Numerical Linear Algebra

Dr. D. N. Pandey
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 30
Vector Norms- II

Hello friends, welcome to our lecture. In a previous lecture we have discussed some properties of vector norm and in this lecture we extend our discussion and discuss some properties based on vector norms and with the help of vector norms we try to define what is matrix norm here. So, in previous lecture we have defined norm like this.

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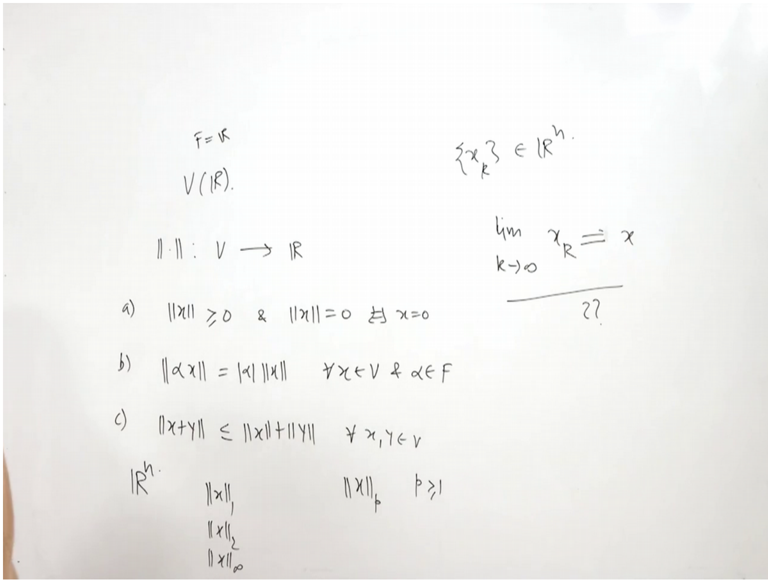

So, here we have vector space v on real scalar say R and this norm function is defined as a function from V to say R we satisfy the following property that norm of x is greater than equal to 0 here and norm of x is equal to 0 if and only if x is equal to 0. And b property is that alpha norm of alpha x is equal to modulus of alpha norm of x here for every x belongs to V and alpha belongs to scalar field F here, F is equal to R here right. And c property which is known as triangle inequality property, so norm of x plus y is less than equal to norm of x plus norm of y for all x comma y belongs to v here.

So, this is what we define as norm here and in previous class we have shown that and on R n we can define several norms which is known as one norm. So, we can say l 1 norm and 2 norm l 2 norm and then infinity norm and we have also defined p norm for p greater than or equal to 1 and we have seen that on R n each one of these norms are equivalent to each other. And in today's lecture we want to show that in terms of norm how we define that convergence of vectors. So, it means that here we want to show that if we have a sequence say x n, or say x k in R n then what do we mean by saying limit k tending to infinity this x k will is equal to what. So, how we can define this limiting criteria?

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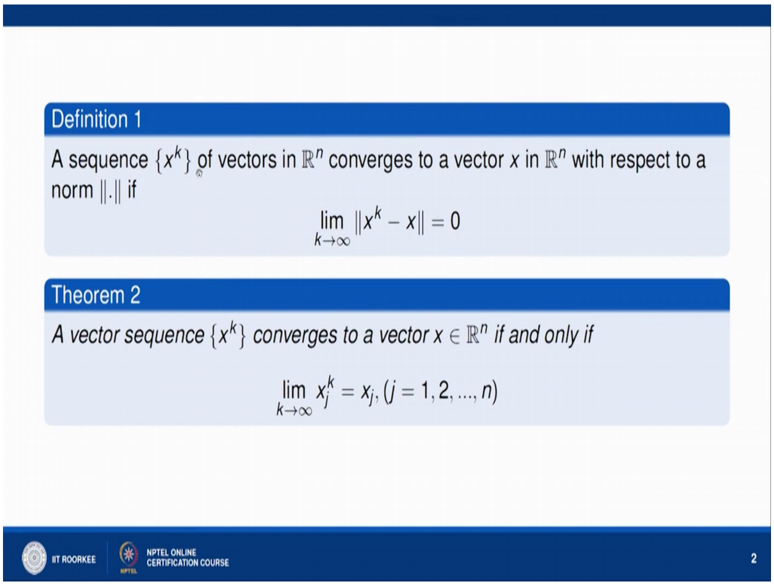

So, for that we define convergence in this way. A sequence x k of vectors in R n converges to a vector x in R n with respect to a norm if limit getting into infinity norm of x k minus x is equal to 0 here.

Now, one important result here that a vector sequence x k converges to a vector x in R n if and only if limit k tending to infinity x k j is equal to x j, here x k j is the jth component of x k and x j is the gth component of x. So, what we want to prove here that if x k which is this, this is a sequence in R n sequence of vectors in R n and here we define that limit k tending to infinity x k converges to x in norm means that limit k tending to infinity norm of x k minus x is equal to 0 here.

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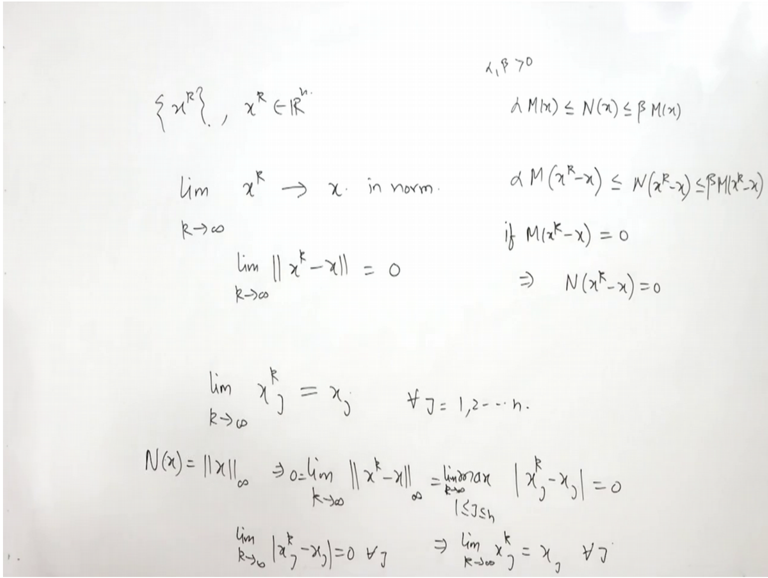

And what we want to prove here that this implies that that component wise this converges. So, it means that if we look at the jth component of each x k. So, jth component of x k will converge to jth component of the limit vector that is x here. So, this we wanted to prove here.

