Numerical Linear Algebra
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Lecture - 26
Conditioning and Condition Numbers- II

Hello friends. This is my second lecture on conditioning and condition number 2. Let us, in the previous lecture we discussed what is meant by a conditioning and the condition number and we did some examples also. Let us continue that discussion further and discuss some more examples. Let us say suppose we want to solve this system of linear equations alpha x plus y equal to 0 x plus alpha y equal to 1, alpha x plus y equal to 0 and x plus alpha y equal to 1. Now, we have to discuss the conditioning of computing x and y for this system.

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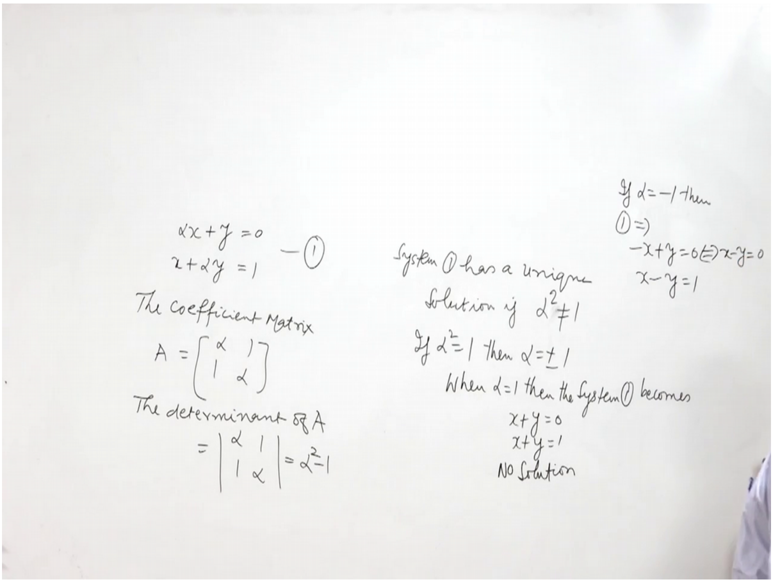

Now, we know that when we have a system of linear equations say n equations in n unknowns then the system has unique solution provided the determinant of the coefficient matrixes nonzero. So, here the coefficient matrix is alpha 1, 1 alpha. So, the determinant of the coefficient matrix is let us say this matrix is A, the determinant of A is equal to alpha 1, 1 alpha which is equal to alpha square minus 1. So, if the determinant is not equal to 0, then this system has a unique solution. So, when alpha square the system the system 1 has a unique solution unique solution if alpha square is not equal to 1.

And if alpha square is equal to 1 then let us see what happens? If alpha square is equal to 1 then there are 2 possibilities, either alpha is equal to 1 or alpha is not equal to alpha is equal to minus 1. So, then alpha is equal to plus minus 1. We can see here if alpha is equal to 1 then the given system becomes x plus y equal to 0 and x plus y equal to 1 we cannot find any 2 real numbers x and y such that x plus y is equal to 0 and simultaneously x plus y equal to 1.

So, there is no solution when alpha is equal to 1. Now, if alpha is equal to minus 1 then what happens? Alpha is equal to minus 1 then we have then 1 becomes minus x plus y equal to 0 and x minus y equal to 1. Now, you can see here if you multiply this equation by minus 1 this is same as x minus y equal to 0. So, we have the system 1 becomes x minus y equal to 0 and x minus y equal to 1. Again such number such real numbers are not possible, so no solution.

