of minus 1, and modulus of minus 1, modulus of minus 9, and modulus of 2, so it is coming out to be 9. Similarly, you can find out any p norm in particular if we take p as 5, then you can calculate p norm of this vector as this minus 1 to power 5 plus minus 9 to power 5 plus 2 to power 5 whole to power 1 by 5. And if you calculate, it is coming out to be 9.0010. And this also indicate one fact that if p tending to infinity, then p norm of u is tending to infinity norm of u. So, if you look at in this example we have consider one norm, two norm, infinity norm and p norm. So, here likewise we want to consider the matrix norm, we have considered one norm and infinity norm, we want to consider say two norm also. So, that we are going to consider in coming lectures in this lecture only.

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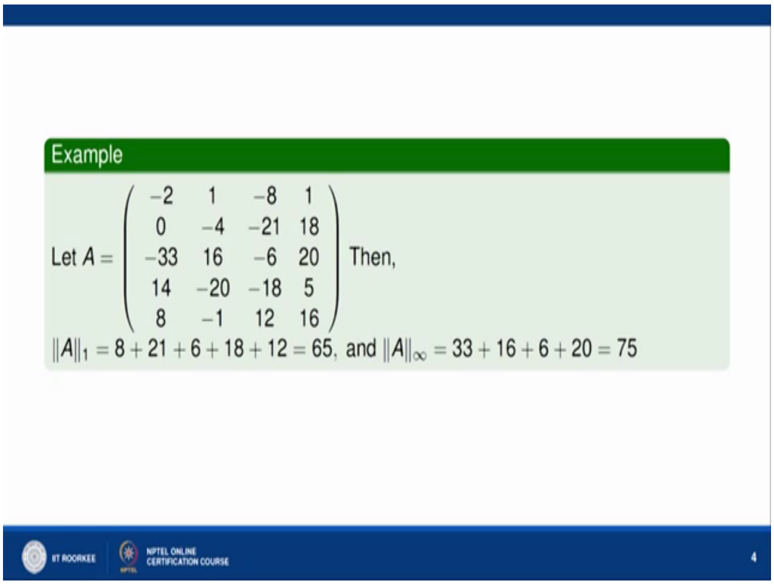

So, now consider a matrix A and given by this 5 by 4 matrix. So, here we have 5 rows and 4 columns. And we want to find out say one norm and infinity norm. So, let us see here. So, if you look at infinity norm, infinity norm is what maximum row sum norm. So, here, here you have to look at the sum of all the rows, and choose the maximum out of this. So, here if you look at the first row is basically what minus 2 1 minus 8 1, so we are we are working with the absolute values. So, first row sum is basically 2 plus 1 plus 8 plus 1. Similarly, you find out same sum of this modulus values of this, this and this and so on.

So, if you look at this infinity norm of A is given by this third row here. So, if you look at the third row, and if you take the sum of all the modulus value here, it is coming out to be 33 plus 16 plus 6 plus 20, and it is coming out to be 75. And if you find out the sum of all the absolute value in the remaining rows, it is coming out to be less than 75. You can check the first row has somewhat 4 12. So, first let me write it here.

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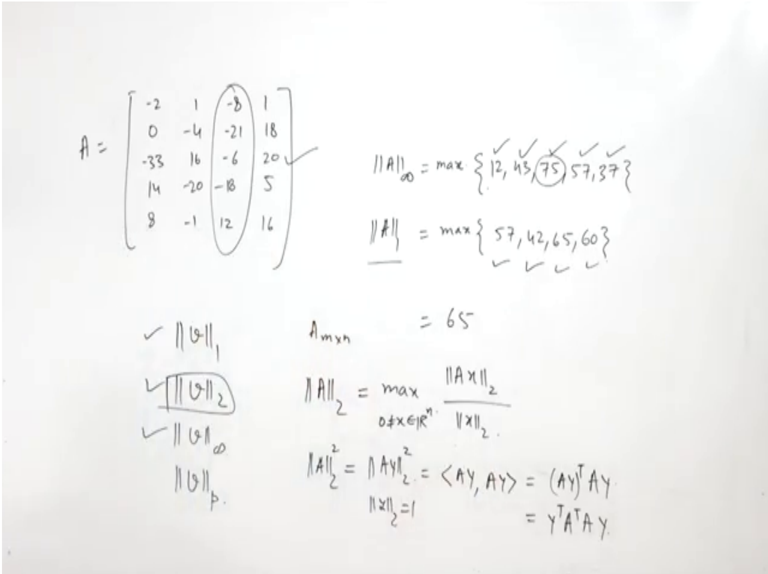