So, for that we try to see here that this convergence criteria is independent of norms in R n, we are talking about R n. So, in R n whatever norm will we want to take we can take and that follows from the equivalent, equivalency of any norms in R n, we have seen that if we take any 2 norms say m and n then it is equivalent. So, it means that they exist alpha beta nonzero, in fact, positive that alpha M x less than or equal to N x less than or equal to beta times M of x. So, here this x k minus x can be written as alpha M x k minus x less than or equal to N x k minus x less than or equal to beta M x k minus x.

So, if this x k converges to x in M norm. So, it means that if M x k minus x is tending to 0 if this value is equal to 0. So, by sandwich theorem we can say that n times x k minus x is also equal to 0. So, here M and N are any 2 arbitrary norm defined on R n. So, it means that if this convergence is in with respect to 1 norm then this shows that converges in any other norms.

So, let us say that take the norm n as infinity norm. So, let us define that n x is infinity norm. So, n x is defined as infinite norm of x and then since x k converges to x in any norm, in particular in infinity norm also. So, it means that 0 equal to limit k tending to infinity, infinity norm of x k minus x that is simply says that x k converges to x in infinity norm.

Now, by the definition of infinity norm it is what? It is nothing, but maximum of x k J minus x J and J is between 1 to n and this is equal to what this is equal to 0. So, it means that maximum maximum x, x k J minus x J, where J is between 1 to n is equal to 0. Now this is possible only when each term is equal to 0. So, it means that for each J from 1 to n x k J is converging to x J. So, it means that this I can write as limit k tending to infinity x k J is equal to x J. Here I have missed one thing here this as limit k tending to infinity. So, here this follows only when.

So, this is true only when limit k tending to infinity x k J minus x J is equal to 0 for each name and this implies that limit k tending to infinity x k J is equal to x J for each J here, right. So, this show that if x k converges to x in norm in R n then it this implies that that component wise also convergence follows. So, it means that jth component of x k will converge to jth component of the limit factor.

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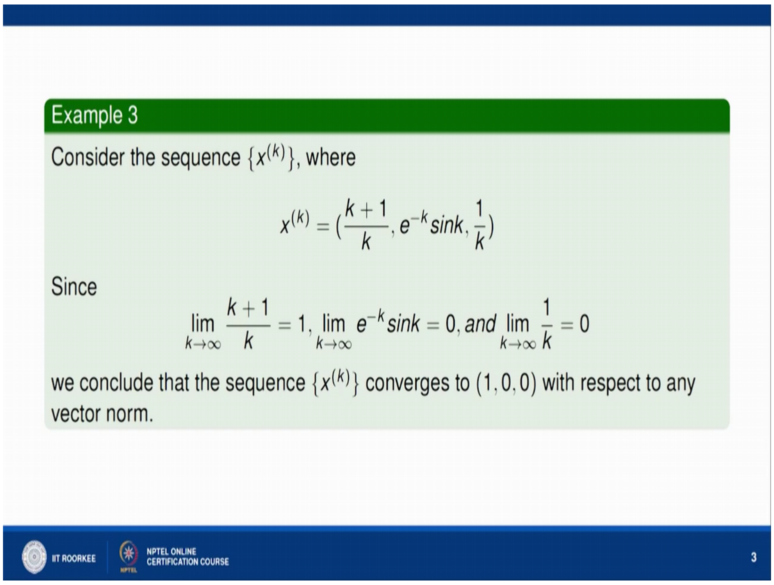

So, using this move to example number 3 here. Here we consider the sequence x k where x is given by k plus 1 divided by k comma e to power minus k sine k comma one by k. So, this is a sequence in R 3 and we want to find out say the limit vector of x k in R 3.

So, if you look at we have seen that if x k converges to sum x then component wise convergence follows. So, it means that this first component if you find out say limit, limit k tending to infinity k plus one divided by k will converge to 1, second component e to power minus k sine k will converge to 0 and last one by k will converge to 0. So, it means that this, this will converge to 1, this will converge to 0, this will converge to 0. So, it means that x k will converge to a limit vector whose first component is 1, second component is 0, last component is 0. So, it means that x k converges to vector 1 0 0 in R 3. And this convergence is happening in any vector norm. So, that is one nice application of the definition, nice application of this theorem 2.

Now, move on to next, next let us define what is a matrix norm. So, to define matrix now we try to recall that set of all m cross n matrix which is a vector space of all real m cross n matrices this can be identified with the Euclidean space are m cross and let us see how it is.

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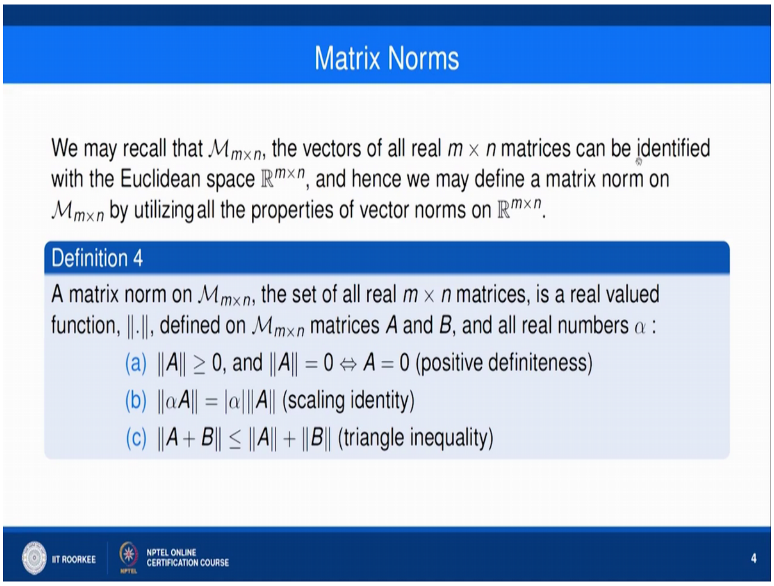

So, here we can say that set of all m cross n matrix over R can be identified with elements of R to power m n.