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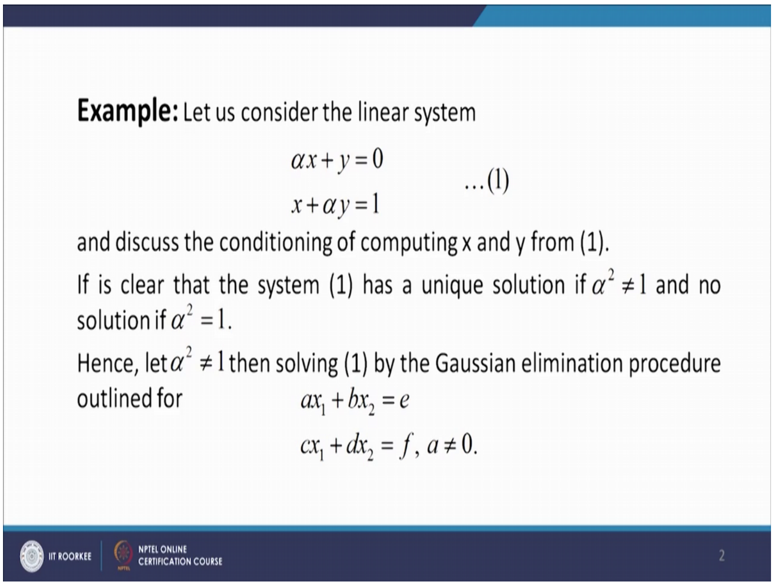

So, when alpha square is equal to 1 the system has no solution. So, let us discuss the situation when alpha square is not equal to 1.

Now, we have earlier discussed in the previous lectures we have discussed how to solve the system of equations ax 1 plus bx 2 equal to e, cx 1 plus dx 2 equal to f, when a is not equal to 0 by using the Gaussian elimination procedure there we are taken m equal to c by a.

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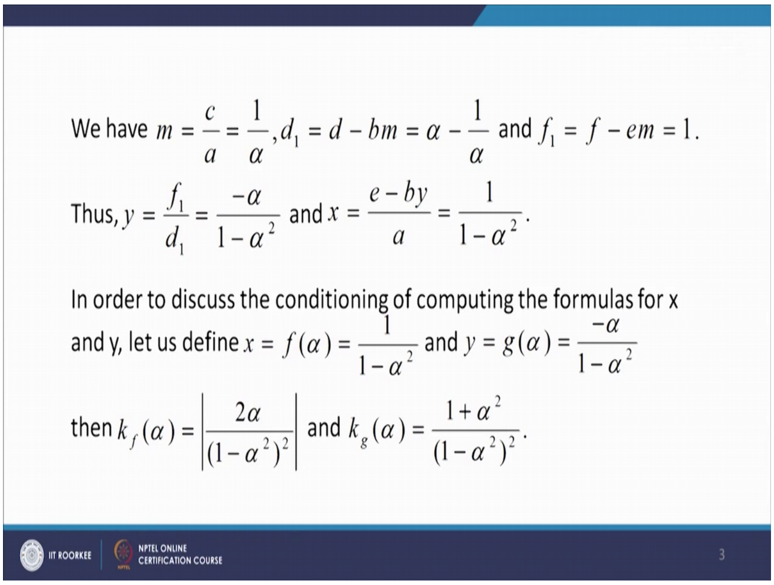

So, let us apply that Gaussian elimination procedure and assume that alpha square is not equal to 1. Then m is equal to c by a, so here c is equal to 1, a is equal to alpha. So, m is equal to 1 by alpha; d 1 equal to d minus d m substituting the value of m and d n b we get alpha minus 1 by alpha, that is the value of d 1. And y is then is equal to f 1 over d 1, so, and f 1 is equal to f minus m which is equal to 1. So, y is minus alpha over 1 minus alpha square and x is equal to e minus b y over a, which is 1 over 1 minus alpha square.

Now, in order to discuss the conditioning of computing the formulas for x and y let us put x equal to f alpha and y equal to g alpha. Then f alpha will be equal to 1 over 1 minus alpha square and g alpha will be minus alpha over 1 minus alpha square. So, let us then compute the condition, condition numbers for x and y.

