Numerical Linear Algebra

Dr. D. N. Pandey
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 29
Vector Norms – I

Hello friends, welcome to the lecture. In this lecture and in coming few lectures, we will discuss about vector and matrix norm. So, let us first start with the vector norms.

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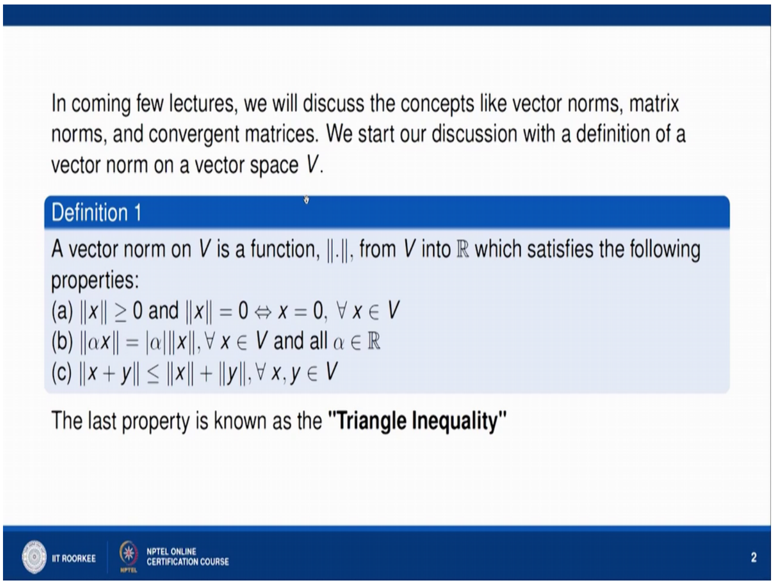

So as I discussed, in the coming few lectures, we will discuss the concept like vector norms, metric norms and convergent matrices. So, we start our discussion with a definition of a vector norm on a vector space V. So, V is a vector space. Let us say, dimension of V is given as n, some finite number is given here. So, we define our vector norm as, a function from V into R, we satisfy the following properties.

So, first property is that norm of x is non-negative and norm of x is equal to 0,if and only if x is equal to 0, and this norm of x is non- negative, for every x belonging to V, and b part says that norm of alpha x is equal to modulus of alpha times norm of x, for every x belongs to V and for all alpha belonging to scalar field, that is we have taken as R here and c part is that norm of x plus y is less than or equal to norm of x plus norm of y, for every x, y belongs to vector space V. So, this c property, which is very useful property and this property is known as triangle inequality.

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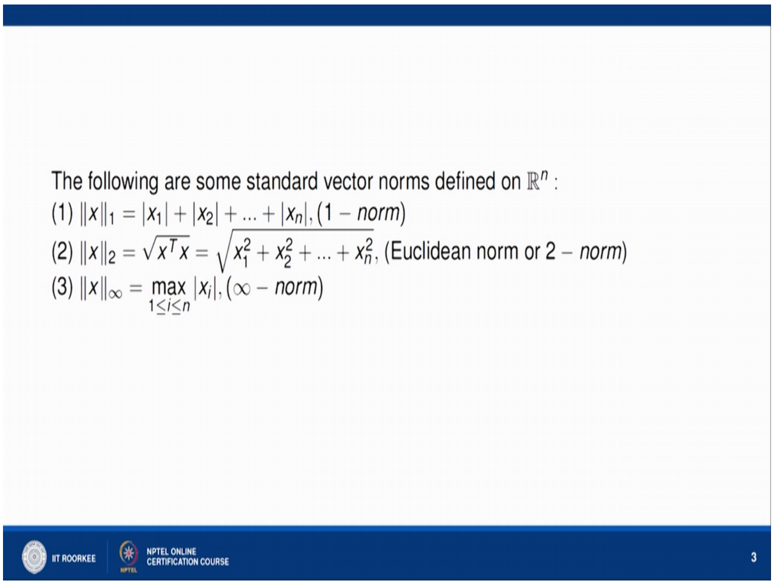

So, let us take certain examples of vector norms on R n. So, in vector norms on R n examples are 1 norm, that is norm of x is equal to modulus of x 1 plus modulus of x 2 plus norm of x n. So, this is known as 1 norm or sometimes, it is also called as L l norm, little one norm. Second is 2 norm that is, norm of x is equal to under root x transpose x which is given S and x 1 square plus x 2 square plus x n square. So, this is usual norm which is known as Euclidean norm or 2 norms. Similarly, we can define the infinity norm; infinity norm is given a norm of x is defined as maximum i is from 1 to n modulus of x of i.

So, in other books you may find this as little L1 norm, little L2 norm, and little L infinity norm. So, these are certain examples on R of n Rn here. So now, we try to show that these norms are actually norms, it means that it satisfy these properties a, b, c. So, we want that these three norms 1, 2, 3 are actually norms on R n.

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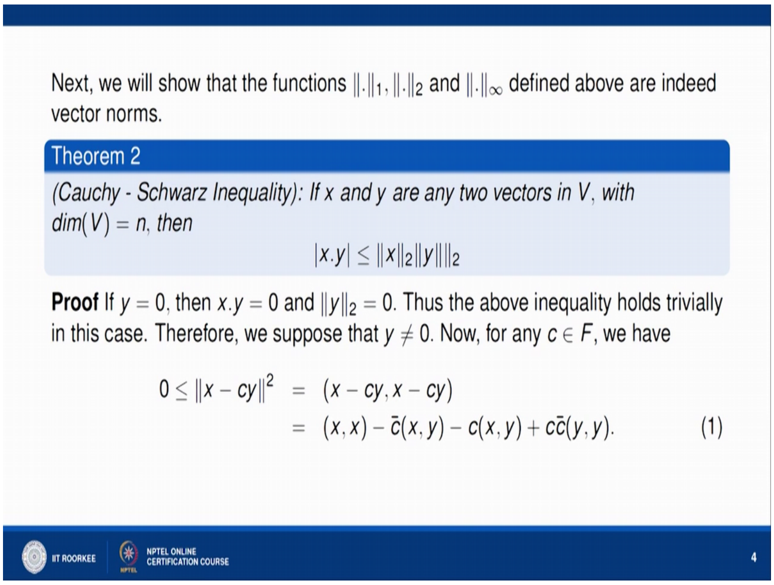

So for that, we try to prove one property which is known as, "Cauchy Schwarz Inequality'' which says that, if x and y are any two vectors in vector space V, with a dimension of V is equal to n, then modulus of x dot y is less than or equal to 2 norm of x into 2 norm of y. Here, x dot y represent inner product between x and y here. So, let us prove this simple property.