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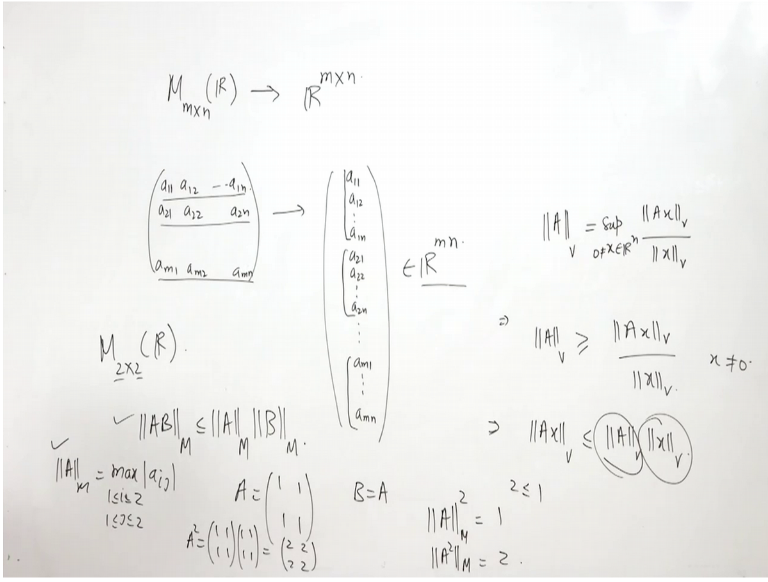

So, how we are identifying here if you look at any matrix of size m cross and this can be written as a column vector like this. So, what we are doing here we are writing the first row as the column here and second row is next column here and so on. So, here basically we writing here a 11 to a 1 here 1 n here, and then look at the second row, second row is also written here like this and so row wise we are writing as a column.

So, last is what a m1 to a m n is given by this. So, this can be written as a column vector in R m n size. So, here corresponding to every matrix here we can write out write a vector in R 2 power m n here. So, we already know how to define vector norm. So, it means that with the help of definition of vector norm now we can define the matrix norm here also. So, here we define matrix norm as follows.

So, a matrix a matrix norm on m m cross and the set of all real m cross n matrices is a real valued function defined on this matrices A and B and all real numbers alpha we satisfy the following 3 properties. First property is that norm of A is greater than or equal to 0 and norm of A will be 0 only an if and only if A is equal to 0. And this property is known as positive definiteness and second property is that norm of alpha A is equal to modulus of alpha norm of A and this is known as a scaling identity here alpha is a scaling factor here. See, says that norm of A plus B is less than or equal to norm of A plus norm of B which is known as triangle inequality. So, first thing is what? We define this matrix norm with the help of vector norms on R power m right.

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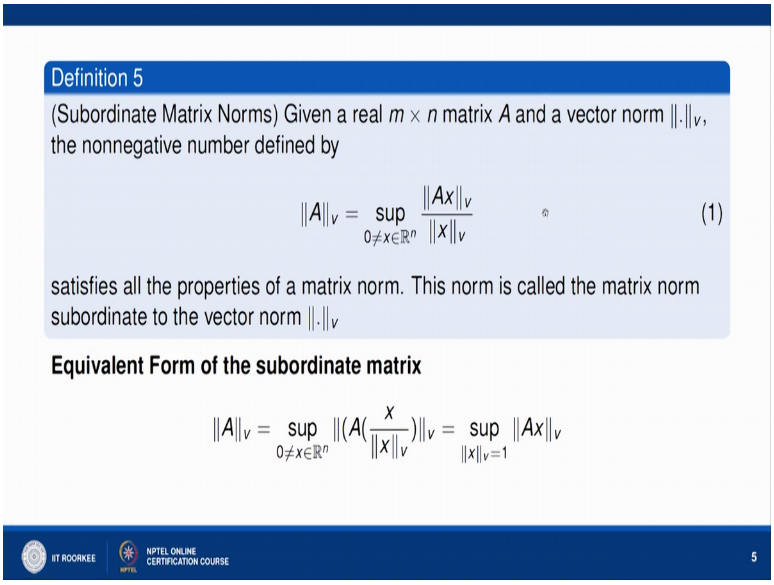

So, now let us define another norm which is known as subordinate matrix norm. So, how we define it? Given a real m cross n matrix A and a vector norm v the non negative number defined by this norm of A is equal to supremum of norm of x v divided by norm of x and this supremum is taken over all those x belonging to R n which are nonzero. So, if you take the supremum here then this supremum is known as v norm of A and we call this norm is induced by vector norm of R n here.

So, we try to show that it satisfy all the properties of matrix norm and this norm is called the matrix norm, subordinate to the vector norm or we can say that this norm is called as matrix norm induced by vector norm v. And we can simplify this one we can rewrite this form one in the following 2 way. Here what we have done here since this is just a constant vector. So, you can take it inside here.

So, using this second property here scaling identity we can take norm of x into inside and we can say that norm v norm of a is equal to supremum taken over all x nonzero vector of R n and it is A divided A x of divided by norm of x. And if you look at this is what? Here we can take that all those x whose v norm is 1 then this is nothing but supremum of A norm of A x and this supremum is taken over all those x whose v norm is equal to 1. So, either we use this form all these forms it is equivalent to each other.