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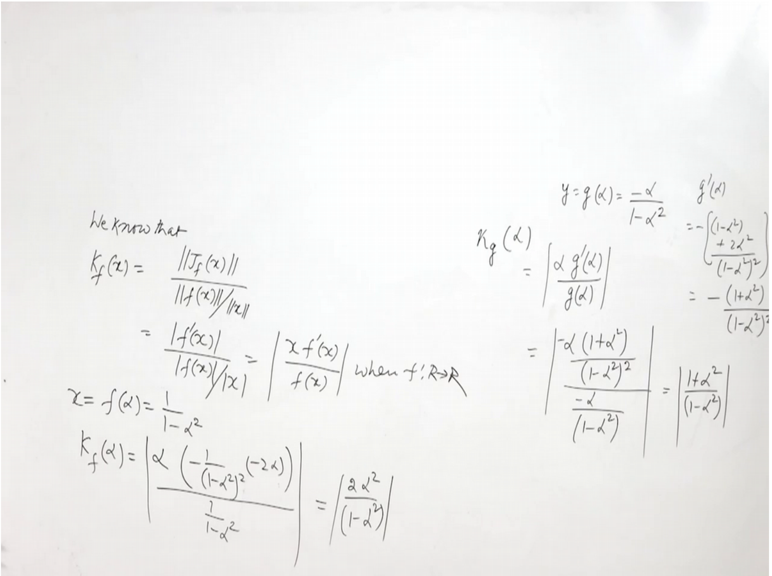

We know that k f x is equal to norm of j f x divided by norm of f x divided by norm of x, when f is a mapping from r into r then this becomes mod of f prime x and the denominator is mod of f x over mod of x or we can say mod of x f prime x divided by f x, when f is a function from r into r in the one dimensional case.

So, let us find. So, here what happens? The condition number we have putting x equal to f alpha and f alpha is 1 over 1 minus alpha square. So, conditioning number for, condition number for f will be k f alpha equal to derivative of f with respect to alpha, so 1 over 1 minus alpha square. So, this will be equal to this is f prime alpha alpha into f prime alpha f prime alpha will be minus 1 upon 1 minus alpha square whole square into minus 2 alpha, divided by f alpha that is 1 over 1 minus alpha square mod of this.

And what is this? This is equal to 2 alpha square divided by 1 minus alpha square; x into f prime x divided by f x; x is equal to alpha f alpha is equal to 1 over 1 minus alpha square. So, alpha times derivative of f alpha will be minus 1 upon 1 minus alpha square whole square into minus 2 alpha and then we divided by f alpha which is 1 over 1 minus alpha square now this is 2 alpha square divided by one 1 alpha square. So, we have 2 alpha square divided by 1 minus alpha square.

And then k g alpha g we have defined equal to g alpha and y was equal to minus alpha over 1 minus alpha square. So, this will be equal to again alpha into g prime alpha divided by g alpha mod of this. So, mod of alpha into g prime alpha will be we can differentiate g alpha. So, g prime alpha is equal to minus derivative of alpha is 1, so 1 minus alpha square and then we have here 2 alpha in, 2 alpha 2 alpha square this will be plus alpha because this is minus 2 alpha into alpha. So, plus 2 alpha square divided by 1 minus alpha square whole square. So, this is minus 1 minus alpha square divided by 1 plus alpha square divided by 1 minus alpha square whole square. So, we have alpha into minus alpha into 1 plus alpha square divided by 1 minus alpha square whole square divided by f g alpha. So, minus alpha over 1 minus alpha square; this equal to 1 plus alpha square divided by 1 minus alpha square. So, what we have here is that we have found the condition numbers for both f and g, now let us note the following.

So, f and g are well conditioned when mod of alpha is very very small then very very small, much smaller than 1 because then what will happen this will not become too large f alpha will k f alpha will not become too large and k g alpha will also not become too large if one minus alpha square is not very close to 0. I mean that means, mod of alpha is not very very small I mean very very less than 1 and ill -condition when mod of alpha is nearly equal to 1.