So, let us take that if 1 of x and y is 0 then inner product of x dot inner product of x and y has to be 0 and since y is equal to 0. Then, with the property given here, that norm of y is going to be 0. So, if y equal to 0 then inner product is 0 and 2 norm of y is equal to 0 then this relation trivially holds. So now, let us assume that none of x and y is 0. So, let us assume that y is not equal to 0. So now, take any constant in a scalar field, and we have that 0, less than or equal to norm of x minus c y whole square, that follows from the non - negativity of any given norm.

So, let us assume that none of x and y are 0. So, let us assume that y is not equal to 0. Then, for any scalar c in F, we can say that, norm of x minus c y whole square is greater than or equal to 0. And if we use the inner product structure for this norm, that is this norm of x minus y whole square is written as inner product of x minus c y with the inner product x minus c y, and if you simplify you can get this expression 1. Now, this expression is valid for every constant c in F.

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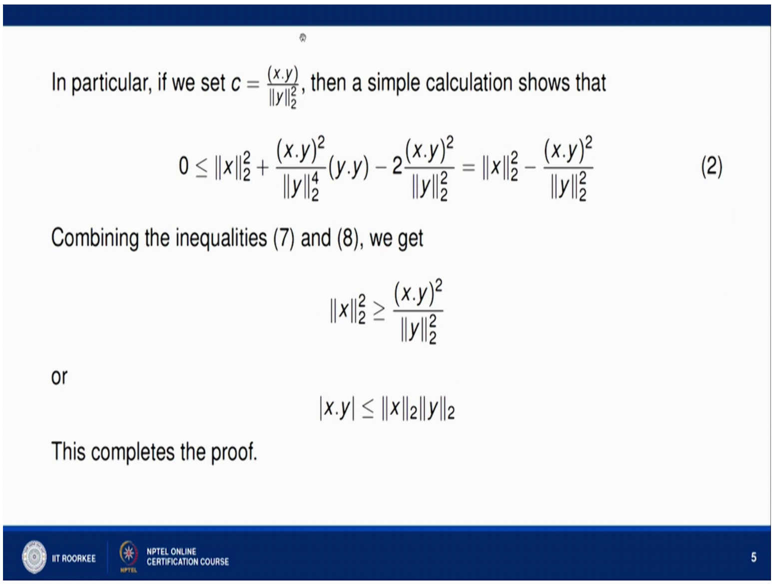

So, in particular take c as inner product of x with y divided by y 2 square. Now, since y is nonzero, then these two norms of y whole square is of course, a non zero. So, we can define this c. So, using this value of c, if you simplify the expression given as n 1 then, we can say that it is written as 0 less than or equal to 2 norm of x whole square plus inner product of x dot y whole square, divided by 2 norm of y to power 4, inner product y with y which is nothing, but 2 norm of y square minus 2 inner product of x dot y whole square divided by 2 norm of y whole square and if you simplify this, this is nothing, but norm of x whole square minus inner product of x and x y whole square divided by y 2 square. So now, this quantity is non- negative.

So, this implies that norm of x 2 whole square and greater than equal to inner product of x dot y whole square divided by 2 norm of y whole square. So, if you simplify this is nothing, but modulus of x dot y is less than or equal to 2 norm of x into 2 norm of y here. So, this completes the proof of this inequality which is known as Cauchy Schwarz Inequality.

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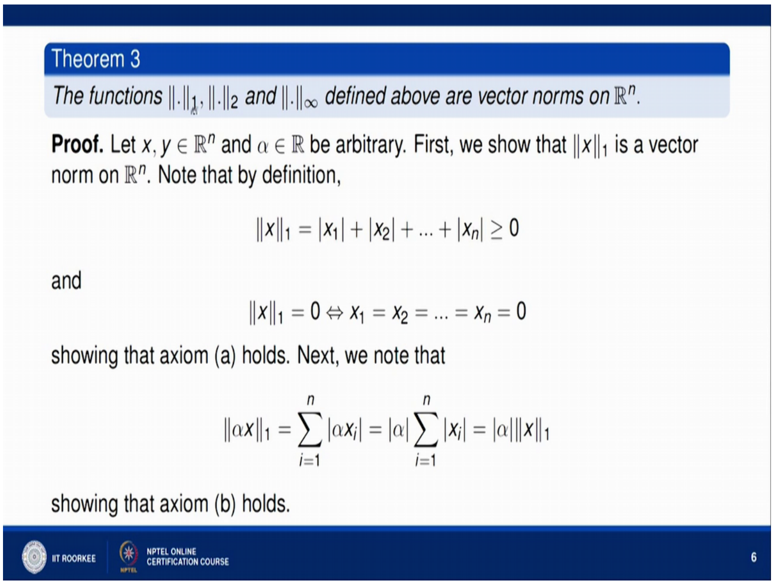

Now, with the help of this, Cauchy Schwarz Inequality, we try to prove the next theorem which says that, the function this norm 1, norm 2 and norm infinity defined above are actually vector norms on R F n. So, contain of this theorem is to show that the function which is defined in second slide, as this actually defines a norm on R n. So, let us start with proving this, that this norm of x 1 which is defined as this modulus of x 1 plus modulus of x 2 plus modulus of x n actually defines a vector norm.

So, let us try to verify all the three properties. So, first property says that this has to be non negative. So, if you look at this, this is what sum of positive sum of non negative number so it has to be non negative. So, norm of x 1 is non- negative and if it is equal to 0, then this implies that this sum is equal to 0. Now, this is what sum of all non negative constants has to be 0, only if each term is equal to 0. This implies that, x 1 is equal to 0, x 2 is equal to 0 and similarly x n equal to 0. So, this satisfies the axiom 1 property. Next, we want to show that if you multiply, if we take the norm of alpha x then, it is nothing but modulus of alpha times norm of x. So, here by definition of alpha x, it is given as summation i is equal to 1 modulus of alpha x i. Now, here this alpha can be taken out and it is written as, modulus of alpha summation i equal to 1 to n modulus of x i which is nothing, but norm of x. Here, 1 norm of axiom. So, this shows that the property 2 is also satisfied.

