Numerical Linear Algebra
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Lecture – 03
Rank of a Matrix

Hello friends, I welcome you to my lecture on Rank of a Matrix. The rank of a matrix is probably the most important concept which we learn in matrix algebra. Let us see how we define the rank of a matrix. Suppose we have a nonzero matrix say, A equal to aij m by n.

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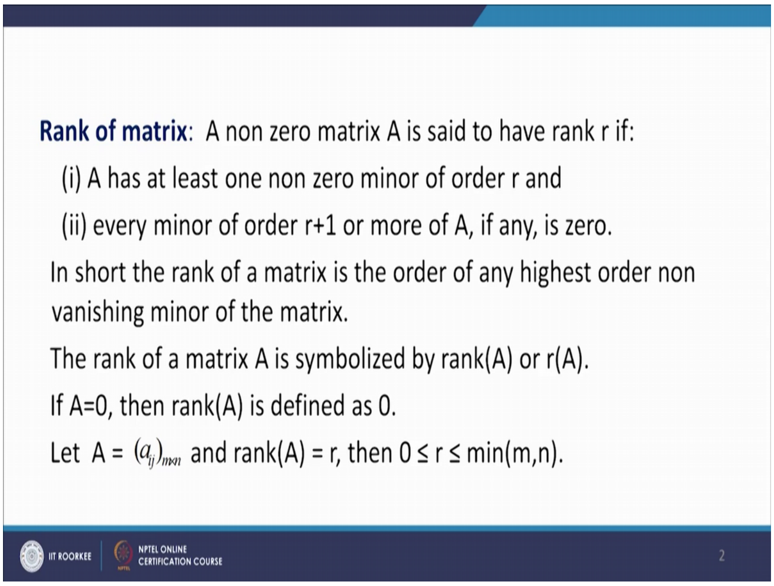

That is the size of the matrix is m by n. Then it is said to have rank r, if A has at least one non zero minor of order r and every minor of order r plus 1 or more of A if any, has is equal to 0.

So, in short the rank of a matrix is the order of the order of any highest order non-vanishing minor of the matrix. The rank of a matrix is implied by rank of A or we write it as r A. So, if let us say A is equal to aij m by n then the rank of A is written as rank of A or r A.

Now, if A is a 0 matrix; A is 0 matrix then rank of A is equal to 0.

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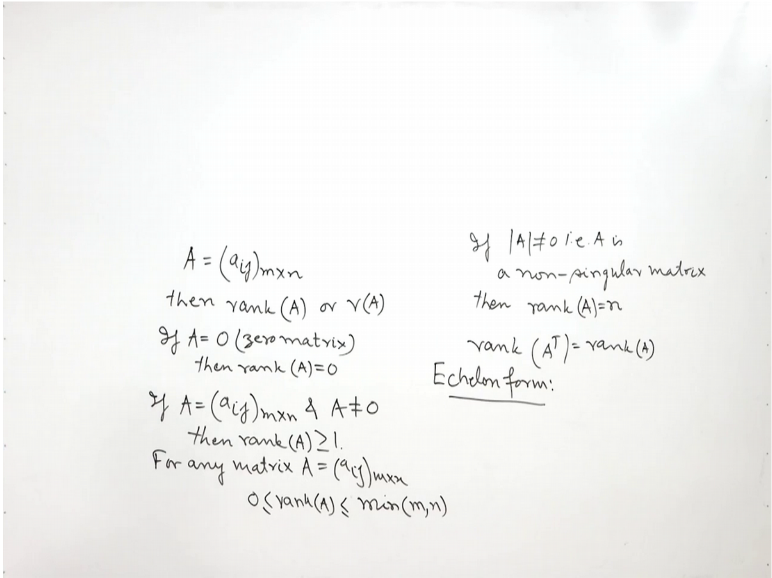

Now if a is any matrix aij m by n, which is a nonzero matrix. Then rank of A is always greater than or equal to 1 ah. For any matrix A equal to aij m by n rank is always bigger than or equal to 0, but less than or equal to in of are minimum of m and n. Which because whichever number out of m and n is the smaller we can have the maximum order square sub matrix of A of that order. Say suppose m is less than n then the largest order square sub matrix that we can have n A will be m cross m.

So, rank of A is greater than or equal to 0 and less than or equal to minimum of m comma n. Now if you take any non singular matrix, if determinant of A is nonzero. That is A is a non singular matrix, then by the definition of rank of A is equal to n. Because the highest order square sub matrix that we can have in A is A itself whose determinant is nonzero so rank of A will be equal to n.

And as I said rank of every nonzero matrix is greater than or equal to 1. If you take the transpose of A, let us say rank of A transpose. Then rank of A transpose is equal to rank of A. This is very easy to see, because if you take any matrix A then the rank of A if it is r they reduced in r by r a square sub matrix whose determinant is nonzero. Now rank of that square sub matrix the determinant of that square sub matrix is same as the determinant of that transpose. So, there exists another square sub matrix ah. The transpose of that whose determinant is nonzero which will be a sub matrix of A transpose.

So, it can be shown easily that rank of A and rank of A transpose are both same. Now let us see how we can find the rank of a matrix. Now the definition of rank which we have seen it depends on calculating the determinants of a square sub matrices ah. And if the rank, if the size of the matrix is very high then it will be very difficult to determine the determinants of the square sub matrices.

