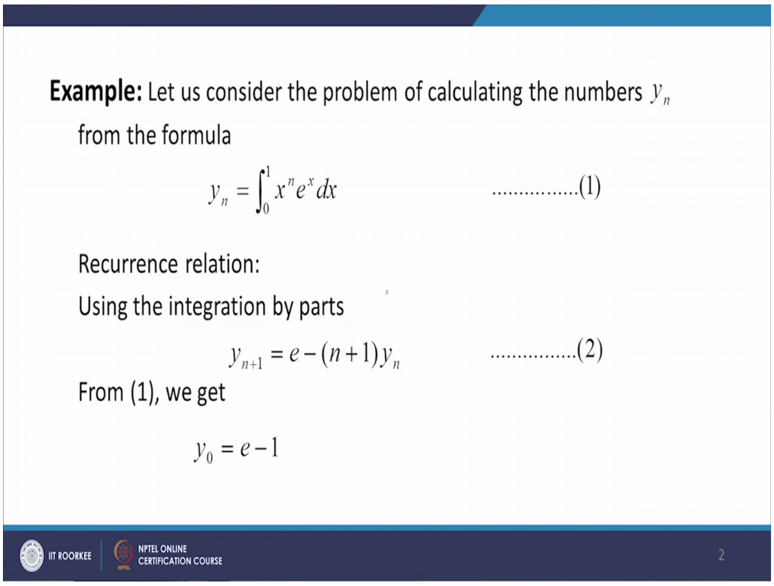
Numerical Linear Algebra
Dr. P. N. Agarwal
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 28
Stability of Numerical Algorithms- II

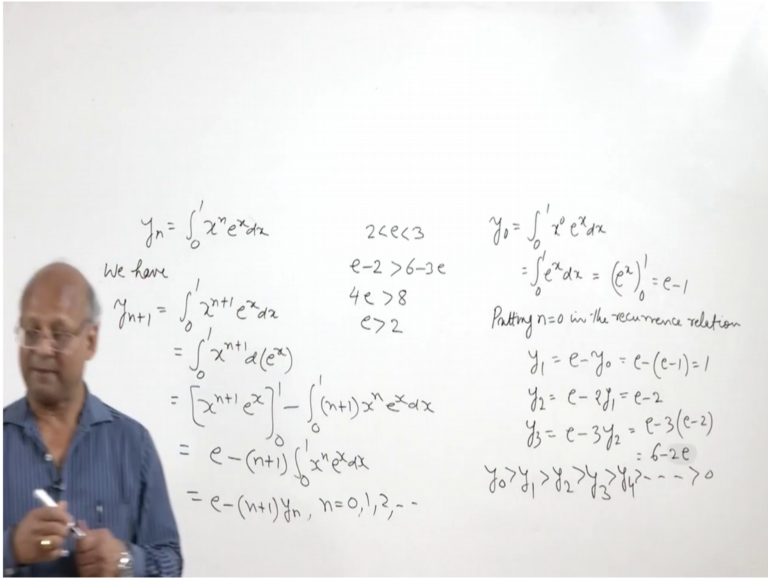
Hello friends I welcome you to my second lecture on stability of numerical algorithms. So, let us study some more examples to make the topic of stability of numerical algorithms clear.

(Refer Slide Time: 00:34)



Let us consider the problem of calculating the numbers y n from the formula y n equal to integral 0 to 1 x to the power n into e to the power x d x.

(Refer Slide Time: 00:43)



First, we shall derive a recurrence relation from this formula, let us integrated by parts. So, we have y n plus 1 y n plus 1 equal to integral 0 to 1 x to the power n plus 1 into e to the power x d x, we can write it as integral 0 to 1 x to the power n plus 1 differential of e to the power x. Now, when we integrate y parts, what we get? X to the power n plus 1 into integral of this d e x which is e to the power x and then we have to put the limits 0 1 minus integral 0 to 1 and we differentiate x to the power n plus 1 with respect to x.

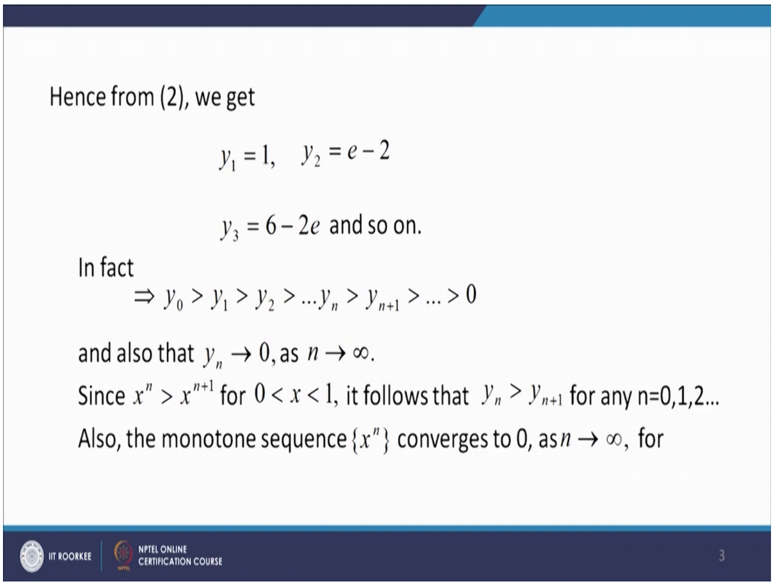
So, n plus 1 times x to the power n into e to the power x d x, now when x is equal to 1, we get this value as e and when x is equal to 0 e to the power 0 is 1, but x to the power n plus 1 becomes 0 because n x values 0, 1, 2, 3 and so on. So, n plus 1 is greater than or equal to 1. So, we get here e minus n plus 1 times integral 0 to 1 x to the power n e to the power x d x, we can write it as e minus n plus 1 into y n where n'th x values 1, 2, 3 and so on.