So, to find out infinity norm of this matrix A, we have to find out say sum of first row. So, if you find out say sum of first row modulus values of first row, so it is plus 2 plus 1 plus 8 plus 1, so it is coming out to be 12. Similarly, if you find out say a modulus values of elements of second row it is come coming out to plus 4 plus 21 plus 18 which is nothing but 43. And similarly, looking at the row, then sum of all the modulus values of this third row it is coming out to be 75; and fourth row it is coming out to be 57 and 37. So, now, we have calculated the sum of each sum of each elements in a particular row. So, this is the so here if you look at the maximum, maximum is coming out to be 75, which we are achieving at this third row.

Similarly, if you look at the one norm, one norm is maximum column sum matrix norm. So, here in particular for calculating one norm you consider it on the columns here. So, look at the first column, and try to find out the sum of the absolute values of this element. So, it is coming out to be 2 plus 33 35 plus 4 14 49 plus 8 57. So, the first column sum is 57, second column sum is going to be 42 and 65 and 60, you please calculate, this is not very difficult thing. So, if you look at what is the maximum thing, maximum thing is coming out to be 65 which you are achieving at this third column here. So, it means that one norm of A is going to be 65 here. So, in this particular example your one norm is coming out to be 65 and infinity norm is coming out to be 75.

Now, as we have pointed out that in vector norm popular norms are one norm and two norm and infinity norm and then for any p. So, generally these are popular, these three are popular. So, we have already calculated the subordinate matrix norm corresponding to one norm and infinity norm. Let us consider the subordinate matrix norm corresponding to v. It means that we want to consider the two norm of A which is given as maximum and maximum of zero nonzero x belonging to R n, here A is m cross n matrix and over A x two norm of A x divided by two norm of x m. So, we want to find out that what is this quantity right.

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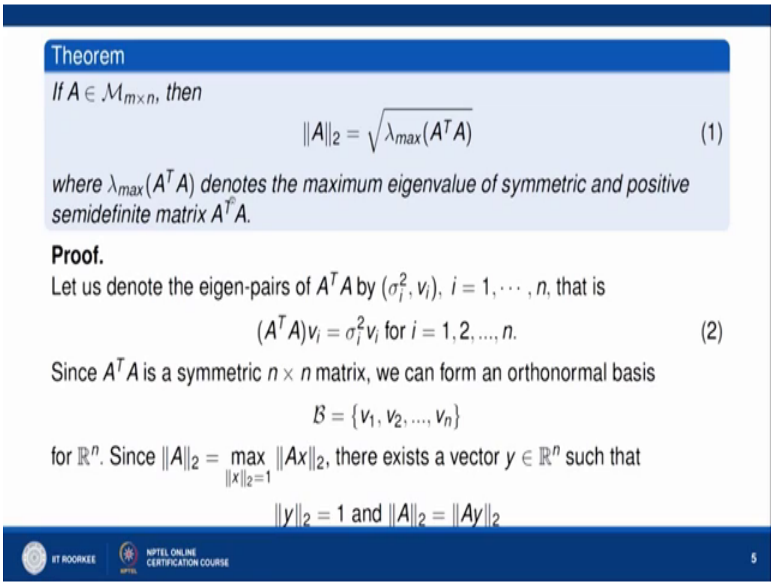

So, this theorem is related to two norms of matrix. So, here it says that if A is a m cross n matrix then two norm of A is given by under root lambda max of A transpose A, where lambda max A transpose A denote the maximum Eigen value of symmetric and positive semidefinite matrix A transpose A. So, first thing we have to note it down that this A transpose A is a symmetric matrix right. So, it means that Eigen values are in fact it is diagonalizable; and one more thing that it is a positive semidefinite matrix. So, it means that positive semidefinite matrix means all the eigenvalues are nonnegative. So, it means that under root makes sense is it ok.