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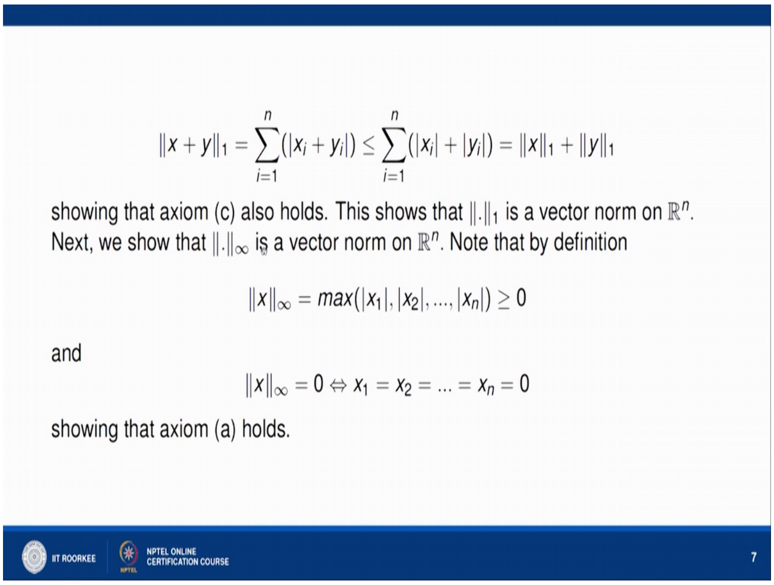

Similarly, the last property which is triangle inequality property, we say, that the norm of norm of x plus y is given as i equal to 1 2 n summation modulus of x i plus y i. Now, this can be written as less than or equal to summation I equal to 1 2 n modulus of x 1 plus modulus of y 1, this is Triangle Inequality use for modulus function. So, if you simplify this, this is written as summation i equal to 1 2 n modulus of x 1 plus summation I equal to 1 2 n y 1. So, this is nothing, but norm of x plus norm of y. So, this simplify that norm of x plus y, 1 norm of x plus y is less than or equal to 1 norm of x plus one norm of y. So, it means that the last property c is also true. So, this implies that this function is actually a vector norm on R n. We call this as 1 norm or little L 1 norm on R n. Similarly, we want to show that, this function defines a vector norm on R n.

So, let us proceed, as we did for 1 norm. So, first thing is that it is non negativity. So, by the definition of norm of x infinity, norm of x is given as maximum of modulus of x i. So here, since it is maximum of non negative numbers so, maximum has to be non negative. So, this property is trivially true. Now, if we equate this quantity to 0, so, this implies that maximum of modulus of x i is equal to 0. So, maximum modulus of x i is equal to 0, means each one is equal to 0. So, modulus of x i is equal to 0, means all x i has to be 0. So, it means that x 1 equal to 0, x 2 equal to 0 and similarly x n is equal to 0. So, this proves and satisfies the first axiom a.

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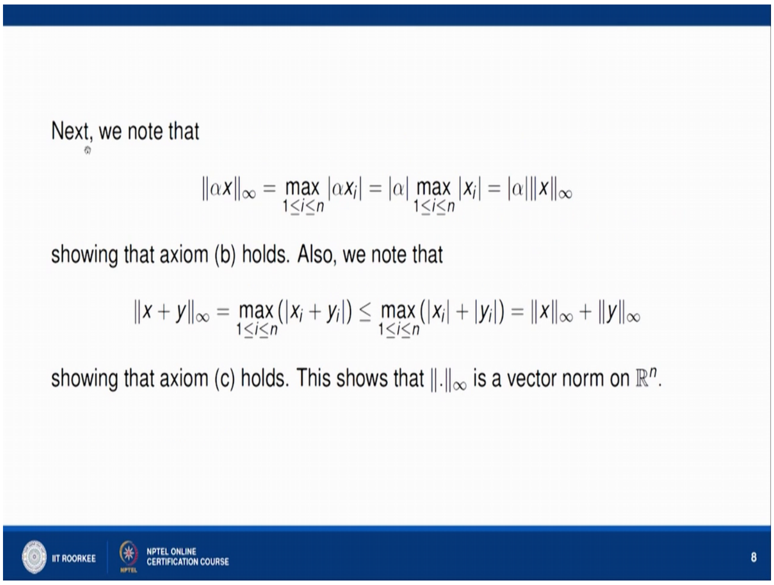

Similarly for second, that is, infinity norm of alpha x, which is by definition is nothing, but maximum of alpha x i, i is between 1 to n. So here, this can be written as modulus of alpha into modulus of x i. So, modulus of alpha is free from this suffix i. So, this can be taken out and this is nothing, but modulus of alpha into maximum of x i, where i is running from 1 to n. So, this is nothing, but infinity norm of x. So, it means that infinity norm of alpha x is given by modulus of alpha infinity norm of x. So, this proves the second property and it means that this function satisfies the second property.

Now, coming to the last property that is, Triangle Inequality. Let us consider, the infinity norm of x plus y which is by definition, maximum of modulus of x i plus y i, i is from 1 to n. So here, let us utilize the property of Triangle Inequality for modulus function. So, this is nothing, but this is less than or equal to modulus of x i plus modulus of y i. So, operating, taking maximum of this, can be written as what modulus of x i plus y is less than modulus of x i plus modulus of y i and which is further less than or equal to maximum of i, i is between 1 to n modulus of x i plus modulus of y i, then you can take the maximum. Here also, you can say that this is true. Now, this is nothing, but maximum of x i, i is from 1 to n. x is infinity norm of x plus maximum of modulus of y i, i is between 1 to n is nothing, but infinity norm of y. So, this satisfies the last property which is known as Triangle Property. So, this implies that this infinity norm, actually satisfies all the properties a, b, c and hence, it defines a vector norm on R n.