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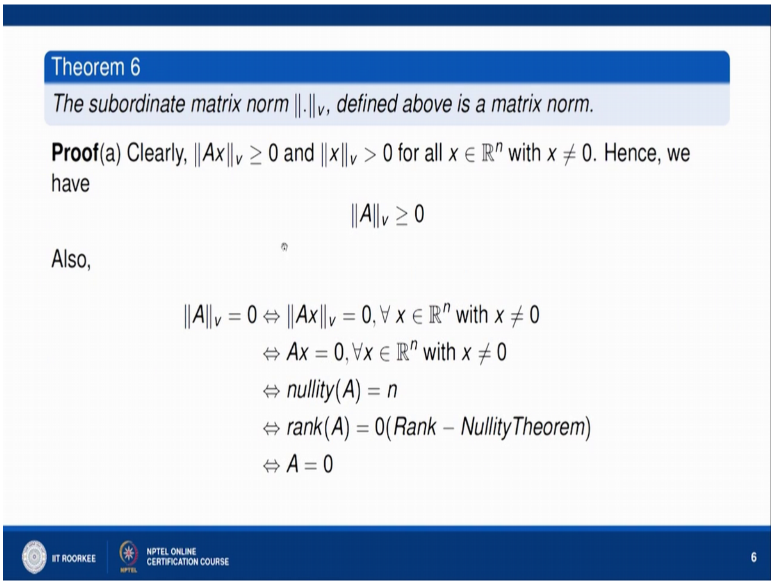

Now, as we have pointed out that this definition says that it is a matrix norm let us prove this that the subordinate matrix norm defines above is a matrix norm. So, we have to satisfy, we have to show that the subordinate matrix norm satisfy all the 3 properties. So, let us prove the first properties.

First property says that your matrix norm is non-negative and it is equal to 0 only when A is equal to 0. So, first non negativity is obvious because v norm of A is ratio of v norm of x and v norm of x and if you look at this is a non negative since it is a vector norm in m cross 1 R n basically and it is vector norm in R n. So, here this is non-negative, this is non-negative by the property of vector norms and we can say that this v norm of A will also be non negative.

Now, looking at the second part that if we norm of A is equal to 0 means that that supremum of this v norm of x divided by v norm of x is 0. So, supremum is 0, so it means that this has to be numerator has to be 0 for all x in R n. So, v norm of x is equal to 0 for every x in R n with x nonzero. So, this implies that since it is a vector norm. So, this implies that A x has to be 0 for every x belongs to R n. So, here these 2 this implies is or this implies this because this v is vector norm. So, this is A x equal to 0 for every x belongs to R n. Now, A x equal to 0 means null space of A has dimension n or we can say that nullity of A is equal to n. Now, we have already seen a rank nullity theorem for linear the similar kind of result also follows for matrix.

So, if you use the line nullity theorem for matrix we can say that if nullity of A is equal to n, then rank of A has to be equal to 0. So, rank of A equal to 0 if I and only if A is A 0 matrix, it means that what we have shown here that if v norm of A is equal to 0 subordinate matrix norm is equal to 0. So, this implies that matrix A is equal to 0. Now, here each step is if and only if. So, we can say that both way it follows. So, subordinate matrix norm is 0 implies, imply and implied by that A is equal to 0. So, this prove the part A.

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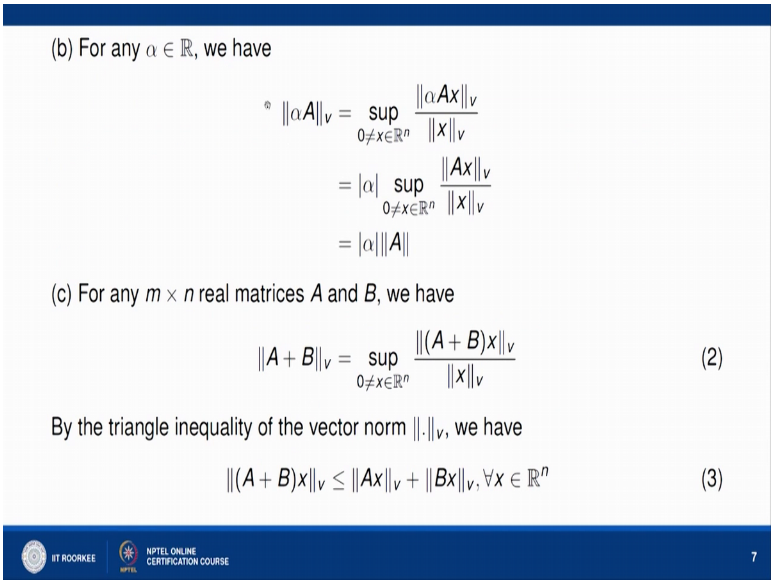

Now, looking at the part b, that looking at the subordinate matrix norm of alpha A which is nothing, but supremum of alpha norm of alpha x divided by norm of x and since it is a vector norm. So, using the property of vector norm this can be written as modulus of alpha and norm of A x and this is nothing, but subordinate matrix norm of A. So, we can write it like this. So, this proves the part b.

Now, look at the part c. So, for that let us take 2 real matrices A and B of size m cross n and try to find out say matrix about in a matrix norm of A plus B. So, which is defined as supremum over all nonzero vector x norm of A plus B operating on x divided by v norm of x here.

Now, here this can be simplified as triangle inequality of vector norm. So, triangle inequality of vector norm says that A x plus B x is less than or equal to v norm of A x plus v norm of B x. So, divided by v norm of x we can write it like this that subordinate matrix norm is less than or equal to supremum of over nonzero x norm of x divided by norm of x plus supremum over nonzero x norm of b x divided by norm of x and this is nothing but matrix norm of A induced by vector norm v and this is nothing, but vector matrix norm of B induced by vector norm v here.