So, now with the help of these information let us denote the Eigen pairs of A transpose A by sigma i square v i for i equal to 1 to n. And this we can write it because it is a symmetric matrix, and symmetric matrix is diagonalizable. So, it means that we have full n linearly independent eigenvectors available to us. So, so it means that Eigen pairs are given by sigma i square v i. So, it means it satisfies this following equation that is A transpose A v i is equal to sigma i square v i for i equal to 1 to n. Now, here again we are writing the same thing that since A transpose A is a symmetric n cross m cross n and a n cross n matrix we can form an orthonormal orthonormal basis B from v 1 to v n. First thing is that these v 1 to v n are l i and then with the help of gram Schmidt orthogonalization process, we can consider an orthonormal basis for R n. So, here B is the say obtain orthonormal basis for R n.

So, now by the definition of two norm of A it is given by maximum for two norm of A x when two norm of A x is given as 1, so that is a definition of two norm of A. So, it means that they exist a vector y in r n such that two norm of y is 1; and two norm of A is given by two norm of A y. So, that is with the help of definition we are getting that this two norm is achieved somewhere for some values say let us say some vector y here.

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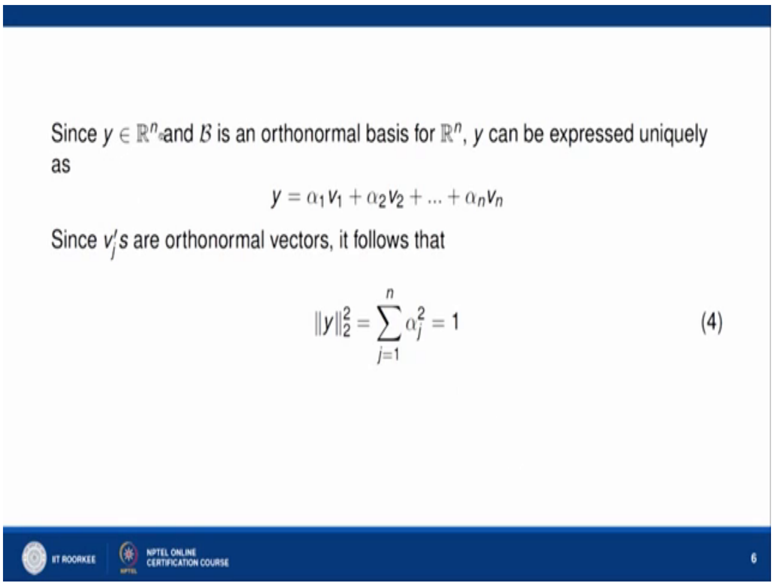

So, now looking at here since y belongs to R n, and B is an orthonormal basis for R n, y can be expressed uniquely as in terms of v i's. So, y can be written as alpha 1 v 1 plus alpha 2 v 2 plus alpha n v n. Now, we already know that these features are orthonormal vectors.

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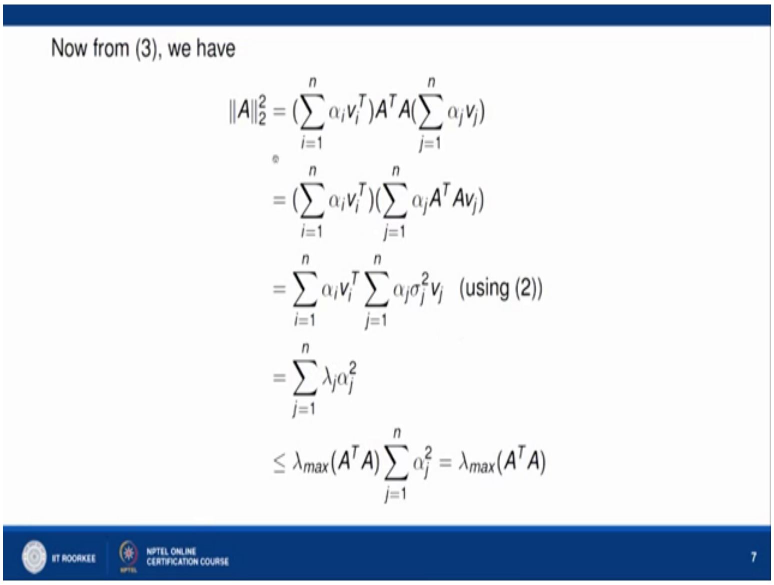