Now, we will need y naught. Y naught can be calculated from the given equation. So, we can put here n equal to 0, 1, 2, 3 and so on. Now we will in order to calculate y 1, we need the value of y naught. Y naught will be calculated directly from the given formula. So, y naught from the given formula is integral 0 to 1 x to the power 0 into e to the power x d x which is equal to integral 0 to 1 e to the power x d x and you know that integral of e to power x is e to the power x. So, e to the power x 0 1 which is equal to e minus 1; so, y naught is equal to e minus 1. Now let us calculate y 1 from this recurrence relation..

So, put n equal to 0 putting n equal to 0 in the recurrence relation we get y 1 equal to e minus y naught and this is equal to e minus e minus 1 which is equal to 1. So, y 1 is equal to 1, again, we can calculate use the recurrence relation to calculate y 2, y 2 equal to putting n equal to 1 y 2 equal to e minus 2 times y 1 e minus 2 times y 1 means e minus 2 because y 1 is equal to 1 and y 3. Similarly, we can calculate y 3 equal to e minus n equal to 3 we are putting. So, 3 times y 2 and this is e minus 3 times e minus two. So, this is 6 minus 2 into e and we know that the value of e is 2.7 something. So, y naught is greater than 0.

And then y 1 is 1. So, y naught is greater than y 1 and y 2 is e minus 2.

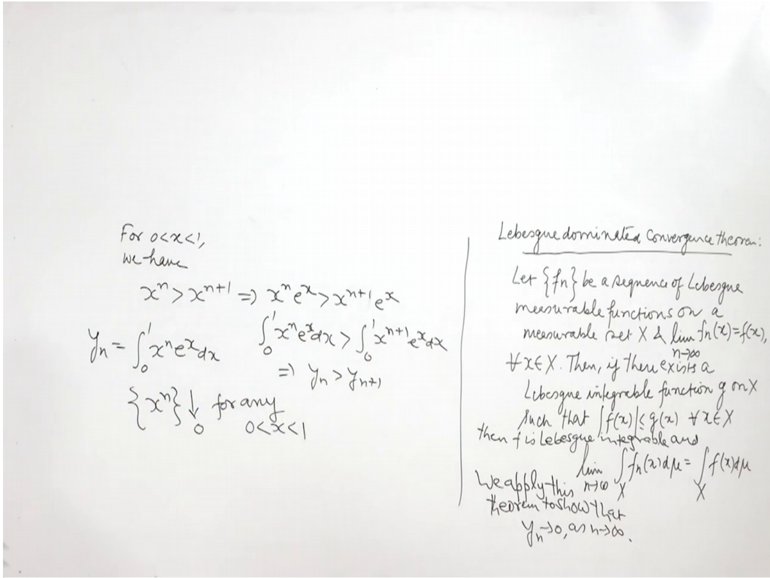
(Refer Slide Time: 05:10)



So, y 1 is greater than y 2 and then you can also see that y 2 is greater than y 3 because see y naught y naught is greater than y 1 y naught is e minus 1 and e minus 1 is greater than 1 because e is greater than 2. So, then y 1 is greater than y 2 y 1 is greater than y 2 because 1 is greater than e minus 2 or you can say 3 is greater than e 2 is less than e less than 3, we are making use of this. So, y 1 is greater than y 2 and y 2 is greater than y 3 and similarly, we can get y 4 greater than like this and greater than 0.

So, the y n is a monotonically decreasing sequence positive real numbers and y 1 goes to 0 as n goes to infinity, how, let us see, how it follows we have seen that y naught is greater than y 1, y 1 is greater than y 2, y 2 is greater than y 3, but this will not lead us to the general inequality y n greater than y n plus 1. So, to see that y n is greater than y n plus 1, let us note that x n is greater than x n plus 1 whenever 0 is less than x less than 1 ok.

(Refer Slide Time: 06:34)

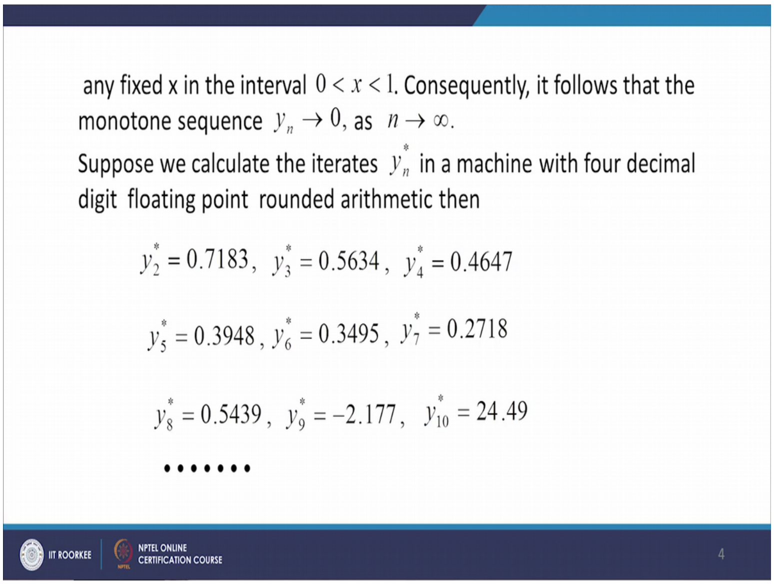


So, for 0 less than x less than 1, we know that we have x to the power n is greater than x to the power n plus 1 and so y n; y n is equal to integral 0 to 1 x to the power n e to the power x d x.