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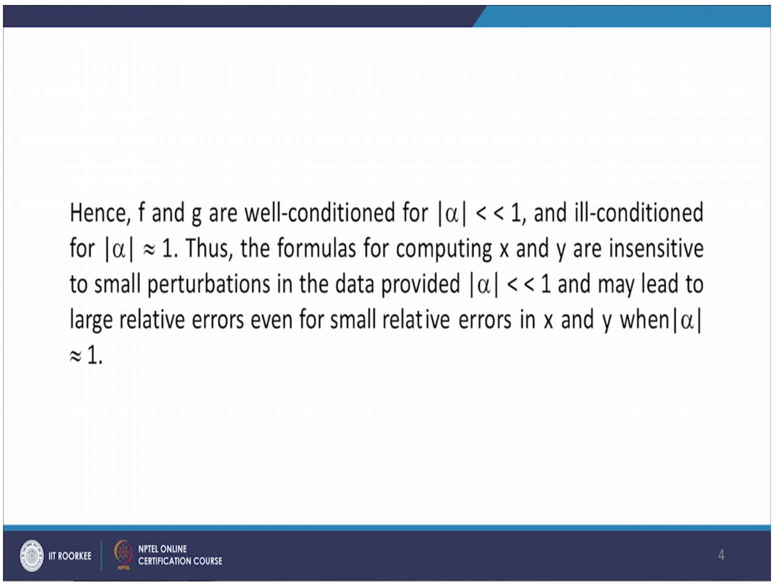

So, the formulas for computing x and y are insensitive to small perturbations in the data provided mod of alpha is very very less than 1 and may lead to large relative errors even for a small relative errors in x and y when mod of alpha is nearly equal to 1.

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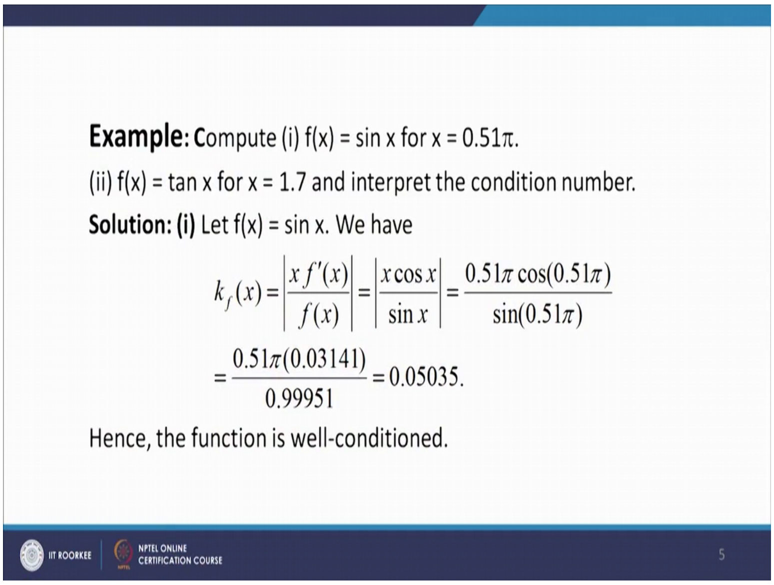

Now, let us compute f x equal to sin x for x equal to 051 pi and f x equal to tan x for x equal to 1.7 and interpret the condition number. So, when f x equal to sin x k f x is equal to mod of f x into f dash x divided by f x which is equal to x into cos x divided by sin x, when we put the value s of x cos x and sin x we get the value 0.05035 and which is not, which is very small. So, we can say that the function is well conditioned.