So, when the size of the matrix is very high, we use another method to find the rank in which we be reduced the matrix to row echelon form. And then count the number of nonzero rows in that. The number of nonzero rows in the matrix in row echelon form gives us the rank of the given matrix. So, let us see; what do we mean by echelon form. A matrix is said to be in echelon form if the number of zeros preceding the first nonzero entry of a row increases row by row till only 0 rows remain. Then the matrix is said to be in echelon form.

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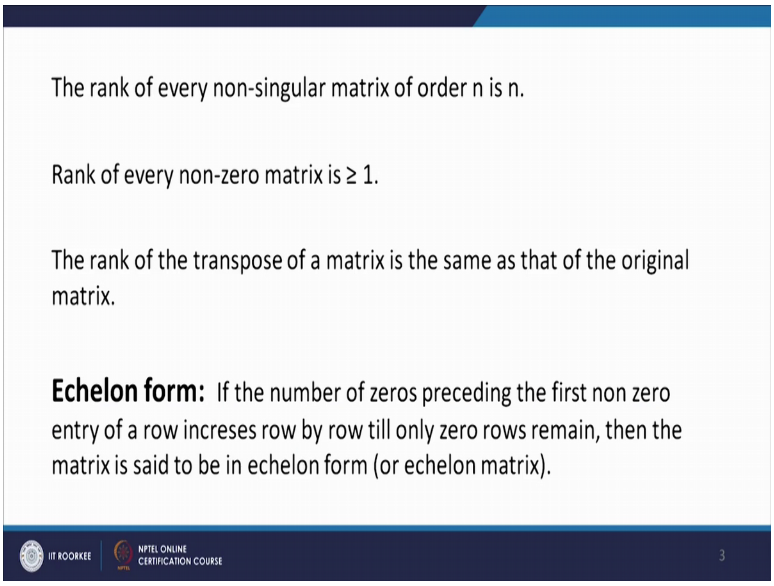

Let us see the examples of such matrices. Let us look at the first example 1 minus 1 0.

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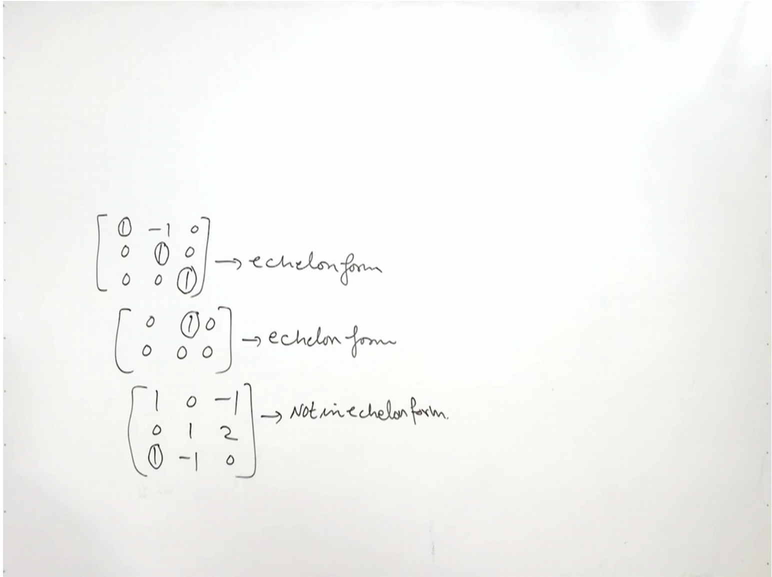

Then we have 0 1 0 we have 0 0 1. This matrix is in echelon form because the number of 0s preceding the first nonzero entry increase row by row in the. Let us start from the first row. In the first row the first nonzero entry is 1, and there is no 0 preceding it so number of 0s preceding the first nonzero entry in row is 0.

In the second row the first nonzero entry is 1, and there is only one 0 preceding it. So, number of 0s preceding the first non-zero entry in the second row is 1. That means, in the previous row number of 0 preceding the first non entry was 0; in the second row number of 0 preceding the first nonzero entry is 1. So, they are increasing in the third row the first nonzero entry is 1 and there are two 0 preceding it. So, number of 0s preceding the first nonzero entry is now 2.

So, number of 0 preceding the first nonzero entry increase row by row and therefore, this matrix is in echelon form. Similarly it was take another matrix say 0 1 0. In this matrix in the first row this is the first nonzero entry which is 1 ah. The there is only one 0 preceding it in the second row all are 0. So, it is a 0 row. So, number of 0s increase row by row till only 0 rows remain. So, it is also an echelon matrix or it is in echelon form.

But if you take a matrix like this, then here the first in the first row first nonzero entry is 1 there is no 0 preceding it. So, number of 0 preceding the first nonzero entry is 0. In the second row first nonzero entry is 1, there is one 0 preceding it. So, number of zero preceding the first nonzero entry increase row by row. But in the third row first nonzero entry is 1 there is no zero preceding it. So, there is number of zero preceding the first nonzero entry is 0. So, number of 0 preceding the first nonzero entry are not increasing row by row; so, it is not in echelon form.