So, when you multiply by e to the power x because e to the power x is always positive, what we will get x to the power n into e to the power x is greater than x to the power n plus 1 into e to the power x when we integrate it over the interval 0 1, we arrive at this which is y n greater than y n plus 1. So, from the fact that x n is x to the power n is greater than x to the power n plus 1 we arrive at the inequality y n greater than y n plus 1 for any n equal to 0, 1, 2, 3 and so on.

Now, we also notice that the sequence x to the power n the sequence x to the power n is a monotonically decreasing sequence of positive real numbers for any x which lies in the interval 0 1 open interval 0 1. So, this goes to 0 and for any x which lies in the open interval 0 1.

(Refer Slide Time: 08:18)

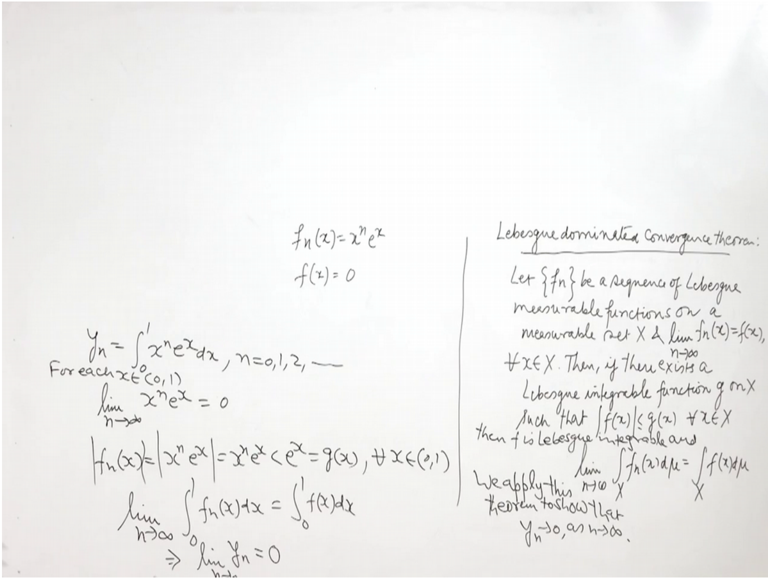


Consequently, it follows that the monotone sequence y n goes to 0 as n goes to let us see how it follows here we will need the Lebesgue; Lebesgue dominated convergence theorem.

Let us see, what does it say this theorem let f n be a sequence of Lebesgue measurable function un-measurable set see x we can take complex valued Lebesgue measurable functions here. So, un-measurable set x and limit n tends to infinity f n x is equal to f x for every x belonging to x then if there exist a Lebesgue integration function g on x such that mod of f x is less than or equal to g x for all belonging to x, then limit n tends to infinity e integral over x f n x d mu is equal to integral over x f x d mu where mu is the major on x. Now then here, we have one more thing, then f belongs to l 1 mu that is f is Lebesgue integrable then f is Lebesgue integrable limit n tends to infinity integral over x f n x d mu is equal to integral over x f x d mu.

Now, let us see, how we apply this theorem here we will apply this theorem to show that y n goes to 0 as n goes to infinity. Now, let us see, what is y n y n is the given to the integral 0 to 1 x to the power n into e to the power x d x.

(Refer Slide Time: 12:08)



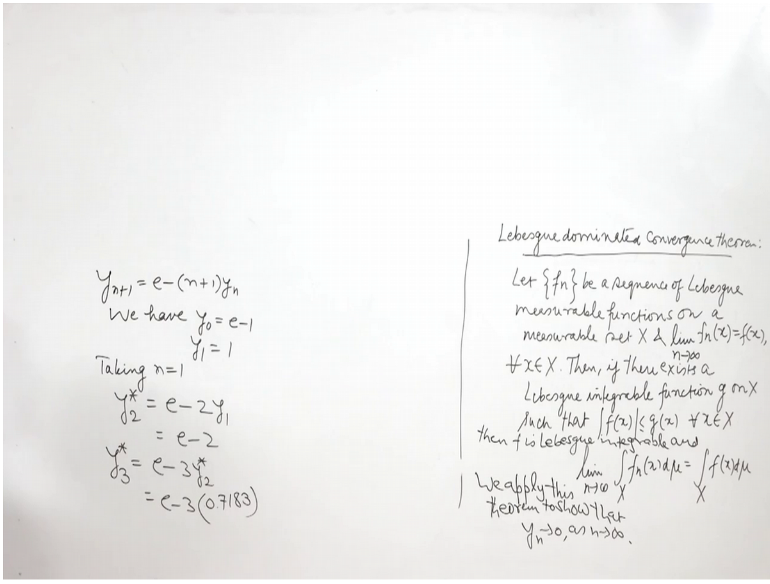
Since x to the power n into e to the power x for n equal to 0, 1, 2, 3 and so on are continuous functions over the interval 0 1. So, they are Lebesgue measurable on the interval 0 1; the interval 0 1 is measurable set and limit n tends to infinity x to the power n into e to the power x goes to 0 for every x belonging to for every 6 for each x belonging to the open interval 0 1 for each x belonging to the open interval 0 1 limit n tends to infinity x to the power n into e to the power x is equal to 0.