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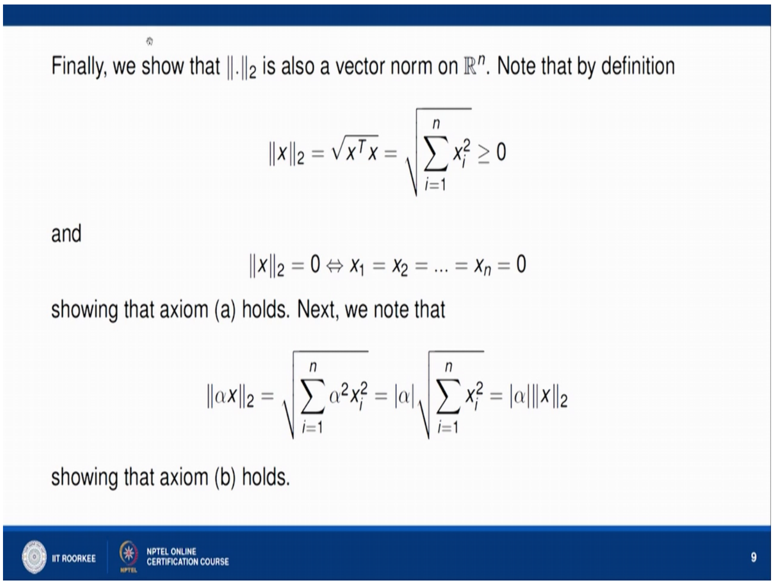

So, coming to the 2 norm of x. So, by definition 2 norm of x is given by under root x transpose x, which is nothing but under root i is equal to 1 to n x i square and since by definition this is going to be non-negative. So now, if it is equal to 0, equated to 0 this implies that summation i is equal to 1 to x i square is equal to O and this is nothing, but sum of all positive numbers. So, all non-negative numbers. So, this can be 0 only if each term has to be 0. So, this means that all x i is equal to 0, all x i is 0.

So, this shows that this norm of x satisfies the first property. Coming on to second property, that 2 norm of alpha x is given by i equal to 1 to n alpha square x i square, whole square root and this can be alpha square is free from this index i. So, this can be taken out and it is written as modulus of alpha times under root i, equal to 1 to n x i square. So, this is nothing, but 2 norms of x. So, this means that norm 2 of alpha x is given by modulus of alpha and 2 norms of x, which shows and satisfies the second property b.

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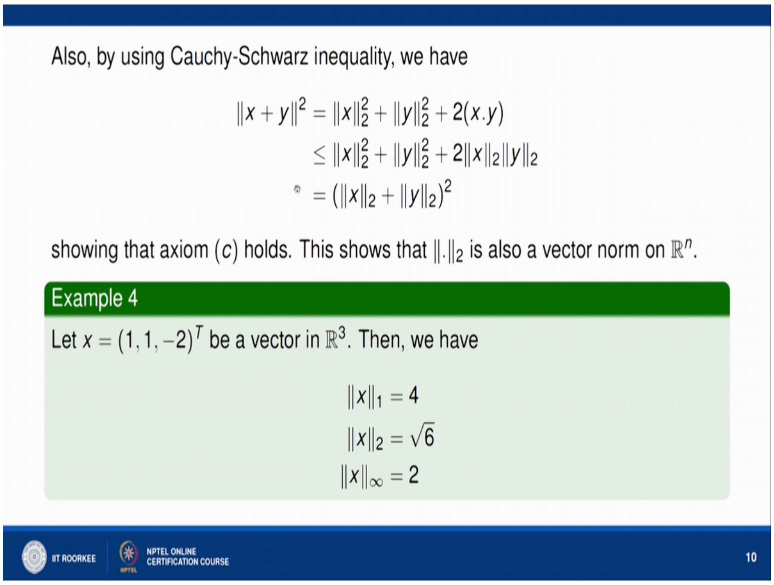

The last property, here, we use the Cauchy Schwarz Inequality, which we have just proved and for that you find out, say, square of this norm of x plus y whole square which, is given as to norm of x square plus norm of y square plus 2 times x dot y which is inner product of x and y. Please remember here, we are using only real scalar fields, that is why we are writing here, two inner products of x and y, if it is, say complex inner product then it is two times real of x dot y.

So, let us stick to real scalar field. So now, these two inner products of x dot y is less than or equal to two times norm of x into norm of y and if you simplify, and look at this, this is nothing, but norm of x plus norm of y whole square. So, you just simplify it, it can be given as that norm of x plus y is less than or equal to norm of x plus norm of y, we say that it satisfies the last triangle inequality property. So, this shows that these two norms are also vector norms on R n. So, on R n we have seen three norms: - 1 norm, 2 norms and infinity norm. Let us find out one example and try to find out these three norms. So, let us take a vector x, 1 1 minus 2 in R 3 and then we try to calculate 1 norm, 2 norm and infinity norm.

So, if you look at the 1 on 1 norm is basically what sum of all the modulus value or absolute value of here. So, if you find out say, absolute value here is 1 1 and 2. So, sum will be what - 1 plus 1, 2 plus 2 here. So, that gives you 1 norm of x. So, 1 norm of x is going to be the sum of absolute values of content. Here, coordinates is given by 4.

Similarly here, 2 norms of x will be what- 2 norms will be summation of x i square. So, summation of x i square means what? 1 square plus 1 square plus 2 square. So, that is going to be 4 plus 1 plus 1 means under root 6. So, 2 norm of x is going to be under root 6.

Now similarly, infinity norm of x will be what? Maximum of x i. So, maximum of modulus of x i will be maximum. So, here it is modulus of x 1 is 1 modulus of x 2 is 1 modulus of x 3 is 2 here. So, maximum will be 2 here. So, infinity norm of x is given as 2. So, here for this particular vector which is given as 1 1 minus 2.Your 1 norm is 4 2 norm is root 6 and 3 infinity norm is given as 2 here. So, as we have defined these 3 norms, 1 2 and infinity.