Let us now discuss how we can reduce a given matrix to an echelon form. So, for that we will be doing the elementary row operations on the given matrix. And let us see what are how we define the elementary row operations. So, let us say; let us say we have the matrix A which is of size m by n Ri be its ith row. So, Ri is equal to ai 1 ai 2 and so on ain.

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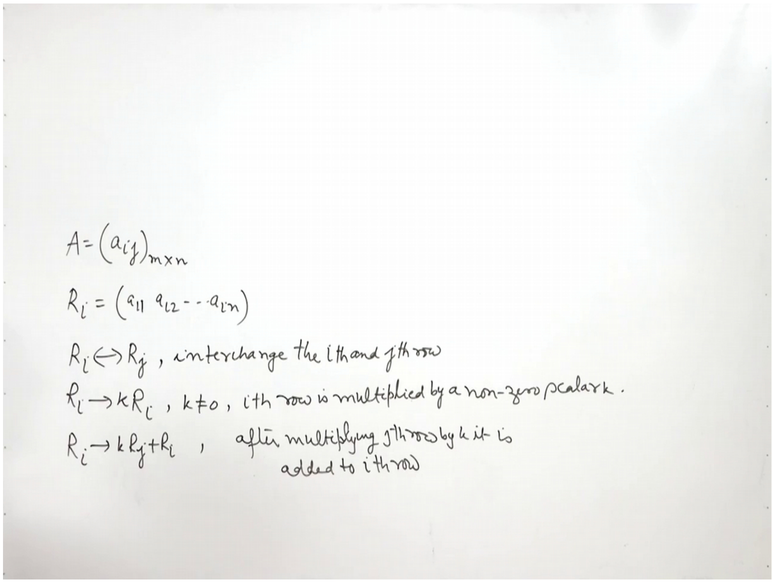

So, Ri denote the ith row here of the matrix A and Rj denotes similarly the jth row. So, first operation that we can perform on the matrix A is that r Ri goes to Rj Rj Rj goes to Ri. So that means, we interchange the ith and jth rows. So, so in that is interchange the ith and jth row. Now the second one is Ri goes to k times Ri their k is a nonzero scalar.

So, ith row is multiplied by a nonzero scalar k. And the third one is Ri goes to kRj plus Ri, where k is any scalar. So, what we do here is that we multiply jth row by k and add the resulting row to the ith row. So, rth jth row is multiplied by k and its then added to i th row so the new Ri is nothing, but k times Rj plus Ri. So, jth row after multiplying jth row by k it is added to ith row. So, we multiply jth row by k and then added to ith row. So, that is the elementary row operation that we can do.

Now, let us note that each of these elementary row operations do not change the zero or nonzero character of the determinants of the square sub matrices of the given matrix A. If you interchange any two rows, then there is a change of sign. So, the zero or nonzero character of the some minor of the matrix A does not change. Similarly if you multiply any row by a nonzero scalar k, again the value of the determinants will be multiplied by k. So, and k is a nonzero scalar. So, the zero are nonzero character will of the minor will not change and when you multiply elements of jth row by k and A to ith row then the value of the determinant does not change the. So, minor it is a value of the minor does not change.

So, why doing any of these three elementary row operations on a given matrix the zero or nonzero character of the minor of the sub matrix does not change. And therefore, we say that the rank of the matrix A which whatever it was it does not change when we carry out a finite sequence of these elementary row operations on the given matrix.

Now, here we have listed the elementary row operations. Similarly we can write the elementary column operations where we interchange two columns ith column and jth column. So, we replace R by C here. So, Ci is Ci and Cj are interchange and then Ci is I mean be new Ci becomes k times Ci. So, the elements of ki are Ci are multiplied by k. So, elementary column operations can also be done ah. So, so what I wish to say that the rank of a matrix does not change when we carry out the elementary finite sequence of elementary row operations on the given matrix.

So, suppose the matrix B is obtained from a after a finite sequence of elementary row operation on A then B said to be row equivalent to A. And we from here we can say that the two or the row equivalent matrices have the same rank. So, now what is an elementary matrix?

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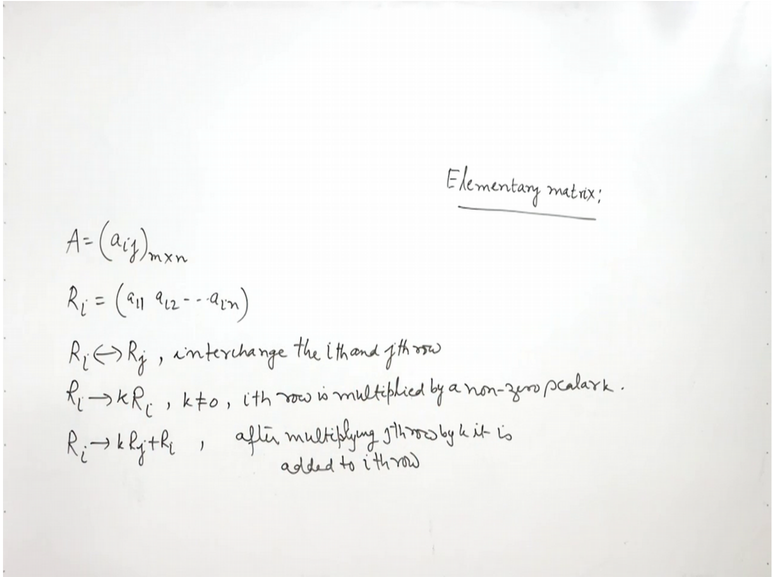

The elementary matrix are defined as it is a matrix obtained from an identity matrix by some elementary row or column operations.