So, we have here f n, we have as sequence as x to the power n into e to the power x x is the interval 1, interval 0 1 limit n tends to infinity f n x is equal to f x. So, f x is equal to 0 here. So, f n x we are taking as x to the power n into e to the power x the limiting function of x which is f x is the 0 function now there must exist Lebesgue integrable function g on x such that mod of f x is less than or equal to g x, let us see, what is that function here. So, f n x is equal to x to the power n into e to the power x. So, mod of f n x is equal to mod of x to the power n into e to the power x and since x belongs to the open interval 0 1, this is all this function and this function both are non negative functions. So, we do not have to put the modulus, we can simply write x to the power n into e to the power x.

Now, since x belongs to open interval 0 1 this is less than e to the power x and. So, you can take e to the power x as equal to g x. So, g x is e to the power x in each power x is a Riemann integral function, what is Lebesgue integral and we also see that and therefore, we can we can say that the function g x is Lebesgue integrable mod of f n x is less than or equal to g to the power g x for all x belonging to 0 1 and therefore, the limiting function f x is Lebesgue integrable f x equal to 0 Lebesgue integrable and limit n tends to infinity by Lebesgue dominated convergence theorem limit n tends to infinity integral over x 0 1 f n x d x d mu will become d x here we are on the real line this is equal to integral 0 to 1 f x d x. Now this is what are limit n tends to infinity.

Now, f n x is f to the power n into e to power x. So, we have limit n tends to infinity y n and f x is equal to 0 integral over 0 to 1 of 0 function is 0 we get y n limit of y n as n goes to infinity equal to 0. So, y n goes to 0 the iterates y n which the values of i n y n which we have calculated from the given recurrence relation they are go to 0 as n goes to infinity now suppose we calculate the iterates y n star in a digital computer with 4 decimal digit floating point rounded arithmetic; that means, we take beta equal to n and t equal to 4, then we will get the value of y 2 star as 0.7183 if you calculate.

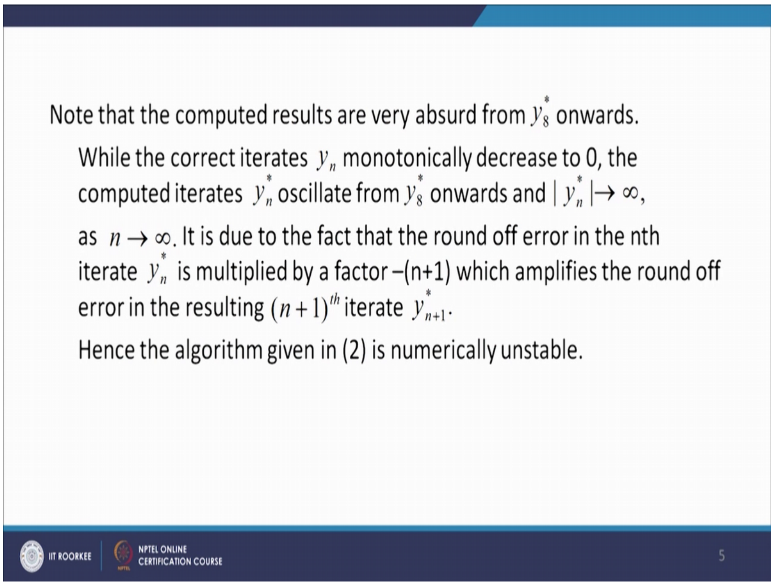
(Refer Slide Time: 16:31)



So, the values of y 2 star y 3 star can be calculated from the recurrence relation y n plus 1 is equal to e minus n plus 1 into y n. So, when we want to calculate we are given y naught equal to e minus 1 and y 1 equal to 1 we have. So, when we calculate y 2 I have take n equal to 1 taking n equal to 1 we get y 2 y 2 we will denote y 2 star here. So, e minus n equals to 1. So, 2 times y 1 and y 1 is equal to 1. So, e minus 2 we get and e minus 2 is equal to 0.7183 when we use 4 decimal digit floating point rounded arithmetic. So, y 2 star is 0.7183.

Now, using the value of y 2 star we can calculate y 3 star here. So, y 3 star is equal to e minus 3 times y 2 star. So, you can put e minus 3 times 0.7183, then we will get y 3 star equal to 0.5634 similarly we can calculate y 4 star, y 5 star, y 6 star, y 7 star, y 8 star. Now after y 8 star, what turns out is that y 9 star becomes negative it is minus 2.177 and y 10 star becomes 24.9. So, the sine starts changing the values of y n star starts fluctuating from here and they grow uncontrollably.