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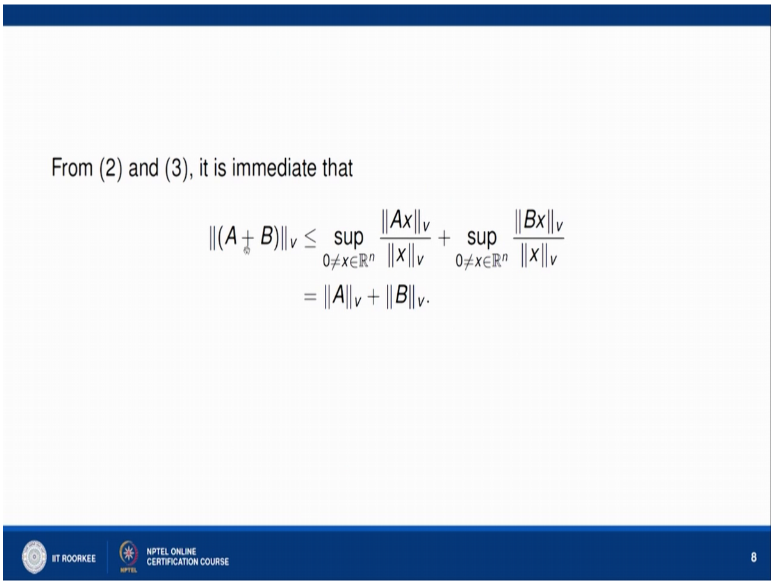

So, this follows the triangle inequality part. So, this shows that this subordinate matrix norm is actually a matrix norm because it satisfies all the 3 properties here.

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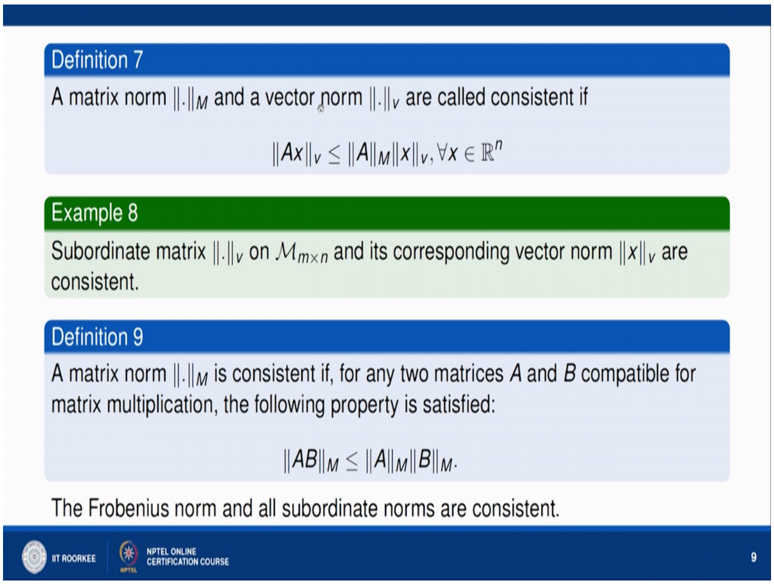

Now, here we also define a consistent norm that a matrix norm and a vector norm are called consistent if we have this relation that v norm of A x it means that vector norm of A x is less than equal to matrix norm of A into vector norm of x and this follows for every x in R n.

Here I am talking about matrix of size m cross n here. So, if we have this kind of relation then we say that matrix norm and vector norm are consistent to each other. Then with the help of previous example or by the definition of subordinate matrix norm we can say that subordinate matrix norm and its corresponding vector norm are consistent to each other. So, how it follows? It follows from the definition itself let us see how it is. So, here we have defined matrix norm like this.

So, matrix norm is defined as supremum norm of A x divided by norm of x v and this is taken over all nonzero vectors of R n. So, this I can say that that supremum, it means that norm of a supremum, norm of a is going to be greater than or equal to norm of A x divided by norm of x here of course, x is nonzero otherwise this derivative is this denominator is not defined. So, this implies that norm of A x is less than or equal to norm of A and norm of here. So, here we say that subordinate matrix norm and vector norm are consistent to each other.

Now, let us move to one more definition we say that any matrix norm is consistent if for any 2 matrices A and B compatible for matrix multiplication, the following property is satisfied. That matrix norm of AB is less than or equal to matrix norm of A into matrix norm of B and we try to show that the Frobenius norm which we are going to define in next lecture in all subordinate norms are consistent in this definition here. And here I just want to tell that not all matrix norm satisfying this property. So, it means that not all norms defined on set of all matrices satisfy this property.

For example, if I take see this is let us say 2 cross 2 matrix over R here and we want to show that matrix norm of AB is less than equal to matrix norm of into A and matrix norm of B here. So, here let us define the matrix norm as say maximum of a ij, i is from 1 to n and j is from 1 to m whatever be the since I am taking m and n as 2. So, here I can say that it is, i is from 1 to 2 and j is from 1 to 2. So, we want to show that this is the norm defined on this, but it actually it does not satisfy this property. For that if we take a as 1 1 1 1, then you can check that, if you take B equal to A then look at the norm of A square and norm of A square. So, what is this quantity? So, if you look at norm of A; norm of A will be what? Maximum of a ij's which is 1 and square is going to be 1 here.

Now, look at the matrix norm of A square matrix, norm of A square will be what? Just find out 1 1 into 1 1 1 1. So, see what you will get? 2 2 2 2. So, here matrix norm of A square is going to be 2 here. So, here we can say that if it satisfy the following property then this implies that 2 is less than or equal to 1 which is really absurd. So, it means that not all norm define on set of all matrices satisfy this property it means that not all norms are consistent norm.

Now, we want to prove in this theorem that the subordinate matrix norm which we have defined earlier is a consistent norm.

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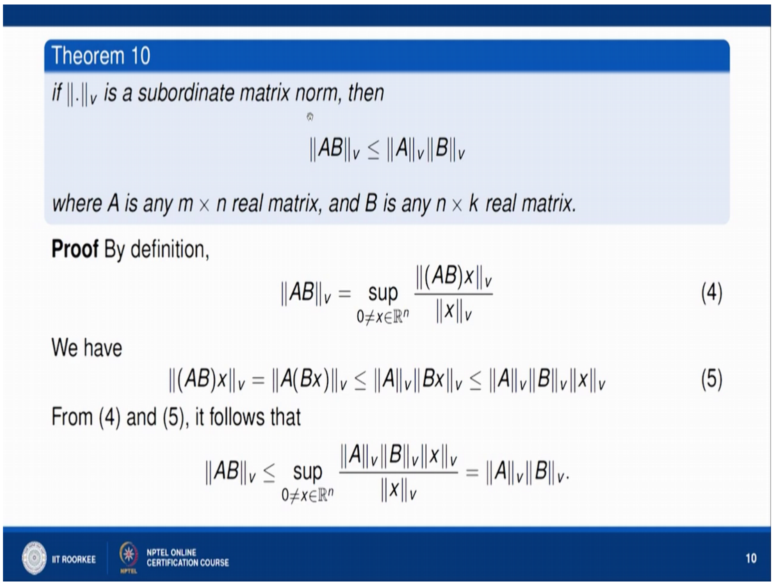

So, it means that it satisfy this property, that the subordinate matrix norm is a consistent norm here. So, to prove this we want to show that v norm of AB is less than or equal to v norm of A into v norm of B here and here since A and B are product is defined. So, it means that A and B are compatible matrix. So, let us say A as is m cross n real matrix and B is n cross k real matrix.