So, we can find out say norm of y as the summation j equal to 1 to n alpha j is square. So, norm of two norm of y square is given as j equal to 1 to n alpha j square equal to 1 now. It is already given that two norm of y is 1. So, this summation j equal to 1 to n alpha j square is given as 1. So, using this information let us find out two norm of A here. So, two norm of A is given as if you look at this thing two norm of A is given by two norm of A y right. So, let us find out say two norm of A y, two norm of A y is given as y transpose A transpose A y.

So, let me write it here. So, here we can write it two norm of A is given as two norm of A y, here norm of y two norm of y is given as 1. So, this I can write it A y. So, if you take this square here, so it is A y with A y and this is inner product on R n. So, this I can write it say A y transpose with A y and this is nothing but y transpose A transpose A y here. So, that is what is written here. So, here two norm of A square is given as y transpose A transpose A y here.

Now, if you simplify here, if you look at this term, so A transpose A operating on summation j equal to 1 to n alpha j v j. So, here using the first equation which is given here as sorry it is equation number two here. So, equation number two says that A transpose A operating on v i is given as sigma i square v i. So, using this equation number 2, we can say that A transpose A applying on this, we can write this as alpha j you can take it out and A transpose A operating on v j. Now, here we can write A transpose A v j as sigma j square v j using two.

So, now if you again simplify this, this is going to be alpha i alpha j v i transpose sigma j square v j. Now, here v i and v j are orthogonal to each other. So, it means that here this will be nonzero only if when j is equal to i. So, for j equal to i we are getting a nonzero value for all other it is a zero value. So, for considering j equal to y, we can write this as summation i equal to 1 to n sigma i square alpha i square and sigma j square.

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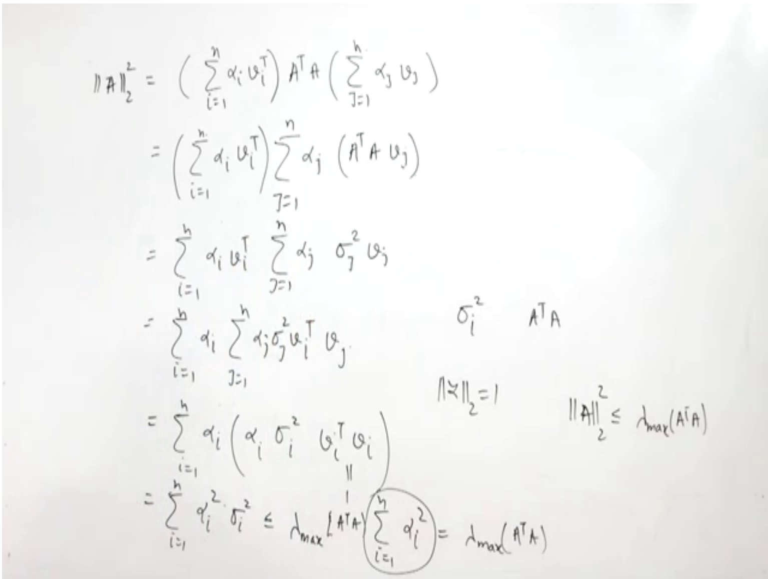

So, here two norm of A square is given by y transpose A transpose A y here. So, if you simplify here. So, this we can write it i equal to 1 to n alpha i v i transpose summation j equal to 1 to n alpha j. Now, we are operating A transpose A on v j. So, A transpose A operating on v j, then we already know that since v j is an Eigen vector corresponding to A transpose A, then it will give you sigma j square v j. So, it means that next step we can write i equal to 1 to n alpha i v i transpose summation alpha j sigma j square v j.