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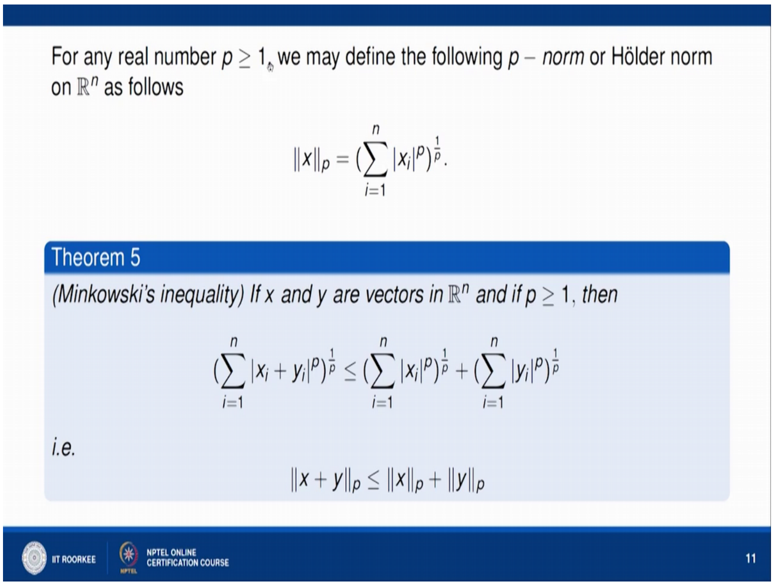

We can define for any real number p, greater than or equal to 1, we may define the following p norms or we can say little 1 p norm, which is defined as norm of p, norm of x is equal to summation I equal to 1 to n, modulus of x i to power p whole power 1 by p and to show that it actually satisfies all the properties listed as a, b, c; we need to know the following inequality, which is known as Murkowski Inequality. So, we say that if x and y are vectors in R n and if p is greater than or equal to 1, then it satisfies the following property. We say that i is equal to 1 to n modulus of x i plus y i to power p whole power 1 by p less than or equal to summation i equal 1to n modulus of x i to power p, whole power 1 by p plus summation i equal to 1 to n.1 modulus of y 1 to power p whole power 1 by p or if you want to write down this in terms of norm, then this is nothing, but norm of x p norm of x plus y and this is nothing, but p norm of x and here p norm of i. So, this simplify that norm of x plus p norm of x plus y is less than equal to p norm of x plus p norm of y which is nothing, but triangle inequality for this p norm and the remaining thing, that it is non- negative and a scaling property that alpha p norm of alpha x is given by modulus of alpha times norm of p. Norm of x is you can trivially prove this (Refer Time: 19:29)

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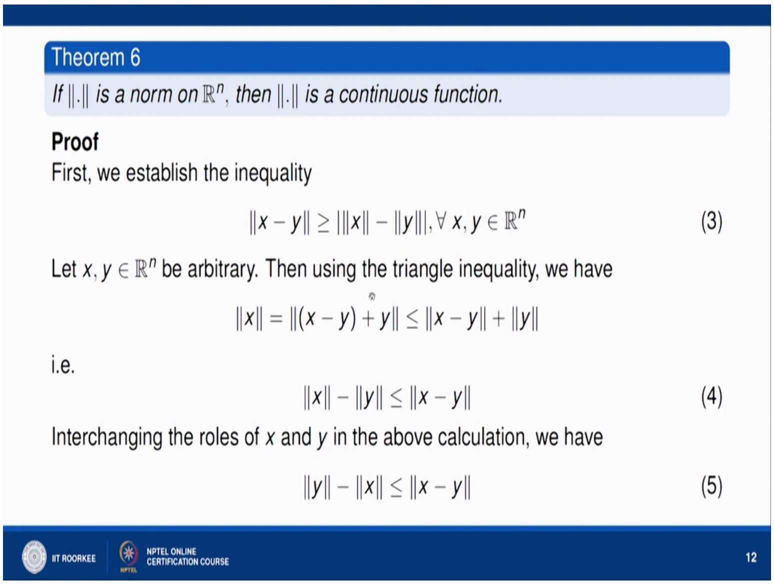

So, moving on, say, next property of norm. So, next property of norm is that, norm is a continuous function.

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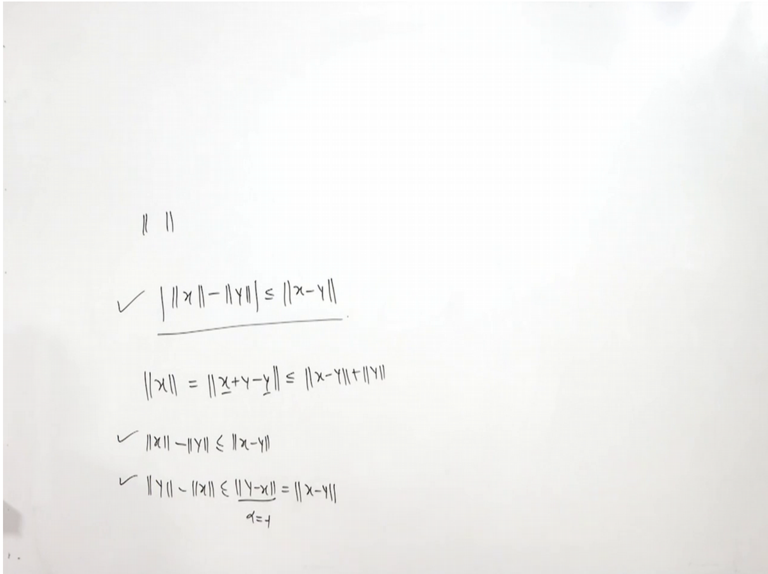

So, to prove that norm is a continuous function, what we try to prove here, first, we try to prove that norm of x minus norm of y modulus of this, is less than or equal to norm of x minus y here. So, this we can prove very easily. Here, we can say that norm of x will be what norm of x can be written as norm of x plus y minus y here. So, this can be written as less than or equal to norm of x minus y plus norm of y here. So here, I can write this as norm of x minus norm of y is less than or equal to norm of x minus y.

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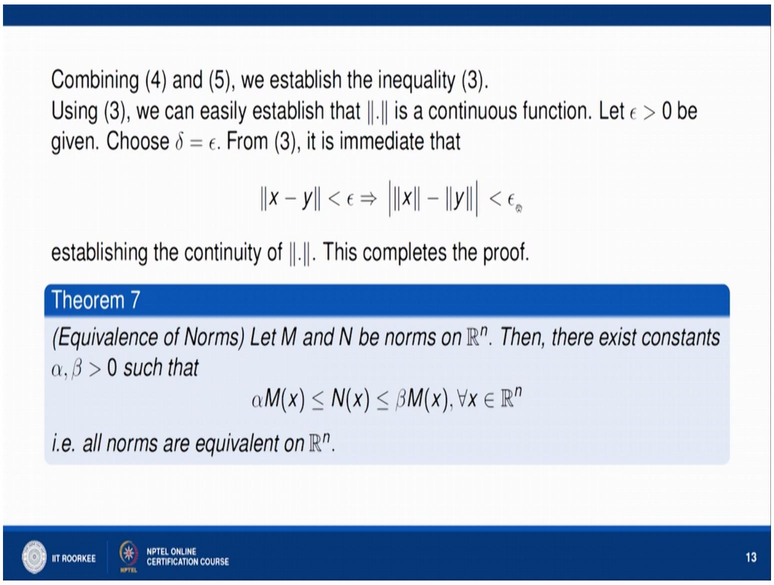