So, by definition v norm of AB is defined as supremum taken over all x in R n, nonzero vectors in R n, ratio of these 2 things that norm of AB x divided by norm of x. Now, look at this numerator here numerator says that v norm of AB x and this can be written as that

A applied on B of x. So, A applied on B of x and then we can say that by the compatibility by the consistency of vector norm and matrix norm which is defined here we can say that this is less than or equal to v norm of A and vector norm of B x. So, here if you look at B x is a vector here right and B x is a vector in R n is it ok.

So, now, again here we use the consistency of matrix norm and vector norm. So, we can say that this is less than or equal to vector norm matrix norm of A into matrix norm of B into vector norm of x here. So, now, using this equation inequality given in 5 in 4 we can say that v norm of AB is less than or equal to supremum over nonzero x in R n and now this is less than or equal to this quantity. So, this divided by this and so it is given by norm of A norm of A into v norm of B. So, it means that subordinate matrix norm is consistence matrix norm. So, that is what we claimed here, in the end of the definition 9 and we have proved here that the subordinate matrix norm is actually a consistent matrix norm.

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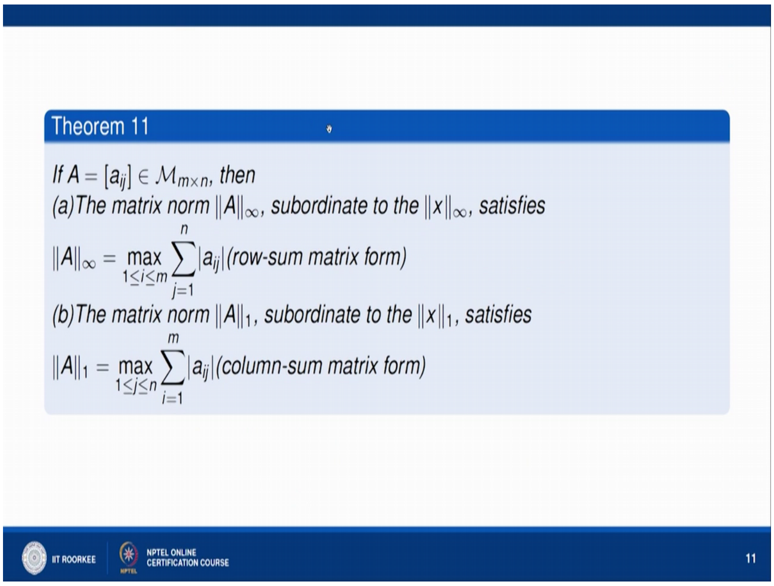

So, now moving on to next result. We have that the matrix norm which is infinity norm of a which is subordinate to infinity norm of x satisfy the following property, that infinity norm of A infinity matrix norm of A is defined at maximum of summation j equal to 1 to n one loss of a ij, i is running from 1 to m and if you look at carefully it means what we are fixing i and taking this sum over j. So, it means that if you look at we fix a row and take the sum of all the element of row and once we have taken row sum then we try to find out a maximum from maximum value of this. So, basically it is matrix maximum row sum norm. So, infinity norm is what? Maximum row sum norm and similarly we can define 1 norm and 1 norm is what, is summation i equal to 1 to n modulus of a ij and maximum is taken over all j between 1 to n.

So, basically here if we look at the previous definition here we have taken sum of all the row element and then finding the maximum of that sum. Similarly here we are fixing j here. So, it means we are fixing a column here. So, fix a column and then take the sum of element given in that column and then try to find out the which one is the maximum. So, that is why we call this one norm of a as maximum column sum norm is it and we want to show that this actually defines a matrix norm.

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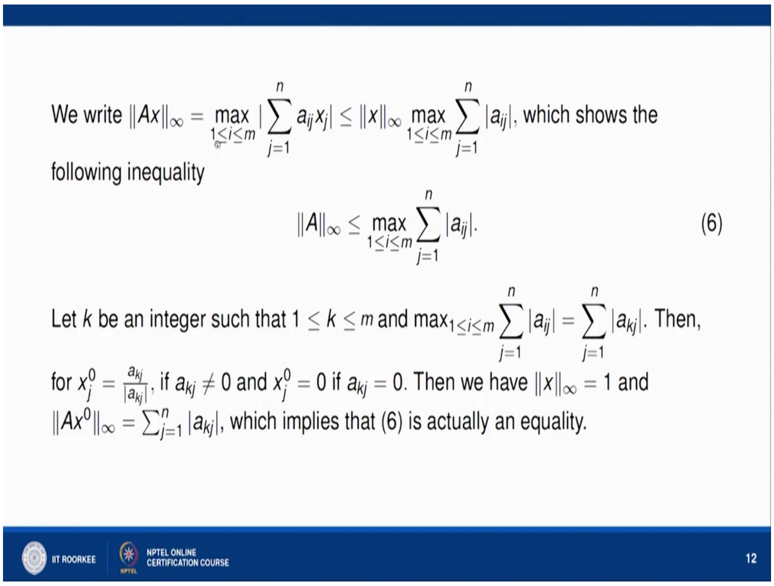

So, here we want to find out say infinity norm of A of x which is which we claim that it is equal to maximum of i running between 1 to m modulus of j equal to 1 to n a ij x j. So, just look at here how it is obtained here.