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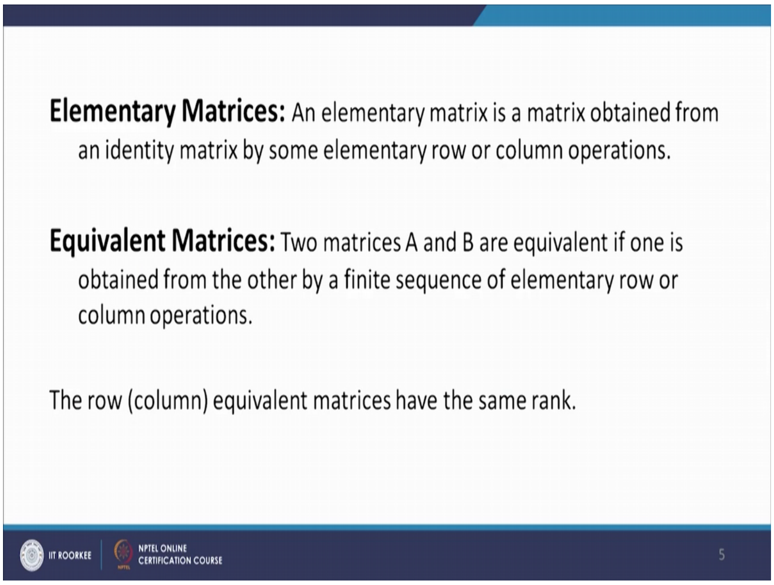

Suppose we are dealing with m by n; n by n matrix identity matrix and we carry out an elementary row or column operation on that then the new matrix that we get that is called as that is called as the as a as an elementary matrix. Now equivalent matrices two matrices A and B are called equivalent if one is a obtained from the other by a finite sequence of elementary row or column operations. As I have said earlier the row and column equivalent matrices have same rank. We have just now seen the matrix B that we get from a by a finite sequence of a elementary row operations ah, has the same rank as the rank of the original matrix.

Similarly, if you do the column operations because column operation row of column operations nothing, but the transpose of the matrix that we get after row operations; so, and rank does not change when you take the transpose of the matrix. So, whether we make of a do elementary row operations or we do column operations to get the ranks that is one and the same thing. So, what we do is we shall be doing elementary row operations on the given matrix to find the rank. Say for example, let us consider this matrix.

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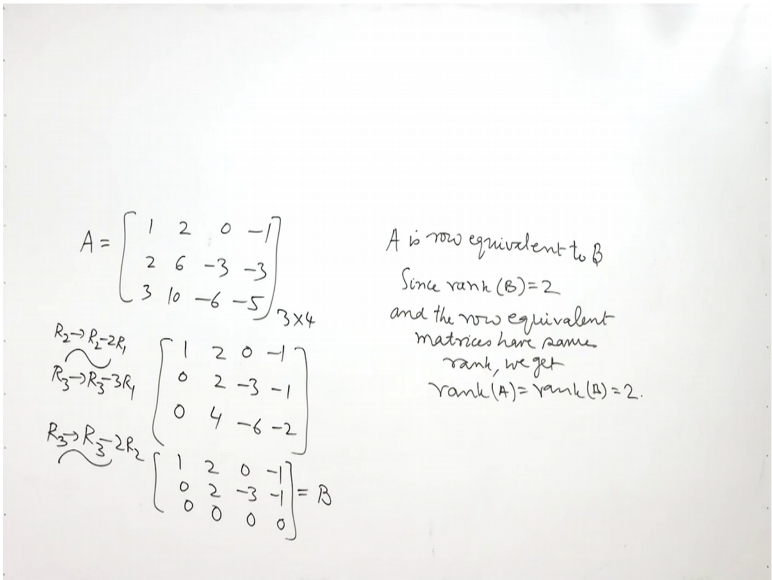

So, let us consider the matrix 1 2 0 minus 1 ah. We have then 2 6 minus 3 minus 3. And we have 3 10 minus 6 minus 5. Suppose we have this 3 by 4 matrix we want to find the rank of this matrix ah. Then what we will do will reduce this matrix A to echelon form. Once it is reduced to echelon form we shall be simply counting the number of nonzero rows in that that gives us the rank.

So, let us say what we do is since in the elementary when we reduce to echelon form. The number of 0s must increase as we go row by row. So, what we do with the help of the first entry here 1. We shall be making two and three 0. So, we can say this equivalent sign. So, we say that R 2 goes to R 2 minus 2 R 1 we multiply, the first row by 2 and subtract on the second row so that second row this 2 become 0. Similarly third row is subtracted by 3 times the first row. So, R 3 goes to R3 minus 3 R 1 and what we get 1 2 0 minus 1 0 and 6 minus 4 is 2, here we will get g minus 3 here we will get two times so we have minus 2 and we subtract minus 2 from here. So, minus 3 plus 2 minus 1 and here we are multiplying 3 and subtracting from here so 0 ah. We are subtracting 6 from 10 so we get 4 then minus 6 here and then we get minus3 into minus 1. So, 3; 3 minus 5 plus 3 so; that means, minus 2.