Now, this can be written for any x and y. So, let us interchange the role of y and x and we can write that norm of y minus norm of x less than or equal to norm of y minus x. Now, this last term is written as norm of x minus y as here alpha is simply minus 1. So, modulus of alpha is nothing, but what. So, if you combine this and this, then we can write down this inequality. Now here, we can say that x is near to y, then norm of x is near to norm of y which shows the continuity of norm function. So, here we have shown that if norm of x minus y is less than epsilon, then modulus of norm of x minus norm of y is also less than epsilon. So, which establishes the continuity of norm and which completes the proof.

Now, next coming on to be very important property that is, equivalence of norm. So, here, we can define equivalence of norm as let M and N be two norms on R n, then they exist constant alpha and beta greater than 0, such that alpha M x is less than or equal to N x less than or equal to beta M x for every x in R n. Now, the content of this theorem is that all norms are equivalent on R n. So, what do you mean by equivalence? Equivalence means, if we can find out alpha and beta to non zero constant, such that the following property whole, then we say that M x and N x are equivalent to each other. Now, we want to prove that all norms are equivalent on R n. So, to prove this, let us come to this.

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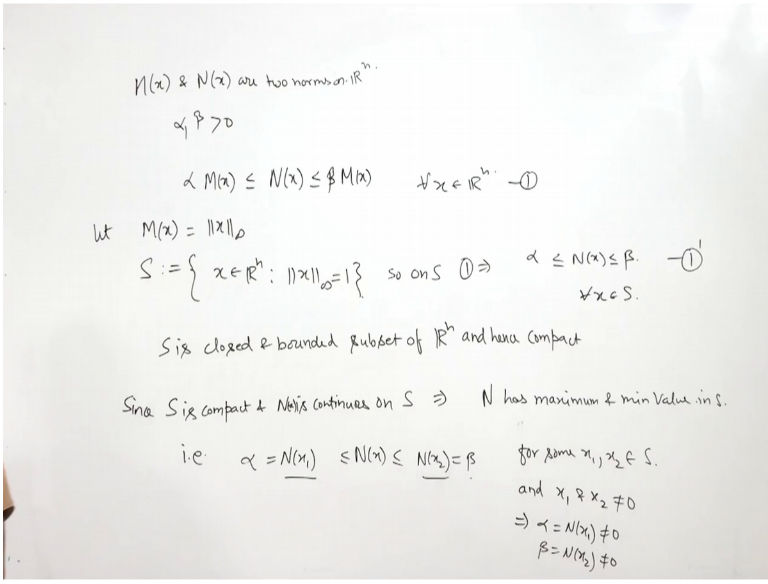

So, here we want to prove that every two norm, defined on R n are equivalent. So, let us take M and N are two norms on R n and we want to show that, these two norms are equivalent. It means that they exist, two constant alpha and beta which are positive and we have relation between M and N like this that alpha M x is less than or equal to N x less than or equal to beta M x for every x in R n. Now here, let us take that this M x is an infinity norm of x defined on R n. So, and if we take this, then we take a set S, a subset of R n, which consists of all those elements in R n, whose infinite norm is equal to 1 here. So, if we take in place of all x coming from this S then, on this S your one reduced to what here M of x is basically what? M of x is infinity norm of x and infinity norm of x on S is basically 1. So, this is one here and similarly this is one here. So, this implies that one reduces set alpha less than or equal to N x is less than or equal to beta for every x belongs to S. So, basically this statement and this statement both are equivalent to each other. So, the only difference is that here x is coming from R n and here x is coming from S.

So now, we already know that this S is closed and bounded. Bounded in the sense that every element here and its norm is equal to 1. So, it is bounded in that sense and it is closed because, if you take this is nothing, but inverse image of singleton set, one which is which proves that it is a closed set. So, S is closed and bounded subset of R n and we know the property, that closed and bounded subset of R n is compact. So, it means that S is a compact set. Now, S is compact and this norm N x is a continuous function on S. So, this implies that every continuous function, makes a minimum a compact set. So, this implies that N has maximum and minimum value in S. So, this implies what? That there exists two values x 1 and x 2 in S, as that N of x is greater than or equal to N of x 1, which is the minimum value of N of x and we call this N x 1 as alpha and less than or equal to N x 2.Here, N x 2 is the maximum value for this. N of x and x 2 is the point of S where this achieves the maximum value. So, it means that N x is bounded between N x 1 and x 2 and we call this N x 1 as alpha and N x 2 as beta here.

Now, we already know that this x 1 and x 2 are coming from S. So, it means that infinity norm of x 1 and infinity norm of x 2 both are 1 here. So, it means that x 1 and x 2 both are non zero. So, it means that the value N x 1 and N x 2 cannot be equal to 0. So, it means that here, we are able to find out two constant alpha and beta which are non zero and satisfy this property, that is, alpha less than or equal to N of x less than or equal to beta means, one dash is true. Now here, we can write this is true for every element of S here. Now, we can write it in general for any value of x. So, we can write down that one is also true. So, it means that alpha times infinity norm of x is less than or equal to N of x plus less than or equal to beta times infinity norm of x here, and this is true for every x in R n. So, this what we have proved here, that N x is equivalent to infinity norm, that is what we have proved here. Now, if we can prove that any norm is equivalent to infinity norm then, if we take any two other norm, then using this relation 1, you can say that, any two norm on R n are equivalent to each other.