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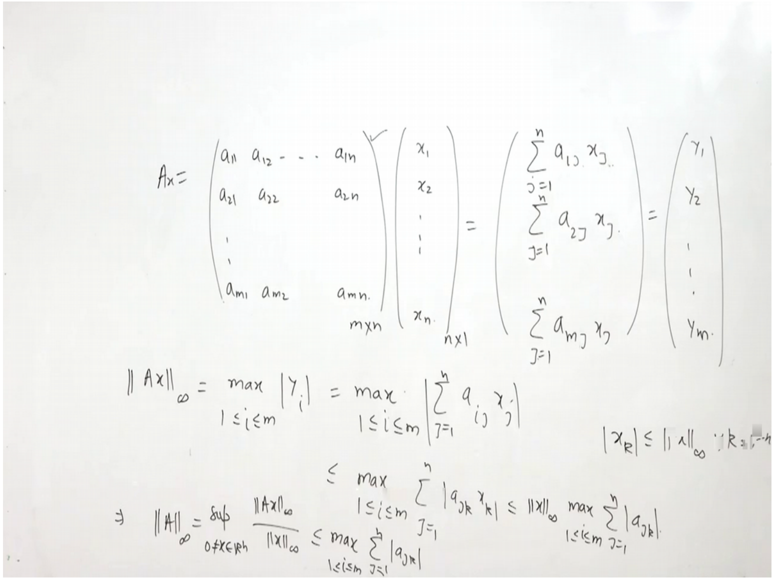

So, we have matrix A we have vector x here. So, matrix A is m cross n, we are writing it here and then it is vector of size n cross 1. We multiply then we have this vector. So, vector is given by that first element of this is given by j equal to 1 to n a ij, a 1 j x j and second component is given by j equal to 1 to n, a 2j x j and similarly mth component is given by j equal to 1 to n a mj x j. So, if we call this vector as vector y 1 to y m then infinity norm of A x is basically infinity norm of this vector. So, infinity norm of this vector will be what? Maximum of its component which over be the maximum of this component we call this as infinity norm of this vector and hence infinity norm of this x. So, we can say that A x; infinity norm of A x means maximum of these y i and i is running from 1 to m.

Now, what is y i here? y i can be written as summation j equal to 1 to n a ij x a here. So, that is what we are claiming here that infinity norm of A x is given by maximum of maximum of summation j equal to 1 to n a ij x j and i is running between 1 to m here and this is less than or equal to this quantity. Here what we have taken? We have taken the modulus inside. So, here we can say that maximum of 1 less than i equal to i less than or equal to m summation j equal to 1 to n modulus of a j k x k here. Just look at here how we have obtained here.

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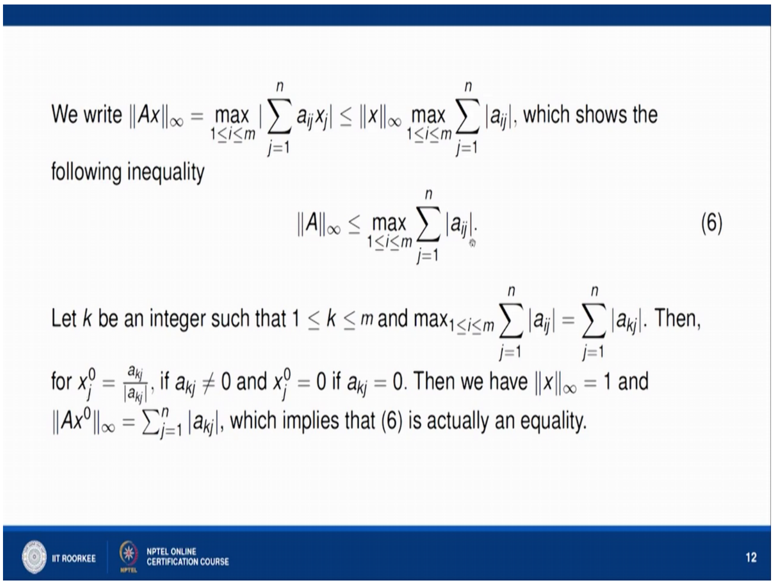

So, if you look at this last quantity here. Now, here this x k is bounded above by, modulus of x k is bounded above by infinity norm of x for each k, for each k here, k is from 1 to n, k is from 1 to say n. So, this implies that this is less than or equal to you take out that infinity norm. So, infinity norm of x and what is left here is writing here maximum of i from 1 to m summation j equal to 1 to n modulus of a jk right.

So, here this implies that by the definition of subordinate matrix norm, norm of A infinity which is equal to supremum of x nonzero belongs to R n norm of A x infinity divided by x infinity, which is given by. So, that this quantity is going to be maximum of i 1 less than or equal to m summation j equal to 1 2 n modulus of a jk is that. So, that is what is given here.

Now, we shows that this inequality given in 6 is true. Now, we want to show that this inequality is nothing, but equality here. So, it means that this maximum value is achieved at some for some vector x here. So, for that let us say that let k be an integer such that this maximum is achieved at this. So, it means that maximum over this quantity j equal to 1 to n modulus of a ij is achieved for kth row. So, it means that, if you look at these are how many rows these are m rows and we can say that we can easily find out the maximum is achieved at what rows. So, let us say that k is the row for which this maximum is achieved. So, maximum of this is achieved at kth row. So, here if you look at infinity norm of x naught, where x naught j is defined here in the slide that x naught j is equal to a kj divided by m modulus of a kj if a kj is not equal to 0 and x naught j is equal to 0 if a kj is equal to 0. Then we can say that infinity norm of x naught is equal to 1 and our claim is that infinity norm of a x naught is equal to j equal to 1 to n modulus of a kj.

Now this is what? This is what we try to say that this is nothing, but maximum of I from 1 to m, j equal to 1 to n modulus of a ij. So, it is nothing but, this is nothing but infinity norm of a. So, it means that equality is achieved for this vector x naught. So, it means that this 6 is not only in equality, but it is an actually an equality because this maximum is achieved for sum x. So, 2, so that this is true let us look at here that infinity norm of x naught is equal to by definition it is maximum of i between 1 to m j equal to 1 to and a ij modulus of x j naught.