Now, after we have made these two entries 0. We see that here the first nonzero entry is 1. In the second row the first nonzero entry is 2. So, number of zero is increasing, but in the third row this first nonzero entry number of 0 is not increasing. So, with the help of this 2 we make this 4 zero and we write, R 3 goes to R3 minus 2 R2. So, that it becomes 1 2 0 minus 1 0 2 minus 3 minus 1 and then 0 0 0 and here we get 0.

Now, we can see that the number of 0s in this matrix are increase go 0 row by row till only 0 rows remain. So, this matrix is an echelon form and the number; there are two nonzero rows in this. So, let us say this matrix is B. So, we can say that A is row equivalent to B. Since row equivalent matrices have same rank; since rank of B is equal to 2 and the row equivalent matrices have same rank, we get rank of A equal to rank of B equal to 2. So, rank of A is equal to 2.

Now so here we describe the first method which is used where we use the minus here, we have said that first we consider the minor of largest order say m which can be formed.

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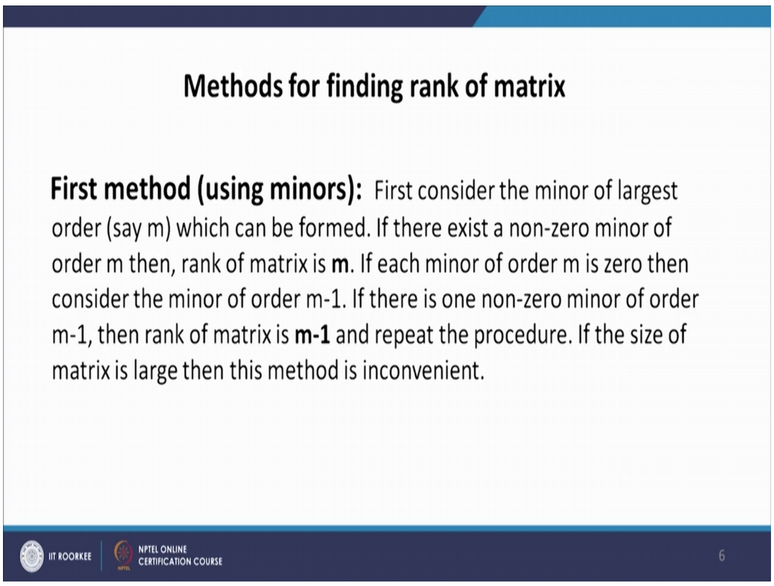

So, like here if you use that first method then this is 3 by 4 matrix. So, the largest order sub matrix that we can get here will be 3 by 3. So, you will have three four 3 by 3 matrices. So, we will find the determinant of all these four 3 by 3 matrices. If there is one 3 by 3 matrix whose determinant is nonzero the rank will be 3 if the determinants of all the 3 by 3 matrices is 0 then we go to 2 by 2 the square sub matrices of this matrix.

Now so we will have to find the determinants of four 3 by 3 matrices and since from this method, we are finding the rank is 2 it will turn out that all 3 by 3 matrices have the determinant 0. Then we go to 2 by 2 sub matrices and 2 by 2 sub matrix you can obtain by deleting say two columns and one rows; so 1 2 2 6, here you can see the determinant is 2 1 2 2 6, this is a 2 by 2 square sub matrix we determinant is a nonzero so rank is 2.

So, so here so here it is not difficult to see that the rank is 2. Because there are only four 3 by 3 sub matrices whose determinants we have to check ah. But when the size of the matrix is very large it is difficult to determine the rank using this method. So, we adopt the second method that is of that is by reducing the given matrix to an echelon form.

Now, here let us look at the properties of rank of matrix. So, and the earlier said for nonzero matrix rank of A is always bigger than or equal to one that is the first property. If A has a nonzero minor of order k ok; if A has a nonzero minor of order k then rank of A will; obviously, be greater than or equal to k.

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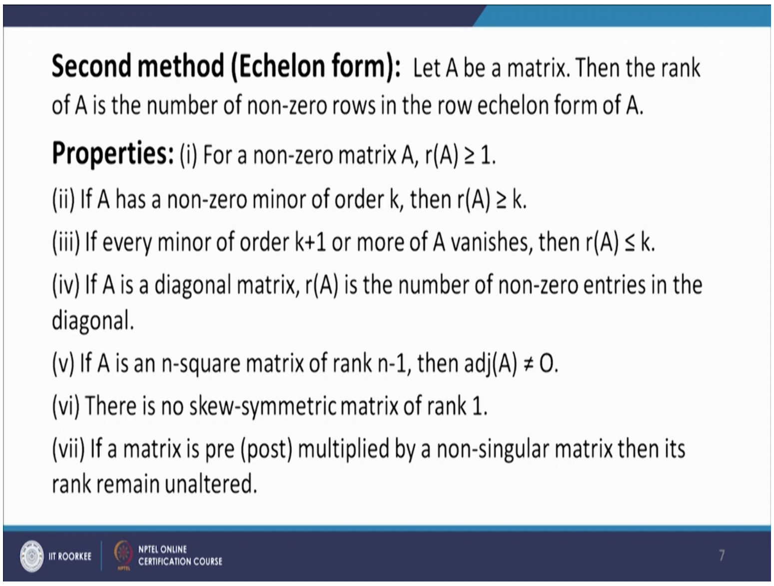

Because there could be a minor of order k plus 1 whose determinant is nonzero.

