Numerical Linear Algebra
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Lecture - 25
Conditioning and Condition Numbers- I

Hello friends. I welcome you to my lecture on conditioning and condition numbers one. There will be two lectures on this topic. This is first lecture, after that we will have second lecture on this topic conditioning and condition number one. Conditioning pertains to the perturbation behaviour of a mathematical problem a problem may be viewed as a function f from X into Y where X and Y are normed vector spaces or you can say normed linear spaces.

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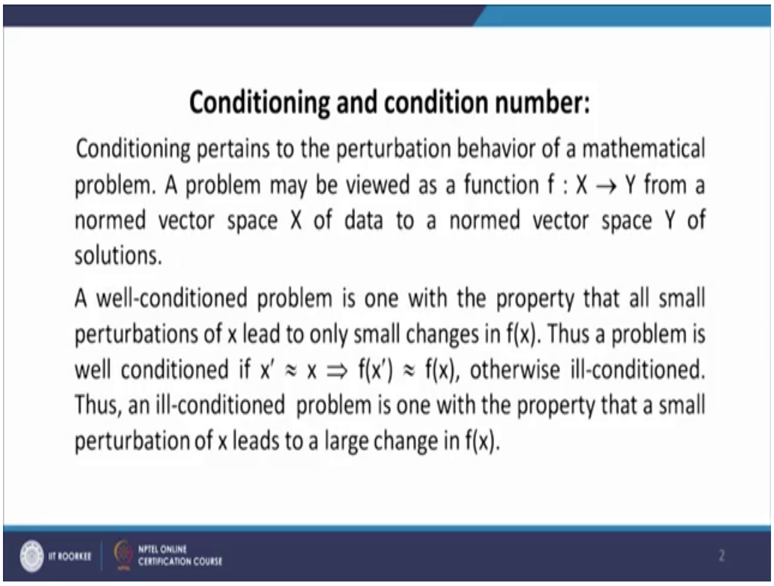

X is the normed vector space of data and Y is the normed vector space of solutions.

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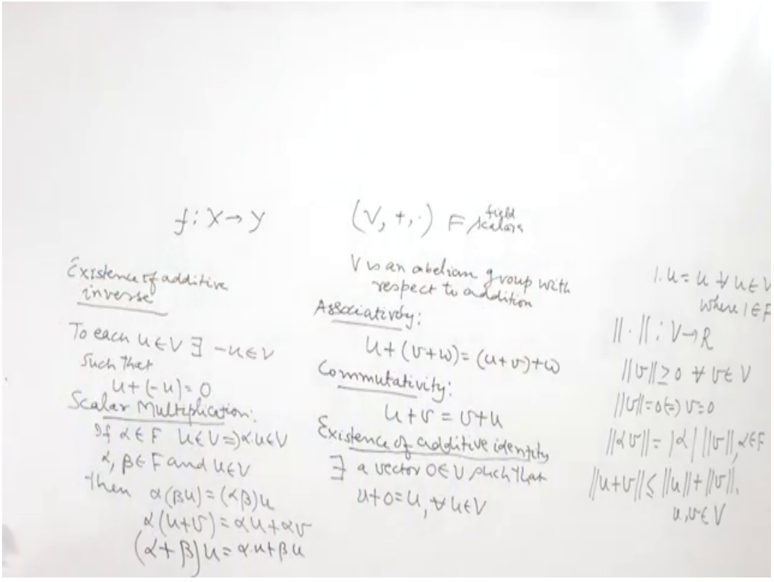

F is a function from X into Y where X and Y are a normed vector spaces, the vector space is also termed as a linear space. So, X and Y are normed linear spaces.

As you know, a vector space is one where we have a collection of objects say be equipped with 2 operations denoted by addition and scalar multiplication and we have a field of scalars v is called vector space with respect to the operations of vector addition and scalar multiplication, if over the field f if it is v is an Abelian group with respect to addition,

Now, addition is a binary operation on v means when X and Y are any 2 vectors in v, then x plus Y belongs to v and corresponding to the Abelian group we have associativity if we take vector u v w in v, then u plus v plus w is equal to u plus v plus w. So, v must be associative with respect to addition, then commutatively where we say if you take any 2 vectors u and v in v, then u plus v is equal to v plus u, then we have existence of additive identity. So, existence of identity must be there; that means, their exist vector which we denote by 0 in v such that u plus 0 is equal to u for all u belonging to v and then we have existence of additive inverse.

So, to each u belonging to v, there must exist a vector denoted by minus u in v such that u plus minus u is equal to 0 vector the additive identity in v and we have. So, if we have all these properties in v, then v said to be an Abelian group with respect to addition and then corresponding to scalar multiplication in this scalar multiplication, what it is if you take a scalar alpha belongs to f and a vector u belongs to be, then alpha into u belongs to f. So, when alpha belongs to f and u belongs to v then alpha into u will be there in v.