Now, here we say that this maximum is achieved for kth row. So, we say that j equal to 1 to n modulus of a kj x j naught, now x j naught is defined as a k j divided by modulus of a kj. So, if you simplify this is nothing, but modulus of a kj whole square divided by modulus of a kj. So, this simplify that it is given by j equal to 1 to n modulus of a kj and if you look at we have already assumed that this is nothing, but the maximum of I between 1 to m summation j equal to 1 to n modulus of a ij and this is this is what we have denoted as infinite norm of A.

So, it means that for this vector x naught this infinity norm of A of x is equal to the infinity norm of A. So, it means that equality is achieved for this x naught. So, it means that the inequality 6 which we have written is achieved is nothing, but equality. So, it means because this is equal for a norm for a vector x naught and defined here. So, this shows that infinity norm of A is defined by this part A m. Similarly we want to define, we will prove for B here.

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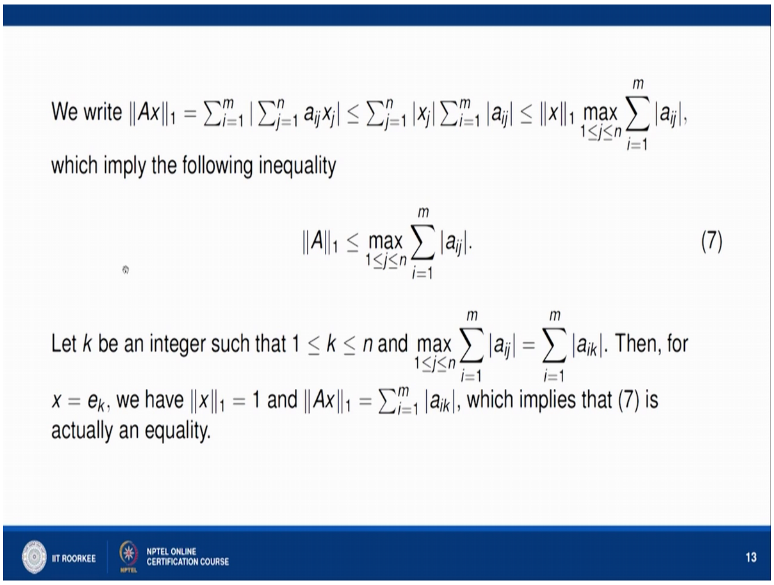

So, here I am quickly writing here that 1 norm of x is given by summation i equal to 1 to m, summation j equal to 1 to n a ij x j. This also follow follows from the fact here that you calculate A x and A x is given by here, now one norm it will be the sum of all these elements here.

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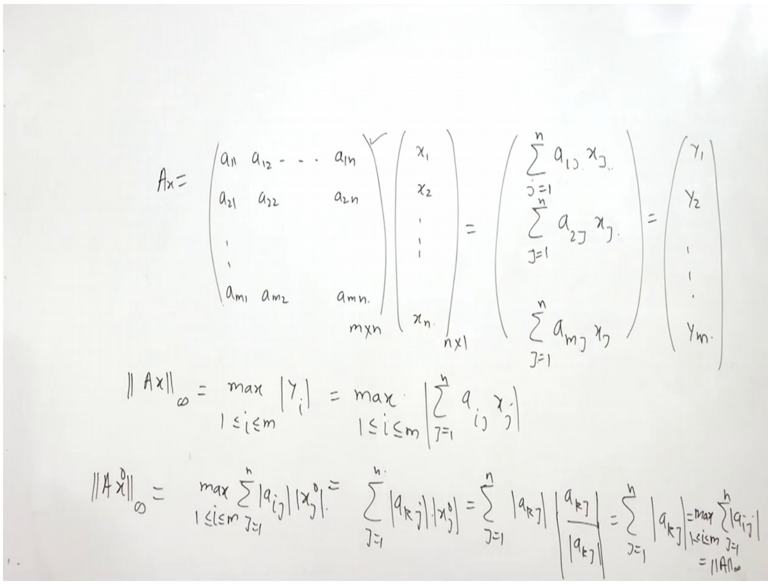

If you find out the sum of all the element here you will say that your one norm of x is defined by this, i equal to 1 to m summation modulus of summation j equal to 1 to n a ij x j and now this is less than or equal to summation j equal to 1 to n modulus of x j, summation i equal to 1 modulus of a ij.

Now, we can say that this is sum here which is if you take these sum here for each j. So, let us take that for each j we can find out some value then take the maximum of all those values. So, it means maximum of j from 1 to n summation i equal to 1 to m a ij we can take it out. So, this if you take that number out then what is left here is summation j equal to 1 to n modulus of x j which is nothing but 1 norm of x. So, we can say that 1 norm of x is less than or equal to 1 norm of x maximum of j from 1 to n summation i equal to 1 to m modulus of a ij. So, which shows that by definition 1 norm of A is going to be less than from this quantity, that is maximum or taking over j from 1 to n j, i equal to 1 to m modulus of a ij.

Now, we want to show as we have shown in first case that this equality will be achieved somewhere. So, here to find out the equality let us see that k be the integer such that this maximum is achieved, maximum is achieved for kth column. So, let us say that maximum from 1 to n, i equal to 1 to m modulus of a ij is achieved for kth column. So, it means if you look at the kth column then column sum is going to be maximum here. So, it is given as i equal to 1 to m modulus of a ik is going to be the maximum of these quantity.

Now, for this k we define vector say x which is given by e k, e k means kth element of basis vector of R n. So, here we say that one norm of x is nothing but 1 and if you equate the 1 norm of A x then it is nothing but i equal to 1 to m modulus of a ik which implies that the equality is achieved for this x. So, it means that this 7 is nothing but inequality is nothing but equality here. So, this proof that 1 norm is actually a norm. So, here what we have shown here that infinity norm is given by this and one norm is given by this. So, infinity norm is what? Maximum row sum norm and one norm is maximum column sum norm.

So, here I will stop here and in next class will, next lecture we will discuss some more properties of matrix norm and some important properties based on this matrix norm. So, here we stop. Thank you for listening us.

Thank you.