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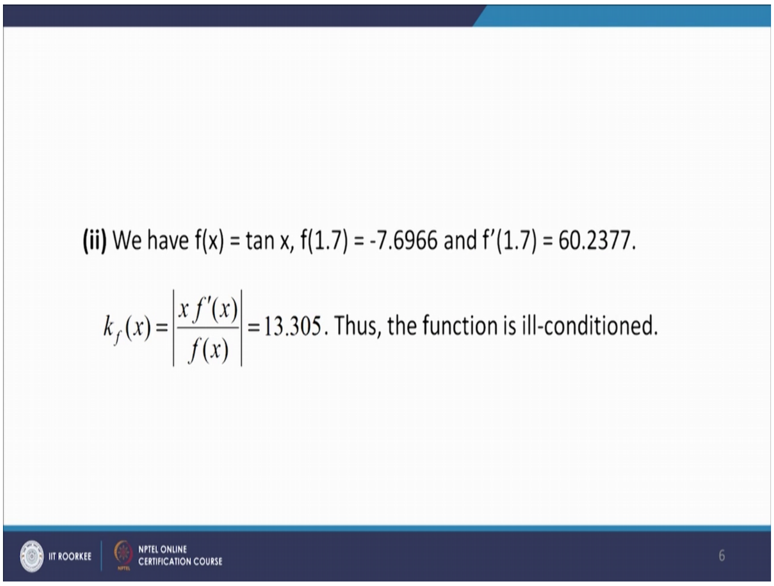

Then we have f x equal to tan x when x is 1.7, f 1.7 is minus 7.6966 and f dash at 1.7 is 60.2377. So, k f x is equal to mod of x into f prime x over f x gives us 13.305 which is too large, which means that the function is ill-conditioned.

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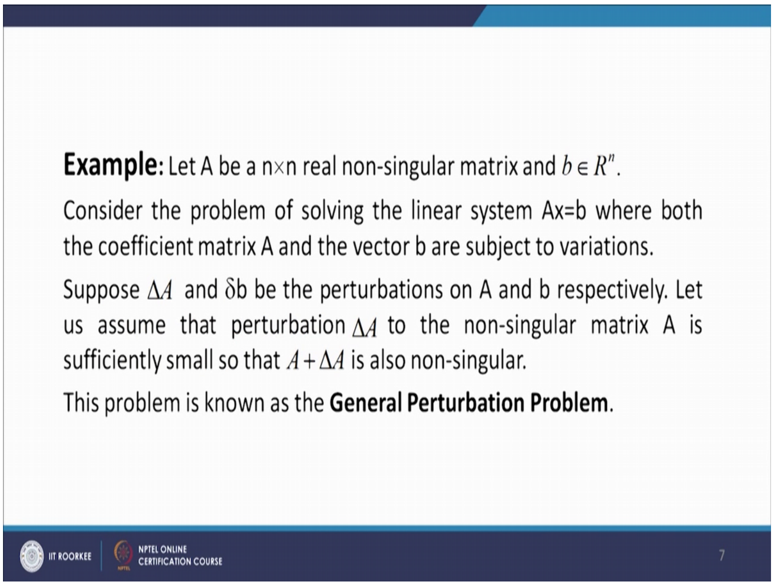

Now, let us consider n by n non-singular matrix, non-singular matrix means the matrix which is invertible or you can say the matrix whose determinant is nonzero and v be a vector belonging to R n. Then consider the problem of solving the linear system Ax equal to b, where both the matrix A and the vector b are subject to variations. Let us suppose that delta A and delta b will be the perturbations on the matrix A and the vector b respectively. Let us assume that the perturbation delta A to the non singular matrix A is sufficiently small so that A plus delta A is also non singular. Such a problem is known as general perturbation problem.