Now, this we here j is from 1 to n right. And if you simplify, then we can write it here that summation i equal to 1 to n alpha i now v i transpose I am taking inside j equal to 1 to n alpha j sigma j square v i transpose v j. Now, this v i's are orthogonal to in fact orthonormal to each other. So, it means that this will be nonzero only when j is equal to i; for all others it is equal to 0. So, taking j equal to i, we have nonzero value and that we are writing here alpha i sigma i square v i transpose v i. Now, v i transpose v i is going to be 1, because these are orthonormal vectors. So, this can be written as summation i equal to 1 to n alpha i square. So, this will multiply alpha i square sigma i square here. Now, this sigma i square is basically what this sigma i square are Eigen values of A transpose A. So, it means that sigma i square sigma i squares are Eigen values corresponding to A transpose A.

So, let us say that take the maximum Eigen value of A transpose A. So, this is bounded above by maximum Eigen value of A transpose A. So, if you take out that maximum value out then what is left here it is summation i equal to 1 to n alpha i square. Now, this value is basically this value is given as one why because two norm of y is coming out to be one. So, there we have already seen that this value is equal to 1. So, it means that this is equal to lambda max A transpose A.

So, what we have seen here that two norm of A square is less than or equal to lambda max of A transpose A right. So, this is one thing that two norm of sorry two norm of A square is less than or equal to lambda max A transpose A. Now, we will prove the other way other inequality.

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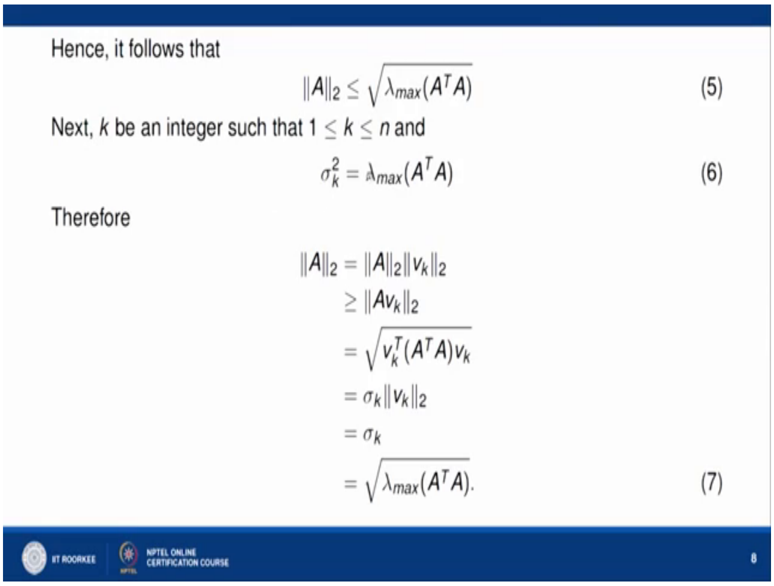

So, to prove other part let us say that this sigma k a square is the value at which we achieve the maximum of A transpose A. So, it means that this sigma k square represent the maximum eigenvalue of A transpose A, where k is between 1 to n. Now, at this you calculate two norm of A, two norm of A is given as two norm of A into two norm of v k right. Since, this we can write because two norm of v k is nothing but 1. So, here we can use consistency of matrix norm, and this I can write as greater than or equal to two norm of A v k. Now, you just calculate two norm of A v k, it is given as equal to under root of v k transpose A transpose A v k; if you simplify it is coming out to be sigma k two norm of v k. Now, two norm of v k is 1, so it is nothing but sigma k. So, it means that two norm of A is greater than or equal to sigma k. Now, what is sigma k, sigma k is under root lambda max of A transpose A.