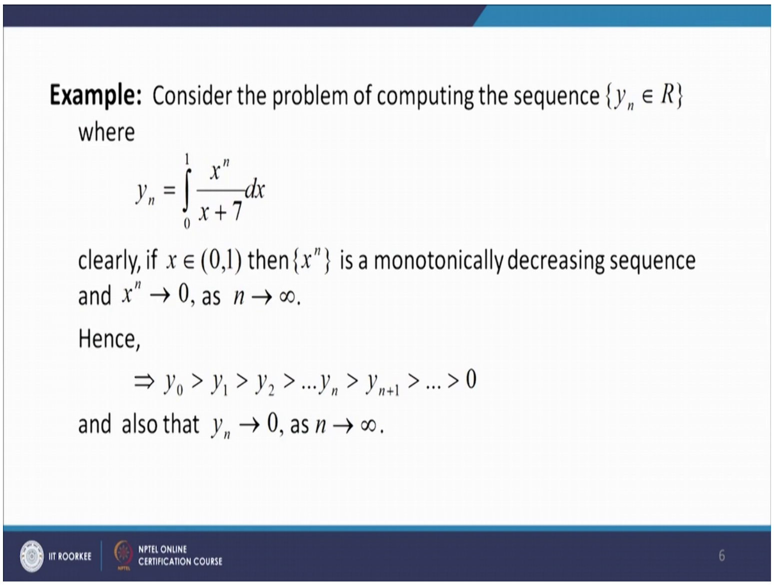
(Refer Slide Time: 18:23)



So, now be the computer results we can see are very absurd from y 8 star onwards y 8 star is 0.5439, then y 9 star becomes minus 2.17. So, 1 7 7. So, the sine changes and then the next iterate is 24.49. So, they become absurd after the starting from y 8 star onwards and the correct iterates y n, we have seen the correct iterates y n go to 0 as n goes to infinity, the computed iterates y n star oscillates from y 8 star onwards and mod of y n star goes to infinity they grow uncontrollably as n tends to infinity.

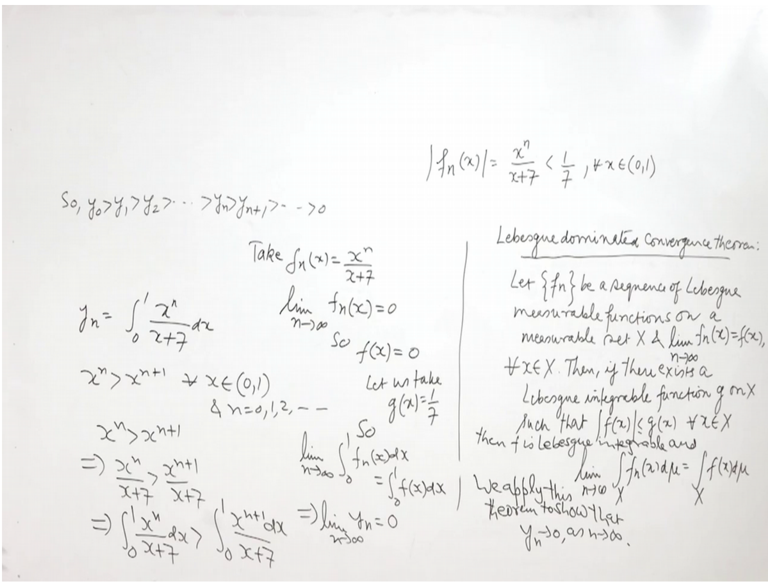
Now this is because of the fact that the round of error in the n'th iterate y n star is multiplied see this is y n plus 1 equal to e minus n plus 1 y n whatever round off error is there in the calculation of y n star here that is get that gets multiplied by minus n plus 1 which amplifies the round of error in the resulting n plus oneth iterate y n plus 1 star and therefore, we can say that this algorithm y n plus 1 equal to e minus n plus 1 into y n is numerically unstable.

(Refer Slide Time: 19:59)



Let us consider one more example where we have another sequence given by y n equal to integral 0 to 1 x to the power n divided by x plus 7 d x now here also id x belongs to open interval 0 1, then x to the power n is a monotonically decreasing sequence and x to the power n goes to 0 as n goes to infinity. So, we can see that y n is equal to integral 0 to 1 x to the power n divided by x plus 7 d x.

(Refer Slide Time: 20:35)



So, x to the power n is greater than x to the power n plus 1 for all x belonging to open interval 0 1 and n equal to 0, 1, 2, 3 and so on. So, what do we notice that y naught y n y n is greater than x to the power n is greater than x to the power n plus 1.

So, this implies that x to the power n over x plus 7 is greater than x to the power n plus 1 over x to the power x plus 7 because when x belongs to open interval 0 1 x plus 1 7 is positive. So, the inequality will not change and this implies integral 0 to 1 x to the n upon x plus 7 d x is greater than integral x to the power n plus 1 over x plus 7 and or we can say y n is greater than y n plus 1. So, y naught; so, we can say; so, we see that y naught is greater than y 1 y 1 is greater than y 2 and so on. So, y n is a sequence of positive real numbers and it is monotonically decreasing.