Now, if every minor of order k plus 1 are more of A vanishes, suppose it so happened that in the matrix say every minor of order k plus 1 or more vanishes. Then rank of A will be less than or equal to k. Because then we will have to look at k by k I mean k or k by k cross k order a square sub matrices ah. And there it may happen that one of them have determinant nonzero. So, rank of A is less than or equal to k.

Now, if A is a diagonal matrix suppose A is the diagonal matrix and r A is the number of nonzero entries in the diagonal ok. So, suppose A is this matrix ok.

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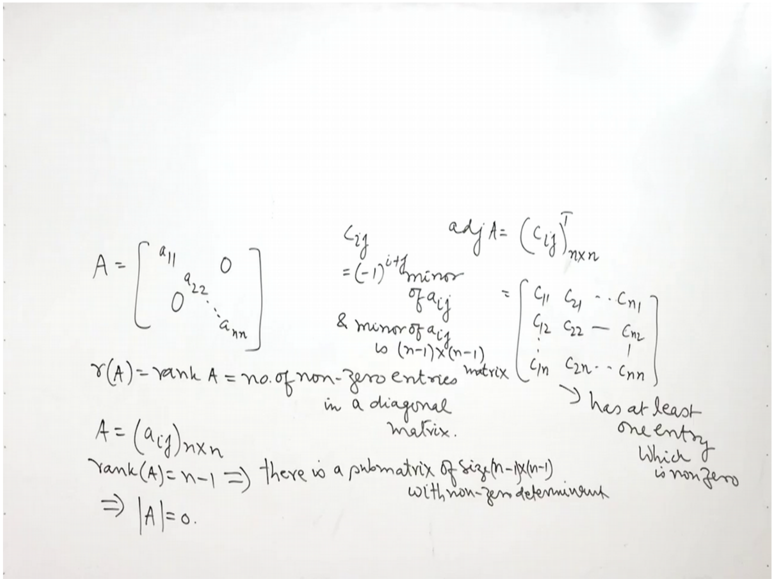

Suppose A is a is diagonal matrix where r A is the number of where the number of nonzero entries number of nonzero entries is nothing, but r A means rank of A r A means rank of A.

So, then the rank of this diagonal matrix is nothing, but the number of nonzero entries in the diagonal. Because whatever nonzero entries in the diagonal we have we consider the square sub matrix of those and then the determinant of that square sub matrix will be product of the diagonal elements. But all the diagonal elements of that that is square sub matrix are nonzero. So, the determinant of that is square sub matrix is nonzero.

So, rank of A is equal to number of nonzero entries in the diagonal matrix; number of nonzero entries in a diagonal matrix. Now if A is a any square matrix of a rank n minus one. Suppose A is a square matrix of order n; n, its rank is n minus 1 suppose we are given a n by n a square sub matrix whose rank is equal to n minus 1 then it will mean that the determinant of A is equal to 0 because a determinant of A is nonzero then the largest order square sub matrix of A is A itself and the determinant is nonzero means rank of A will be n.

. So, it means that determinant of A is equal to 0. Now let us look at the adjoint of A matrix. Adjoint of A is the transpose of the matrix of cofactors. Is the transpose of the matrix of cofactors? So, this is what this matrix is at the type c 1 1 c 1 2 c 1 n ,c 2 1 c 2 2 c 2 n and then we have cn 1 cn 2 and so on cnn .Now let us look at cij ok, cij is the is the a cofactor of the aij element cij is minus one to the power i plus j into minor of; minor of aij. And minor of aij is n minus 1 cross n minus 1 matrix.

So minor of aij is n minus 1 cross n minus 1 matrix since the rank of A is n minus 1. There exists a square sub matrix whose determinant is a nonzero ok. So, this implies that ok. So, that is so n by n n minus 1 cross n minus 1 matrix will exist there with a non vanishing determinant and therefore, there will be at least one element here which will be a nonzero element. So, what we have is in this matrix so this matrix has at least one element at least one entry which is nonzero. And so, adjoint of A is a nonzero matrix.

Now, let us look at the next one there is a no skew symmetric matrix of rank 1; there is there is no skew symmetric. Now we know that suppose A is skew symmetric matrix.