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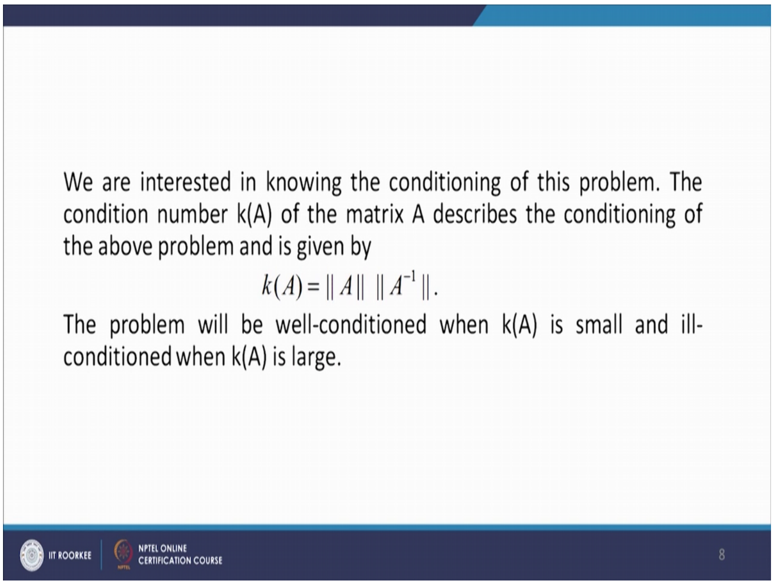

Now, we are interested in knowing the conditioning of this problem the condition number k of the matrix A describes the conditioning of the above problem and is given by k equal to norm of A into norm of A inverse, where norm of A is the matrix norm of A. The problem will be well conditioned when k is a small and ill-conditioned when k is large.

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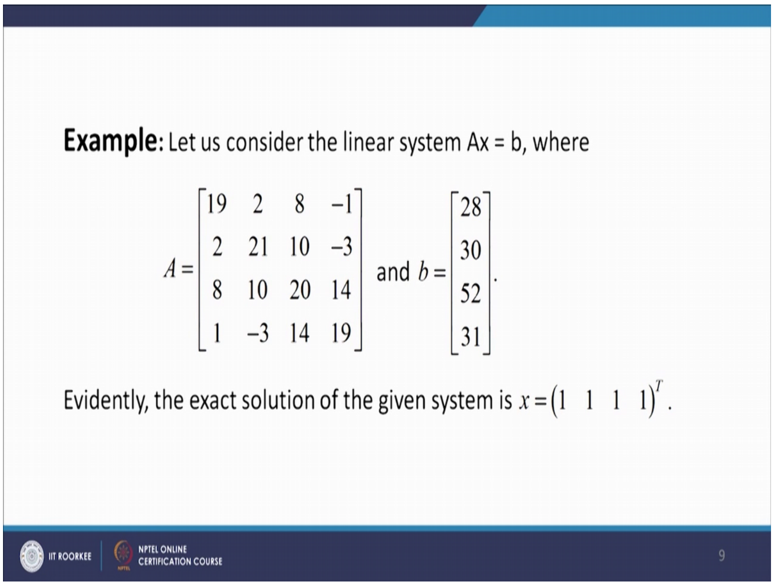

Let us consider the linear system Ax equal to b, where A is the matrix. This is 4 by 4 matrix. The first row is 19 2 8 minus 1; second row 2 21 10 minus 3; third row 8 10 20

14; fourth row 1 minus 3 14 19; and b is the column vector 28 30 52 31. You can see that when you when you consider the system Ax equal to b then the trivial solution of this system is there it is 1 1 1 1.

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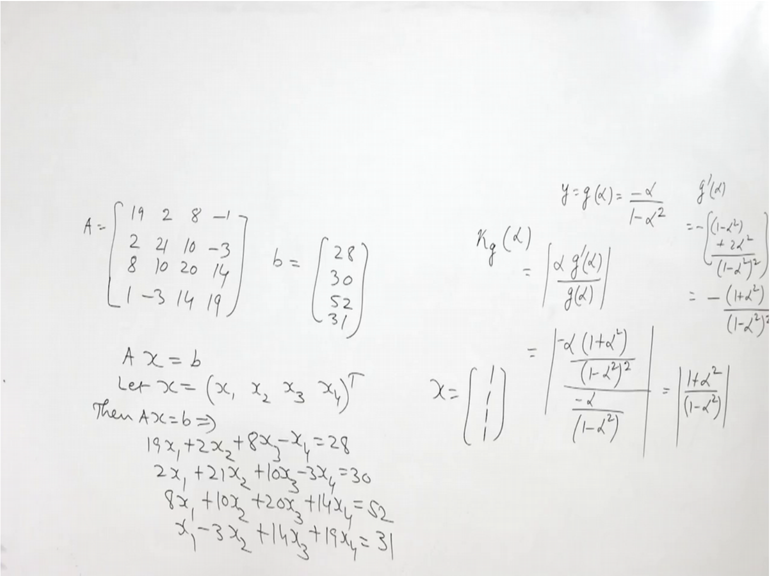