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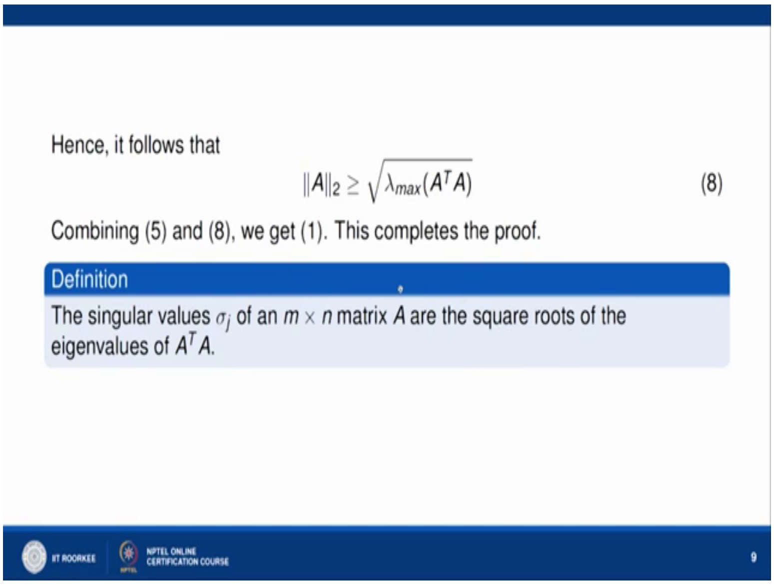

So, if you look at the equation inequality 5 and inequality 7, we can say that your two norm two norm of A is equal to lamb under root under root lambda max of A transpose A. So, this inequality 8 and inequality 5, if you combine these two will get the result that two norm of A is given as under root of lambda max of A transpose A. So, it means that here this matrix subordinate norm matrix norm of A two norm of is given as under root lambda max of A transpose A. So, sometimes this norm is also known as spectral norm.

Now, here this sigma j which is commonly known as singular values so, the singular values sigma j of an m cross n matrix A are the square root of the eigenvalues of A transpose A right. So, that is what we have utilized here. This sigma k are known as singular values of A transpose A.

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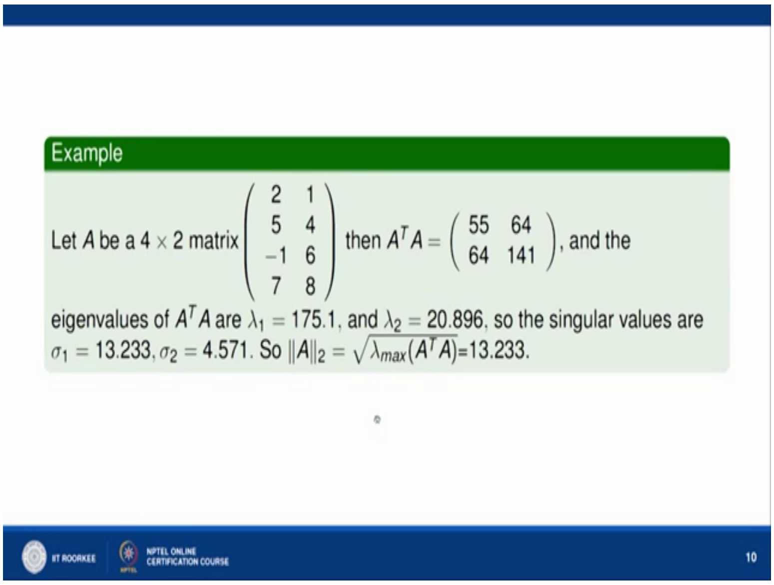

So, now, let us consider one example. So, let A be a 4 cross 2 matrix. It is given here 2 5 minus 1 7 1 4 6 8, then you can calculate A transpose A which is given as 55 64 64 and 141. And we can calculate the eigenvalues of A transpose A here. So, for calculating the Eigen values we have utilize the matlab software and you may also use this matlab software. So, here we have obtained the Eigen values of A transpose A as lambda 1 equal to 175.1; and lambda 2 equal to 20.896. So, singular values which are nothing but square root of lambda i's.

So, first singular values sigma 1 is given by under root lambda 1 which is given as 13.233; and sigma 2 which is under root of lambda 2, it is given by four point five points 4.571. So, two norm of A is nothing but under root of lambda max A transpose A. So, lambda max if you look at it is coming out to be 175.1. So, if you take the square root it is coming out to be 13.233. So, two norm of A is given here 13.233.