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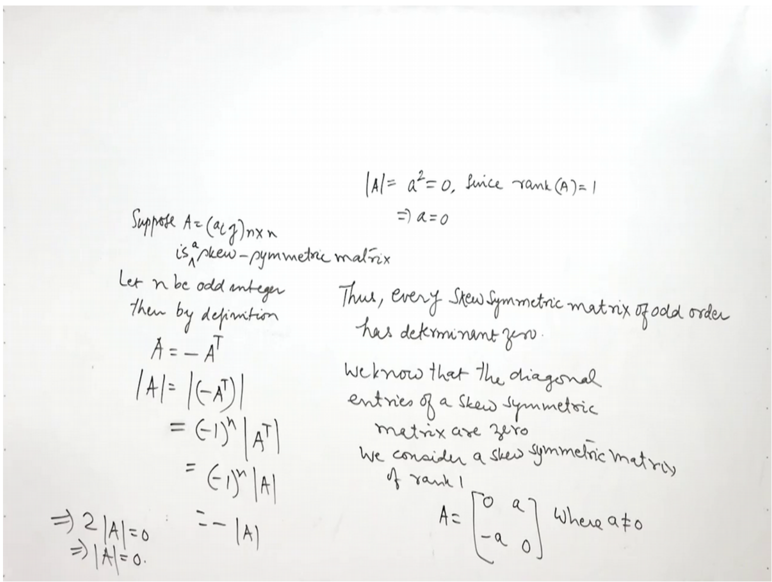

Suppose A equal to aij n cross n be is is a skew symmetric matrix. Then and suppose order of n is odd integer. Let n be odd integer.

So, then by definition of by definition of a skew symmetric matrix A is equal to minus A transpose. If we take the determinant both sides then determinant of A is equal to determinant of minus A transpose. In the A transpose minus 1 is multiplied. So, minus 1 is multiplied to all the n rows of A determinant. This means that this is the equal to minus 1 to the power n into determinant of A transpose which will mean that minus 1 to the power n into determinant of A.

Now, since n is odd this is equal to minus determinant of A. So, what we get is determinant of A equal to minus determinant of A; So, two times determinant of A equal to zero. So, this implies that determinant of A is equal to 0. And this means that all or ordered a square sub skew symmetric matrices have zero determinant. So, thus a skew symmetric matrix every a skew symmetric matrix of odd order.

So, if we have 3 by 3 or 5 by 5 a skew symmetric matrices their determinants are all 0s. So now, we the question is whether whether there exists a skew symmetric matrix of rank 1. A skew symmetric matrix of rank 1 suppose it exists, then the a skew symmetric matrix of order 2 if you consider a skew symmetric matrix of order 1 suppose it exist then it will be of this type because every skew symmetric matrix has diagonal entries 0.

So, let us consider a skew symmetric matrix. Suppose we know that the diagonal entries of a skew symmetric matrix are 0 s, diversity of a skew symmetric matrix are 0. So, let us say suppose we consider a skew symmetric matrix of rank 1 ok. We consider rank 1. So, we considered 2 by 2 type ok.

So, then this will 0 this is 0 if a element is here then it will be minus a here where a is nonzero. These skew symmetric matrix whose rank is 1 because if rank is 1 if a is nonzero then only the rank will be 1, but then what happens is that suppose I call it a matrix then determinant of A determinant of A is equal to a square determinant of a square determinant of A is equal to a square since rank is 1. So, determinant of A must be zero. Since rank of A equal to 1.

So, determinant so determinant of A which is a square must be zero ah. Which implies that a is equal to 0. So, there is a contradiction so we can say that there does not exist a skew symmetric matrix of rank 1. Now if a matrix is pre or post multiplied by a non singular matrix. Let us see how we get that so if a matrix suppose we have a matrix A suppose we have a matrix A if a matrix is pre or post multiplied by a non singular matrix ok.

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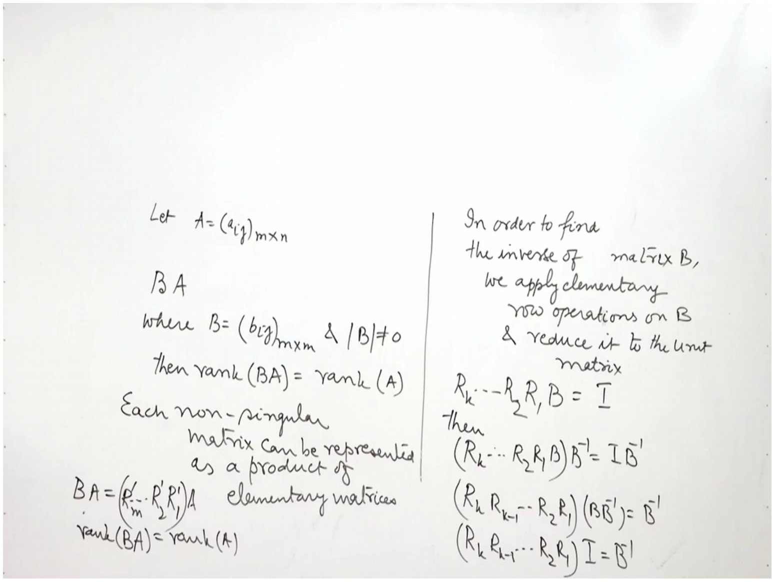

Suppose it is pre multiplied say by say some matrix B. So, B into A where B is we have to take B as bij m by m because this b BA will be only possible when the number of columns of B are same as the number of rows of A. So, B is equal to bij and B is non singular matrix ok. Where B equal to bij and determinant of B is not equal to 0. Then we have to show that then rank of BA is equal to rank of A this is the claim ok.