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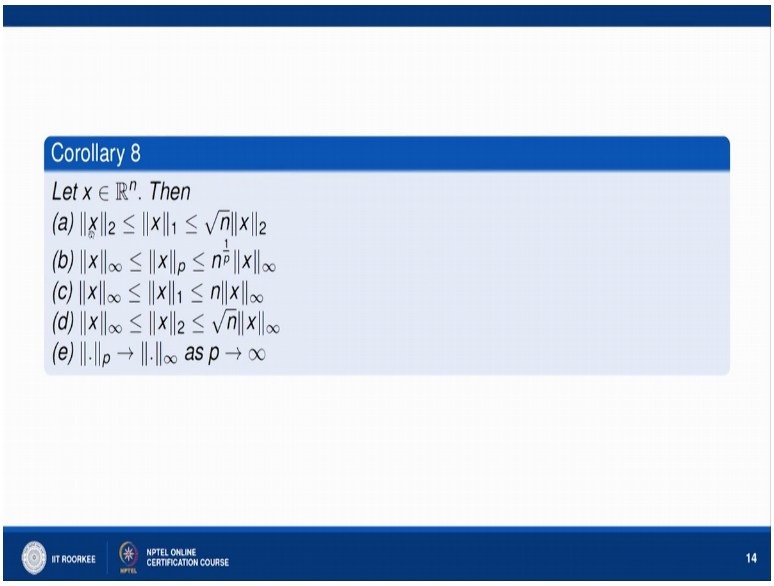

So, now let us take the following corollary which says that, your 1 and 2 norms are equivalent to each other, infinity and p norm are equivalent to each other, and infinity p norm is going to be infinity norm as p tending to infinity. So, this shows the values of alpha and beta here, in first example is, 1 and beta is root of n. So, let us try to prove the following corollary. So, here we want to prove that, two norms of x and 1 norm of x are equivalent here. So, it means that we want to prove the first property.

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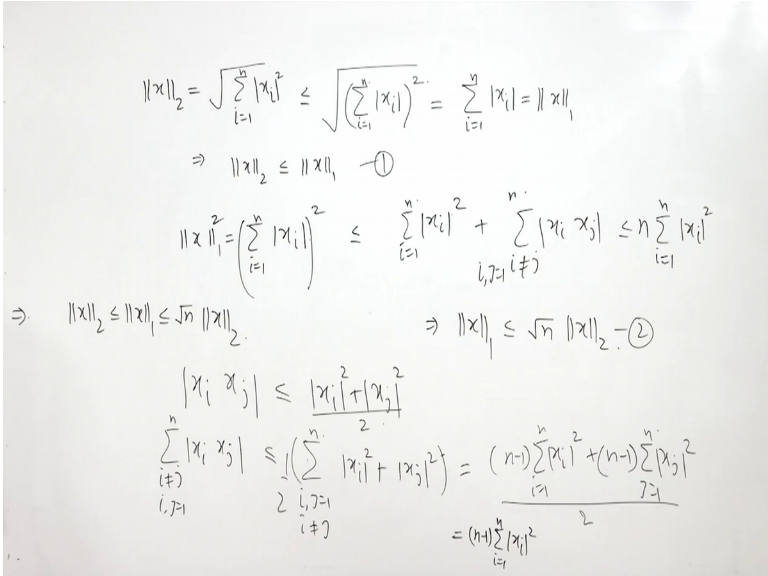

So, let us start with the 2 norms of x. 2 norms of x are given by under root i equal to 1 to n modulus of x 1 whole square. Now, this quantity is less than or equal to under root i, equal to 1 to n modulus of x 1 whole square. So, here it is summation of x 1 square. It is less than or equal to summation of x i whole square. So, this value is bigger than this, now, this can be simplified as summation i equal to 1 to n modulus of x i which is nothing, but 1 norm of x here. So, what we have proved here, that two norms of x are less than or equal to 1 norm of x. Now, we want to prove that norm of x 1 is less than or equal to some constant multiples of 2 norms of x.

So for, that you, just find out one norm of x. So, 1 norm of x, this is a small mistake here, 1 norm of x will be what? 1 norm of x is defined as, summation i equal to 1 to n modulus of x 1. Now, here I am taking the square of 1 norm of x. So, square of this. Now, if you simplify this, then it is written as i equal to 1 to n modulus of x i square, plus here we have, double summation I not equal to j and i j is running from 1 to n modulus of x i and x j. Now, our claim is that, this quantity is less than or equal to n minus 1 times summation i, equal to 1 to n, 1 loss of x i whole square. So, just simplify this quantity here. So, for that, we already know that modulus of x i, x j is less than or equal to modulus of x i square, plus modulus of x j square divided by 2, it is nothing but geometric mean is less than or equal to arithmetic mean.

So, here we are taking the geometric mean of x i square modulus of x i square and modulus of x j square. So, here it is simply g m less than or equal to m. Now, apply the double summation here. So, double summation means i i n j from 1 to n here, i not equal to j modulus of x i x j, is less than or equal to 1 by 2 i j is running from 1 to n i, not equal to j summation i m modulus of x i square, plus modulus of x j whole square. Now, if you look at here, the first component is free from j and right. So, this can be written as n minus 1 times summation i equal to 1 to n modulus of x i square. Why n minus 1 because here, j is running from 1 to n, but j cannot take the value i. So, it means that the possible value here, is less than and it is equal to n minus 1 only. So, j cannot take the value i, rest it can take all the values.

So, it means that here, we have n minus 1 times summation, i equal to 1 to n x i square. So here, we have n minus 1 times i, equal to 1 to n modulus of x i whole square. Similarly, if you look at the second component, this component is free from i. So, if you take the summation, it is again n minus 1 times summation j equal to 1 to n modulus of x j whole square. Now here, if you look at these two terms are different only by say, different indices here. So, if you change the indexes j by r by i then, it is nothing, but the whole thing can be written as n minus 1 times i, equal to 1 to n modulus of x i whole square. So, here we have taken the value. Here, this value is bounded above by n minus 1 times i equal to 1 to n modulus of x 1 square. So, using this value here; so, this value is less than or equal to n minus 1 times I, equal to 1 to n x 1 modulus of x i square. So, using this bond, we can write that this is less than or equal to n times summation i equal to 1 to n modulus of x i square.

Now, if you take the square root on both the sides, we can say that, one norm of x is less than or equal to under root 2 norm of x here. So, if we combine this equation 1 and equation 2, then, we can say that, here norm of x of 2 is less than or equal to norm of 1 norm of x, is less than or equal to root n times 2 norm of x. Here, which proves the first part of corollary. Now, proving for next, let us write here to show that infinity norm and p norm are equivalent.