Let us consider A is equal to 19 2 8 minus 1; and then you have 2 21 10 minus 3, then third row is 8 10 20 14 and then we have 1 minus 3 14 19 and b is 28 30 52 31. So, let us consider the system Ax equal to b.

So, then suppose x has components x 1, x 2, x 3, x 4. Let x be then we, then Ax equal to b will give us the 4 equations 19 x 1 plus 2 x 2 plus 8 x 3 minus x 4 equal to 28. Similarly 2 x 1 plus 21 x 2 plus 10 x 3 minus 3 x 4 equal to 30. We have similarly 8 x 1 plus 10 x 2 then 20 x 3 and 14 x 4 equal to 52; then x 1 minus 3 x 2 then 14 x 3 then 19 x 4 equal to 31. Now, you can see 1 1 1 1, x equal to 1 1 1 1 this is the trivial solution of this system because when you take x 1, x 2, x 3, x 4 all equal to 1 then 19 plus 2 21, 21 plus 8, 29, 29 minus 1 is 28. Here 2 plus 21, 23 plus 10 33 minus 3 is 30 and similarly you can put x 1, x 2, x 3, x 4 equal to 1 in the remaining 2 equations you can see they are satisfied. So, the exact solution of the given system is x equal to 1 1 1 1 transpose.

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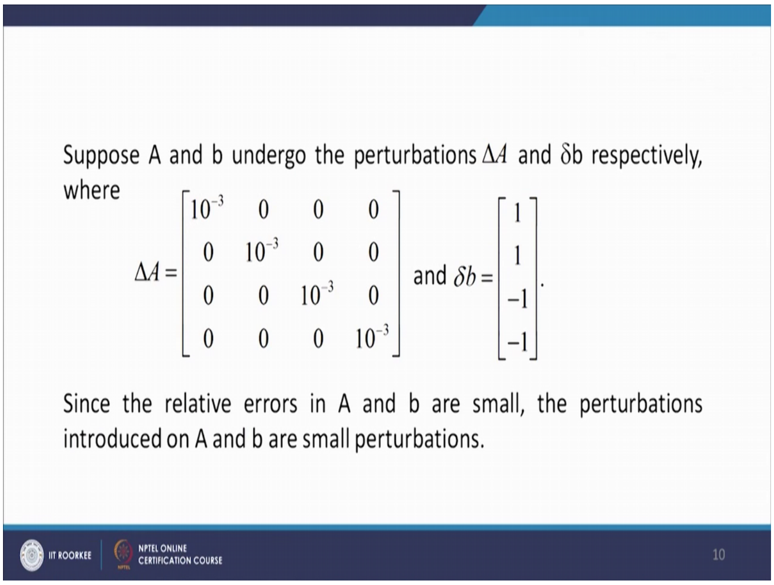

Now, let us suppose that A and b undergo the perturbations delta A and delta b respectively, where delta A is equal to 10 to the power minus 3 0 0 0; then 0 10 to the power minus 3 0 0 and so on. So, we give the, we make a slide change in the diagonal elements of the matrix A, there is no change in the non diagonal entries of A. So, and the change in the diagonal entries is of the order of 10 to the power minus 3.