Now, let us see that suppose we have a matrix A which is invertible ok. Because rank of B is I mean B is a non similar matrix. So, B is invertible how we can find the the the the inverse of the matrix B we can use the formula that B inverse is equal to adjoint of B divided by determinant of B. That is one method. The other method is by applying elementary row operations.

So, what we do is in order to find; in order to find the inverse of B inverse of matrix B we apply elementary row operations on B. We apply a finite sequence of elementary row operations on and reduce it to the unit matrix and reduce it to. So, suppose those elementary row operations are denoted by r 1 r 2 or rk then we will have.

Suppose we do k elementary row operation on the matrix B. Each elementary row operation that is done on B means whatever elementary row operation you want to do on B that elementary row operation you do on unit matrix of size m by m and the matrix that you get that matrix you pre multiply to B. So, for example, you want to interchange the two rows i ith row and jth row of b r one operation is nothing, but interchanging the ith jth row of B then this r 1 matrix will be the matrix where ith and jth row of the unit matrix of size m by m r have been interchanged.

So, that will be r 1 and similarly r 2 ah. Whatever elementary row operation r 2 is here that operation is that elementary matrix is the matrix which is obtained from the unit matrix by doing the row operation that is desired here. So, these are suppose k elementary row operations that are done on B to reduce it to I ok.

So, let us say we have k elementary row operations that are done on this ok. Then Rk r 2 R 1 B B inverse let us post multiply by B inverse. So, what we will get so Rk; so inverse of B ok. Inverse of B is the matrix that is obtained by doing the same elementary row operation; I mean the same elementary row operation in the same sequence on the identity matrix.

So, each non singular matrix each non singular matrix can always be expressed as the product of elementary row matrices. So, here what we have is this matrix B there is a result which says that each non singular matrix can be represented product as represented is a product of elementary matrices. So, so this B BA will is nothing, but it can be written as sum at R 1 dash R 2 dash and so on R m dash.

So, BA is equal to R one this R one dash R two dash (Refer Slide Time: 00:00) m dash which are elementary matrices into A. So, this is now these are nothing, but m and elementary row operations that are done on A. So, since these elementary row operations which are done on A do not change the rank of A, rank of BA will be equal to rank of A. So, the when we multiply a matrix A by a non singular matrix B the rank of matrix does not change because the non singular matrix can all be you can all be can always be expressed as the product of elementary matrices.

So, that is this one and then we have then we have let say row reduced row reduced row echelon form.

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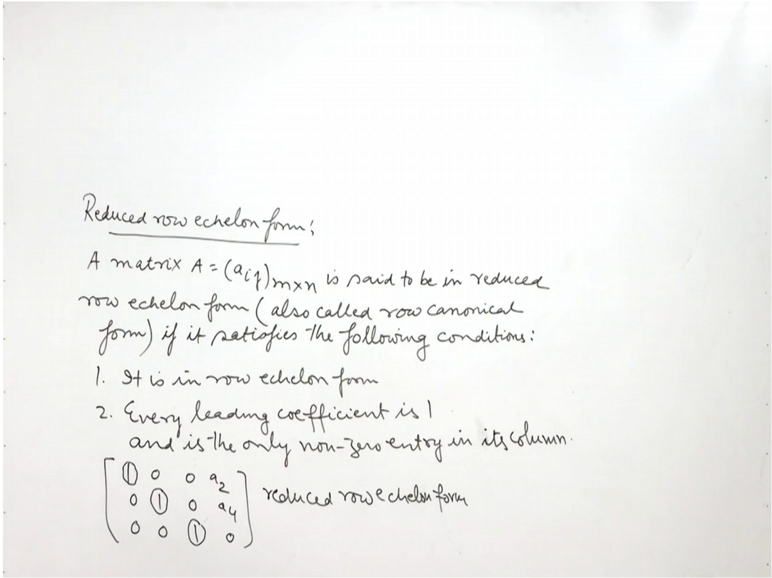

So, let us see what we have defined row echelon form, this is reduced row echelon form. So, a matrix is said to be in reduced row echelon form. Reduced row echelon form also called row canonical form also called row canonical form, if it satisfies the following conditions.

So, first condition is that it is in row echelon form, and the second condition is that every leading coefficient is 1 every leading coefficient is 1 and is the only nonzero entry in its column. So, besides the matrix being in row echelon form be also need to have that every leading coefficient is 1. And it must attend it is the only nonzero entry in its column. Say for example, let us say this 1 0 0 1 we can see here that in this matrix this is the first row. In the first row the first nonzero entry is 1 and it is the leading coefficient.

So, leading coefficient is 1 and it is the only nonzero entry in its column. Then we go to the second row, in the second row the first nonzero entry is 1 it is the leading coefficient and all the other entries in this column are 0s. And in the third row we have 1 here. So, leading coefficient is again 1 and it is the only nonzero entry in its column. So, this this matrix is in row reduced row echelon form. So, when we reduced the matrix to a echelon form we do one more thing that the leading coefficients are made equal to one 1. With after making them equal to 1 we also make the other entries of the in that particular column 0s. So, then we will arrive at the reduced row echelon form. So, with that I would like to conclude my lecture.

Thank you very much.