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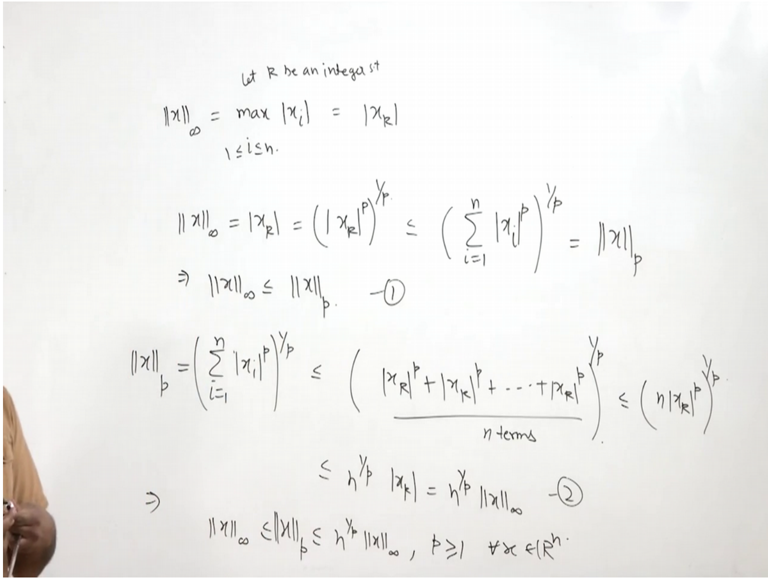

We need to find out alpha and beta such that, we need to prove this relation that no infinity norm of x is less than or equal to p norm of x, is less than or equal to n to power 1 by p infinite norm of x. To prove this, let us say that let k be an integer, say that infinity norm of x which is defined at a maximum of modulus of x i i is running from 1 to n is achieved for I equal to K. So, it means that infinity norm of x, is given by modulus of x k, basically it is maximum values of x i.

So, these are finitely many value x 1 to x n. So, we can say that which 1 is maximum let us say that in k x, k'th value is the maximum. So, we can say that infinity norm of x is given by modulus of x k. So now, modulus of x infinity norm of x is given by modulus of x k.Now, this can be written as modulus of x k to power p, power 1 by p and this, we can write that,this quantity is less than or equal to summation i equal to 1 to n modulus of x i to power p whole power 1 by p. So, in place of modulus of x t to power p, we also add certain other values at its x i, to power p i not equal to k. So here, we say that, this is certainly less than or equal to this quantity and if you look at this quantity is nothing, but p norm of x. So, if you use this, then it is the first inequality. Here, that infinity norm of x is less than or equal to p norm of x.

So, this proves the left hand side of this. Now, to prove this, let us start with p norm of x. p norm of x is defined by i equal to 1 to n modulus of x i to power p, power 1 by p. Now here, we already know that the infinity norm of x is, given by x k, It means that x k is maximum of all these x i. So, let us write down the maximum value for each x i. So, it means that x x 1 is less than s k and x 2 is less than x k and so on. So, it means that, here I can write that, modulus of x i to power p x 1 to power p is less than modulus of x k to power p modulus of x 2 to power p is less than or equal to modulus of x k x k to power p. So, for each, we can write down this thing, and if you look at how many terms we have, we have n terms.

So, it means that we can write down, that this is nothing, but n times modulus of x k to power p power 1 by p. So, this n can be taken out. So, here we can say that this is nothing, but n to power 1 by p modulus of x k and modulus of x k is nothing, but infinity norm of x here. So, this is what n to power 1 by p infinity norm of x. So, if we combine these two and one, we can say that infinity norm of x is less than or equal to p norm of x, is less than or equal to n to power 1 by p infinity norm of x here. So, this simply shows that infinity norms and p norms are equivalent to each other.

Now here, this is true for any p, which is greater than or equal to 1 right. So here, this is true for every p greater than or equal to n and for every x in R n. So, this proves the property b here.

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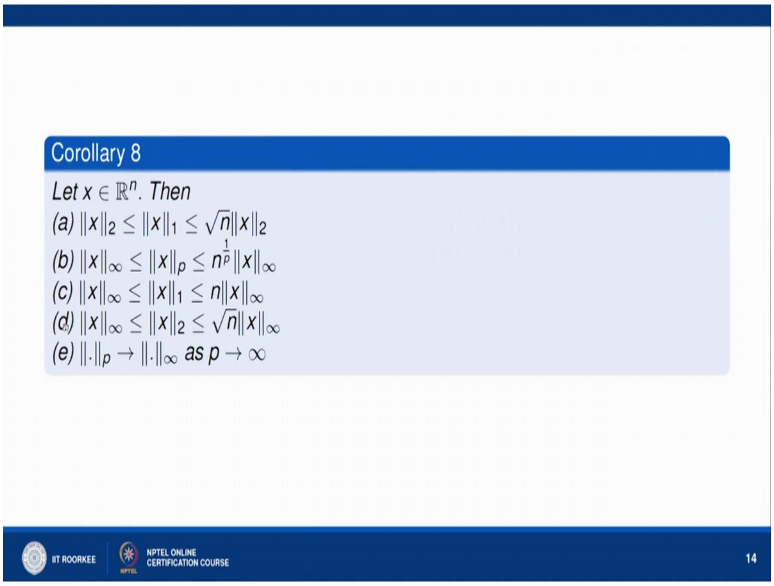

Now, if you look at this c and d are nothing, but the particular case of b here, in c, we take p equal to 1 and in d we take p, equal to 2 and this c and d follows from the part b here. Now, look at the part e. Here, which says that p norm of x is tending to infinity norm of x, as p tending to infinity that, also follows from this b part that here, if we take limit p tending to infinity, then this is independent of p. So, this will keep as it is. So, here limit p tending to infinity norm of p, norm of x is less than or equal to here, limit p tending to infinity n to power one by p and to power one by p will tend to one as p tending to infinity.

So here, by sandwich theorem, you can say that, limit p tending to infinity norm, norm of x is nothing, but infinity norm of x. So, we will also follow from b, and use of sandwich lemma here. So, what we have seen in today's lecture is how to define vector norm and some properties of vector norms and also we have shown that that on R n, every vector norm are equivalent to each other. So, and we have seen in a last corollary that, how this one and two norms are equivalent and how p th norm is equivalent to infinity norm. So, we will stop here and in next class, I will discuss some more properties of vector norm and matrix norm. Thank you for listening us.

Thank you.