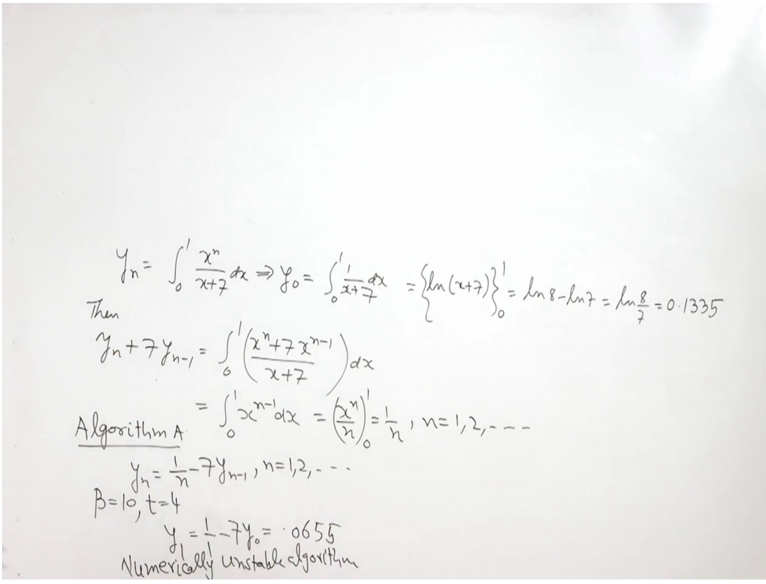
Now, here also we notice that y n goes to 0 as n goes to infinity. Now let us see how y n goes to 0 as n goes to infinity. So, here again we will use the Lebesgue dominated convergence theorem f n x, you take as take f n x equal to x to the power n over x plus 7 then f n x are continuous functions over the open interval 0 1 open interval open interval 0 1 is measurable set and the continuous functions are measurable. So, f n x are f n is a sequence of Lebesgue measurable functions on the measurable side 0 1 and limit n tends to infinity f n x is equal to 0 for each x belonging to open interval 0 1 because x to the power n goes to 0 as n goes to infinity. So, f x is 0 function here f x is the 0 function and there are just we there should adjust a Lebesgue integrable function g on x such that mod of f x is less than or equal to g x.

So, we can see that mod of f n x n is equal to x to the power n divided by x plus 7 because x to the power n over x plus 7 is a non negative functions. So, we can we do not have to put mod now this is less than one by 7 for all x belonging to open interval 0 one. So, we can take g x to be one by seven. So, let us take g x equal to 1 by 7 which is a constant function constant function is continuous function. So, it is Riemann integrable and therefore, it is Lebesgue integrable.

So, g x is Lebesgue integrable function thus we have found a Lebesgue integrable function g x such that mod of f x is less than or equal to g x and therefore, the function f is integrable that is this 0 function is Lebesgue integrable and the limit of f n x as n goes to infinity over the interval 0 1 is equal to integral 0 to 1 f x g x. So, limit n tends to infinity integral 0 to 1 f n x d x is equal to integral 0 to 1 f x d x now 0 to 1 f n x d x is y n f n x is x to the power n over x plus seven. So, we get limit n tends to infinity y n equal to f x is a 0 function. So, we get 0. So, y n goes to 0 as n goes to infinity using Lebesgue dominated convergence theorem.

Now, we shall discuss 2 algorithms to evaluate the value of y n for n equal to 0, 1, 2, 3 and. So, on we shall show that one algorithm is a numerically unstable while the other algorithm is numerically stable. So, what we will do is let us see we can write the given equation like this, we have y n equal to this is given to us. Now from this equation, we see that we can write.

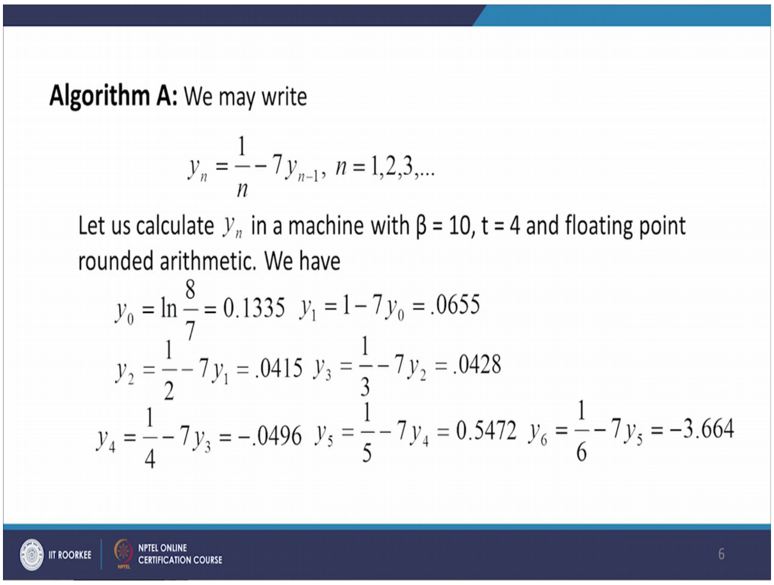
(Refer Slide Time: 26:13)



When we add y and n 7 times y n minus 1, we get integral 0 to 1 x to the power n plus 7 times x to the power n minus 1 over x plus 7 d x and then we can cancel x plus 7 here because x plus 7 is not equal to 0 x belongs to the open interval 0 1. So, x plus 7 is not 0. So, we have integral 0 to 1 x to the power n minus 1 d x and this is equal to x to the power n divided by n. So, we get y n plus 7 times y n minus 1 equal to 1 by n and here n takes values starting with 1 2 3 and so on because of y n minus 1.

Now from the, with the value of y to calculate the value of y 1 from here we need the value of y naught. So, y naught is calculated from the given equation this gives you y naught you put n equal to 0. So, 0 to 1 one over x plus 7 d x. So, this is equal to l n x plus 7. So, l n 8 minus l n 7 or l n 8 by 7, we do the value of y naught, this value of y naught is then use to determine y 1 y 2 y 3 and so on.