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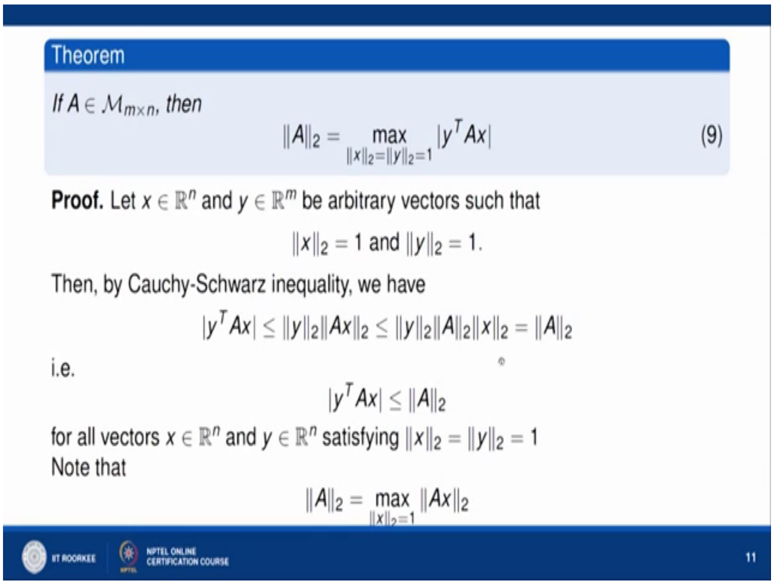

Now, there is an alternative way to calculate two norm of A. So, this is given in this theorem. So, if A is m cro m cross n matrix then two norm of A is given by maximum taken over two norm of x and two norm of y is equal to 1, y transpose A x. Please remember here, here x is coming from umm R n and y is coming from R m. So, let us quickly prove this result. So, here we say that let x belongs to R n, y belongs to R m be arbitrary vectors satisfying the condition here which is it means that two norm of x is 1, and two norm of y is equal to 1. Now, please remember here two norms of x means two norm of x in R n is equal to 1; and two norms of y in R m is equal to 1.

So, then we can apply the Cauchy-Schwarz inequality. So, y transpose A x means inner product of y with A x is less than or equal to two norm of y, and two norm of A x. So, again we can apply the property of matrix subordinate norm. So, two norm of A x is given by two norm of A x is less than or equal to two norm of A and two norm of x. Now, here since two norm of y is given as 1, and two norm of x is given as one, so this is nothing but two norm of A. So, it means that white modulus of y transpose A x is less than or equal to two norm of y. So, this we have achieved.

Now, we want to prove the other way round means that two norm of A is less than or equal to y transpose A x. So, for that we already know that two norm of A is given by two norm of maximum of two norm of A x when two norm of x is given as 1. So, this we already know. So, it means that they exist a vector x in R n whose norm is 1, and two norm of A is achieved on that vector.

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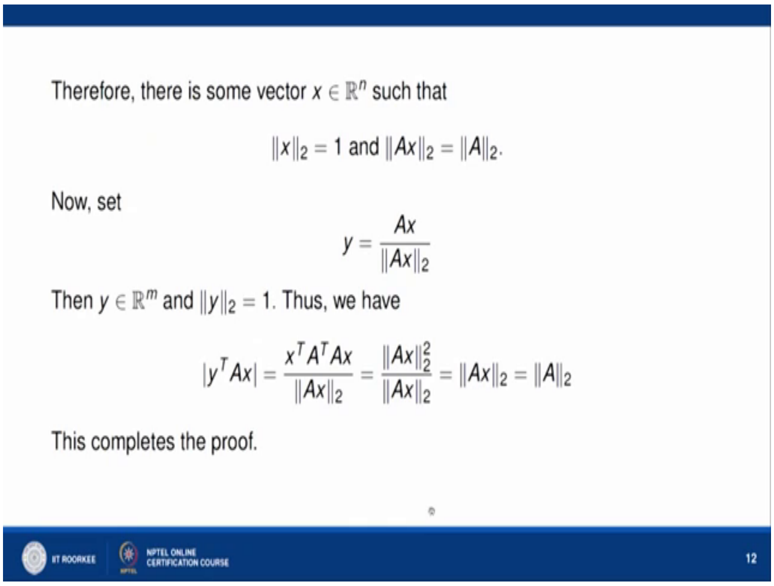

So, it means that they exist some vector x says that two norm of x is 1 and two norm of A x is nothing but two norm of A. So, using this information you set y as a x upon two norm of A x. So, using this x we are defining y. So, if you look at the two norm of y is coming out to be 1. So, you can calculate two norm of y is coming out to be 1. So, using this y you calculate y transpose A x. So, y is given as A x divided by two norm of A x. So, y transpose is going to be x transpose A transpose. So, x transpose A transpose divided by two norm of A x into A x.