So, we have, we are making very small change in the matrix A and delta b is 1 1 minus 1 minus 1. So, in the matrix b we make a change 1 1 here we change by these 2 entries we change by 1 and these 2 entries we change by minus 1 minus 1. Then let us seen what happens to the solution x. So, since the relative errors in A and b are small the perturbations introduced on A and b are small perturbations.

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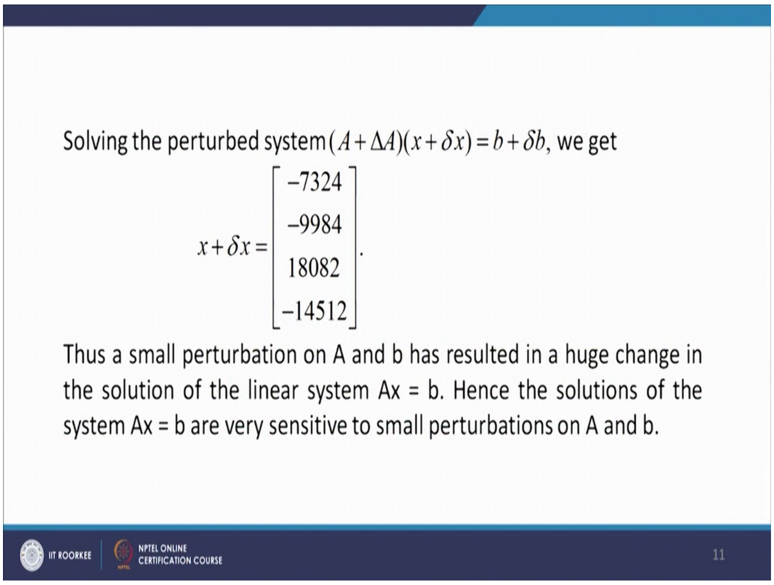

Now, let us solve this perturbed system A plus delta A into x plus delta x equal to b plus delta b by MATLAB we have solved it, it x plus delta x comes out to be this matrix minus 7324, minus 9981, 18082, minus 14512. Thus A small perturbation on A and b has resulted in a huge change in the solution of the linear system Ax equal to b and hence the solutions of the system x equal to b are very sensitive to small perturbations on A and b.

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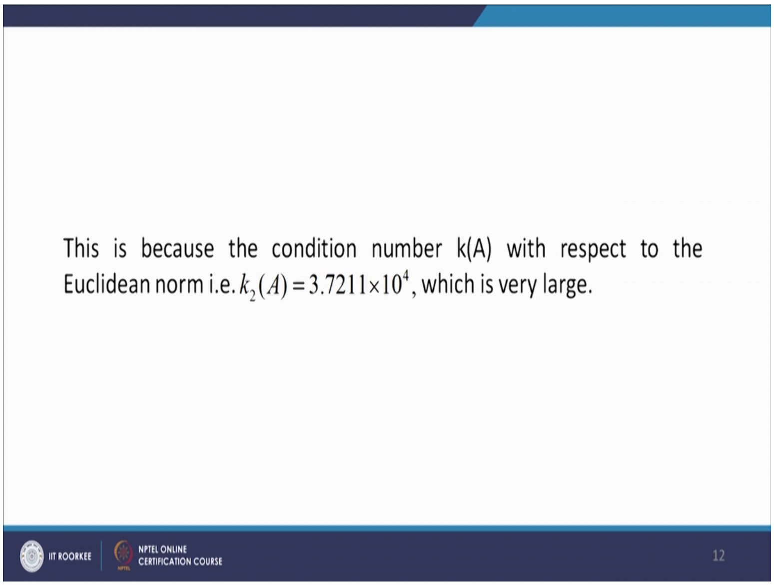

And this is because the condition number k A with respect to the Euclidean norm which also called as the spectral norm of the matrix A. So, the because of that the k 2 A comes out to be 3.7211 into 10 to the power 4 which is very large. So, because the conditioner number is very large, condition number of the matrix is very large there is a huge change in the value of x. So, with that I will close this lecture.

Thank you very much for your attention.