(Refer Slide Time: 28:29)

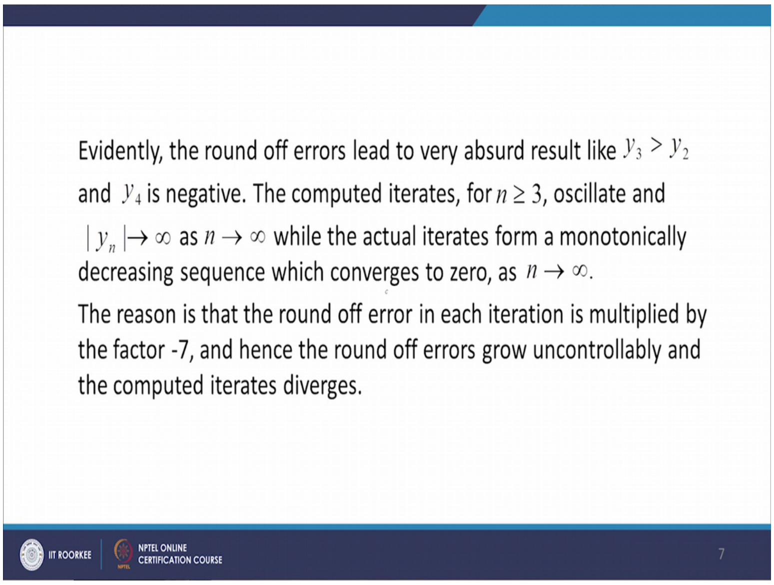


Let us look at algorithm A. So, in this algorithm what we do is we write y n equal to 1 by n minus 7 y n minus 1 where n'th x values 1 2 3 and so on. So, from y n plus 7 y n minus 1 equal to 1 y n we have y n equal to 1 by n minus 7 y n minus 1 and is equal to 1 2 3 and so on. Now let us take beta equal to 10 and t equal to 4 t equal to 4, then we know the value of y naught y naught is equal to l n 8 by; so, y 1 star. So, we will call it as y 1 star y 1 star is equal to 1 by 1 minus 7 times y naught and y naught which is l n 8 by 7 is 0.1335; 0.1335 when we use t equal to 4 the value of l n 8 by 7 is 0.1335. So, let us write y 1, let us write y 1 here, let us y 1 is equal to 1 minus 7 y naught and that comes out to be 0.0655..

So, then we calculate y 2 y 2 is equal to 1 by 2 minus 7 y 1, we put the value of y 1 as 0.0655 to calculate y 2 y 2 comes out to be 0.0415, then y 3 is 1 by 3 minus 7 y 2 which is 0.0428, y 4 comes out to be minus 0.0496, y 5 is 0.5472 and y 6 is minus 3.624. So, you see that starting with y 4, the value of the iterate. In fact, starting with y 3 we see that the value of the y n start fluctuating. So, this is not a numerically stable algorithm and one more, we think we see that y 2 is 0.0415 y 3 is 0.0428. So, y 2 is y 3 is greater than y 2.

While we have seen that the values of y n should decrease n decrease to 0 as n goes to infinity.

(Refer Slide Time: 31:14)

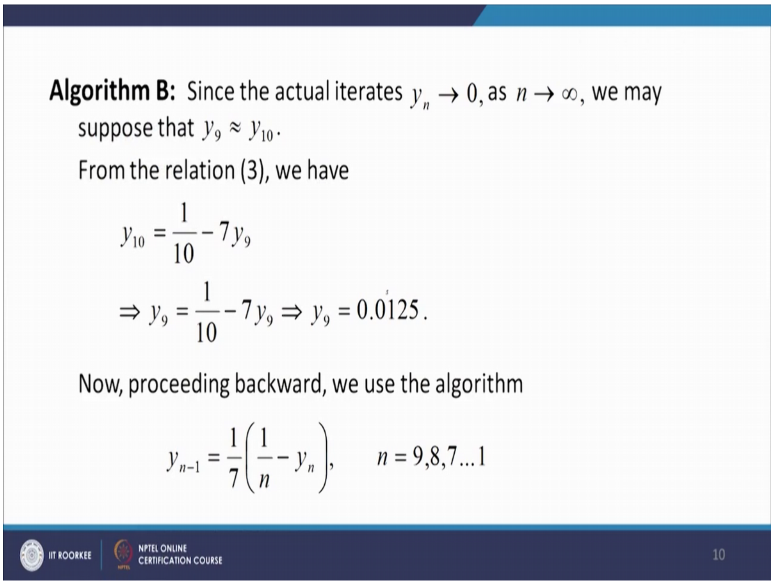


But here y 3 is greater than y 2 and starting with y 3 the values of y n start fluctuating the round off error lead to very absurd result here and the round off errors are magnified actually because of the multiplication of minus 7 2 y n minus one. So, whatever round off error is there in the computation of y n minus 1 that get gets multiplied by minus 7 and is therefore, amplified..

So, y 3 is greater than y 2 y 4 is negative the computed iterates for n greater than or equal to 3 is starts oscillating and the absolute value of y n tends to infinity as n goes to infinity you can see that the numerical the numerical value of the iterates is starts increasing the y 5 is 0.5472, y 6 is numerical value of y 6 is 3.64. So, they go to infinity as n goes to infinity while the actual iterates form a monotonically decreasing sequence which convergence to 0 as n goes to infinity. Now as I said the region is that the round off error in each iteration is multiplied by the factor minus 7 and therefore, the round off errors go grow uncontrollably and the computed iterates diverge. So, this algorithm is numerically unstable this is numerically unstable algorithm.