So, if you calculate this, this is nothing but inner product of A x with A x. So, we can write this as two norm of A x whole square divided by two norm of A x, which is nothing but two norm of A x. And we already know that x is such that that two norm of A x is given by two norm of A. So, it means that y transpose A x is actually two norm of A with such A x and such A y.

So, it means that your inequality which we have considered in here is actually a equality and it is nothing but y transpose A x is given by two norm of A. So, it means that we have proved our theorem that two norm of A is given by maximum of y transpose A x modulus of y transpose A x, where y and x are satisfying these this condition.

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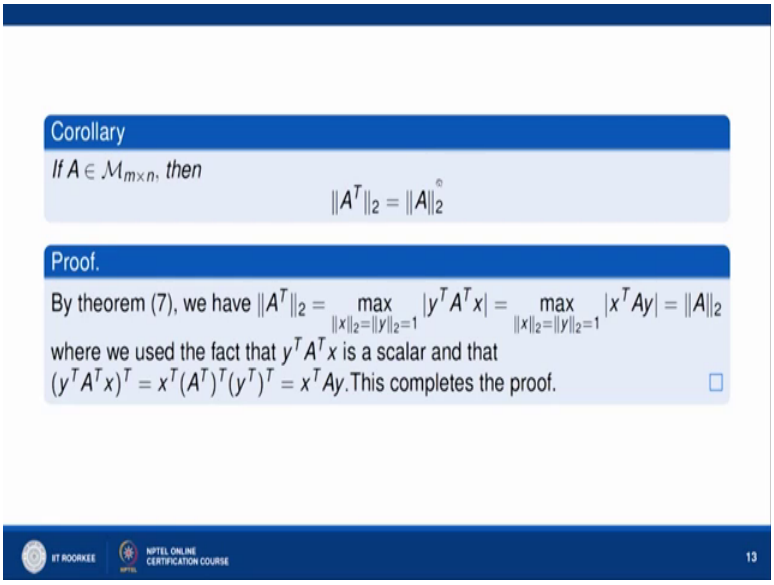

Now, as a corollary, we can easily prove that two norm of A transpose is equal to two norm of A. So, in fact, if you have we have started with that at infinity norm of A transpose is equal to one norm of A here we have seen that two norm of A transpose is same as two norm of A. So, to proving for proving this, we simply observe this fact that two norm of A transpose is nothing but maximum of modulus value of y transpose A transpose x where y and x satisfying this condition.

Now, this is what if you look at this is nothing this value is nothing but a scalar value now, a scalar value is same as transpose of that scalar value. So, this can be written as transpose of this. So, transpose of this is given by x transpose A y. Now, if you look at this is what this is nothing but two norm of A here. So, so here we can say that two norm of A transpose is given by two norm of A here. So, here what we have utilized here that y transpose A transpose x is a scalar quantity.

So, its transpose is same as value this. So, transpose of y transpose A transpose x is given by x transpose A y, which we have utilized here that this value is equal to this value and which is nothing but two norm of A here. So, this we have proved that for two norm two norm of A transpose is same as two norm of A. So, this is not true for infinity norm and one norm here.

So, what we have seen in this lecture that we have seen first of all that infinity norm of A transpose equal to one norm of A. Similarly, we can say that one norm of A transpose is equal to infinity norm of A. And we also define two norm of A means matrix subordinate norm with vector norm two, which we have defined here. And we have discuss certain properties of two norm. And here we will stop with this. And in next lecture, we will discuss one very important norm which is known as Frobenius matrix norm and we will also see certain properties of Frobenius matrix norm and two norm which is known as a spectral norm. So, here we stop.

Thank you for listening us. Thank you.