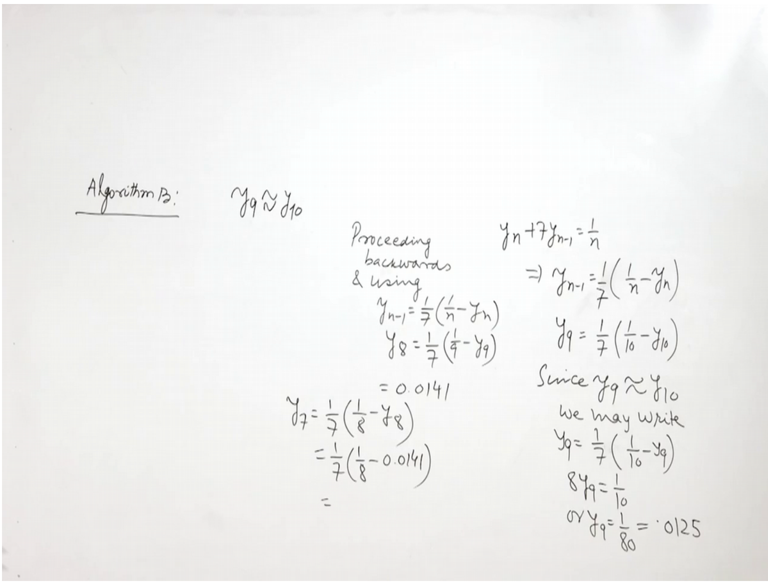
Now, let us see we will write the equation y n plus 7 y n minus 1 equal to 1 by n and we shall see we have a stable algorithm. So, how to get this stable algorithm? So, in the algorithm B, we know that the actual iterates y n go to 0 as n goes to infinity.

(Refer Slide Time: 33:31)



And therefore, we may suppose that y 9 and y 10 are approximately same. So, y 9 is approximately equal to y 10.

(Refer Slide Time: 33:41)



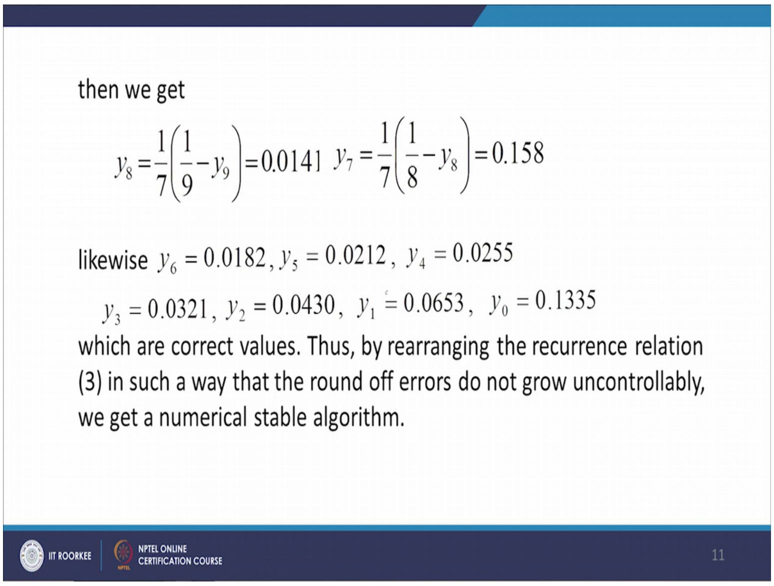
Now, from this relation y n plus 7 y n minus 1 equal to 1 by n from this, what do we notice? We notice that y 10 is equal to 1 by 10 minus we notice that this gives you y n minus 1 equal to 1 by n minus y n multiplied by 1 by 7.

So, y n minus 1 is equal to 1 by 7 into one by n minus y n. So, y 9 is equal to 1 by 7 times 1 by 10 minus y 10. Now what we will do is since y 10 is approximately same as y

9, we can write y 9 equal to. So, this gives you y 9 equal to 1 by 7 times 1 by 10 minus y 9 and so, this is how much 7 y 9 plus y 9. So, 8 y 9 equal to 1 by 10 or we can say y 9 equal to 1 by 80. So, when we assume that y 9 and y 10 are approximately same the value of y 9 comes out to be 0.0125.

Now, what we do is we proceed backwards and use the algorithm. So, proceeding backwards and using y n minus 1 equal to 1 by 7 into 1 by n minus y n, we can calculate y 8, y 7, y 6 and so on. So, y 8 if we want we put n equal to 9 here. So, y 8 equal to 1 by 7 times 1 by 9 minus y 9 and when we calculate y 8 from this equation. So, then y 8 comes out to be 0.0141. Now again y 7 is equal to 1 by 7, 1 by 8 minus y 8 putting n equal to 8. So, y 7 we can calculate. So, 1 by 7 times 1 by 8 minus 0.0141 and when we obtain the value of y 7, it comes out to be 0.158.

(Refer Slide Time: 37:40)



So, likewise we go and calculating the values y 6 is 0.0182, y 5; 0.0212, y 4; 0.0255, y 3 is 0.0321, y 2 is 0.0430, y 1 is 0.0653 and y naught is 0.1335 which are correct values thus by rearranging the recurrence relation 3 the recurrence relation 3 is y n plus 7 y n minus 1 equal to 1 by n in such way that the round off errors do not grow uncontrollably we get a numerically stable algorithms. So, this is a numerically stable algorithm. So, we see that one scheme or one algorithm is numerically unstable while the other algorithm is numerically stable. So, with this I would like to conclude my lecture.

Thank you very much for your attention.