So, v is close with respect to scalar multiplication, then it is satisfy the following 4 properties, if we have alpha beta belonging to f and u belonging to v, then alpha beta u equal to alpha beta into u and then we have the scalar multiplication is distributive over addition alpha into u plus v equal to alpha u plus alpha v scalar multiplication is distributive over vector addition.

And then we have third alpha plus beta into u is equal to alpha u plus beta u and the fourth one is multiplicative identity 1 into u is equal to u for all u belonging to v where one belongs to f the field of scalar.

So, if v satisfies all these properties we say that v is a vector space with respect to addition and scalar multiplication now it is called a normed vector space or a normed linear space, if we further define a function denoted by this from, v into R such that norm of v is greater than or equal to 0 for all v belonging to v norm of v is equal to 0, if and only if v is equal to 0 and then norm of alpha into v is equal to mode of alpha norm of v where alpha is the scalar in f and then we have norm of u plus v less than or equal to norm of u plus norm of v where u and v are any 2 vectors in v.

So, if v is equipped with this function or find v into R then we say that v is a normed vector space. So, here a function a problem may be viewed as a function from a normed linear space x into a normed linear space by where x is the space vector space of data and Y is the vector space of solutions.

Now, a well-conditioned problem is one with the property that all small perturbations of x lead to only small changes in fx; that means, if you make a small perturbation in the input data x, then corresponding to this that there must be a very small change in the value of fx, then the problem is said to be a well condition problem now that that change in the data may be due to an error or it may be done by a. So, some if we if there is a small error in the data input data x then corresponding to that in the value of fx, there must be a small change, then we say that the problem is well conditioned otherwise problem is said to be ill conditioned.

So, this means that a problem is well conditioned if f dash is approximately equal to x implies that fx dash is approximately equal to fx, otherwise, the problem is said to be ill conditioned. So, we can define an ill conditioned problem as the one with the property that a small perturbation of x leads to a large change in fx.

Now, condition number can be are of 2 types absolute condition number and relative condition number.

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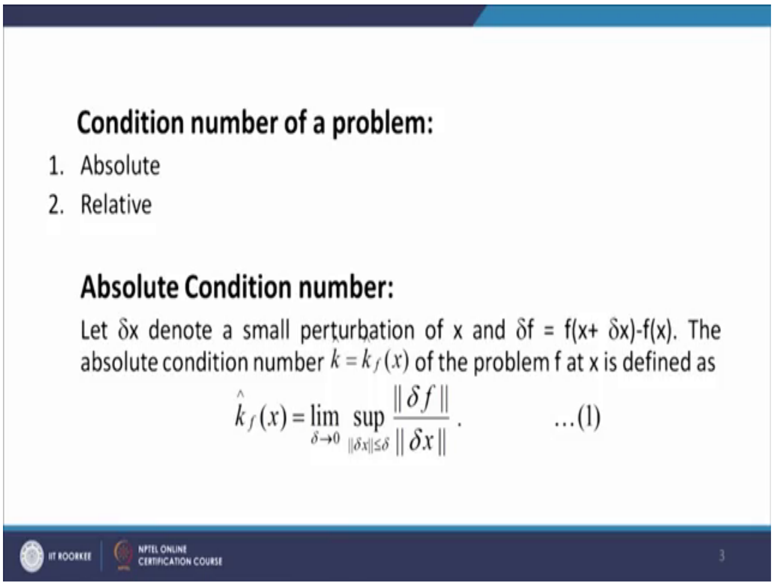

Let us see how we define an absolute condition number let us say delta x denote a small perturbation of x and delta f denotes the change in the value of f corresponding to the change delta x in the value of x. So, delta f is equal to fx plus delta x minus fx, then the absolute condition number k which is k of x because it depends on f as well as x. So, k of x of the problem f at the point x is defined as k of x equal to limit delta tends to 0 supremum norm of delta x less than or equal to delta of norm delta f over norm of delta x.

Now, here when we say norm of delta f norm of delta f is the norm that is the norm in the space Y because norm of because delta f is equal to fx plus delta f minus fx. So, fx from delta x and fx, they are the values of f in the space Y and therefore, norm delta f means norm in the space Y and in the denominator we have norm delta x, this is the change in the value of x this is the input data. So, here we norm by norm we mean that it is the norm in this space x.

So, when there is a very small, I mean when delta is very small then maximum value of the ratio norm of delta f over norm of delta x is defined to be the absolute condition number. So, when delta x and delta f are sufficiently small, we generally write k of x equal to supremum of over delta x norm of delta f over norm of delta x.

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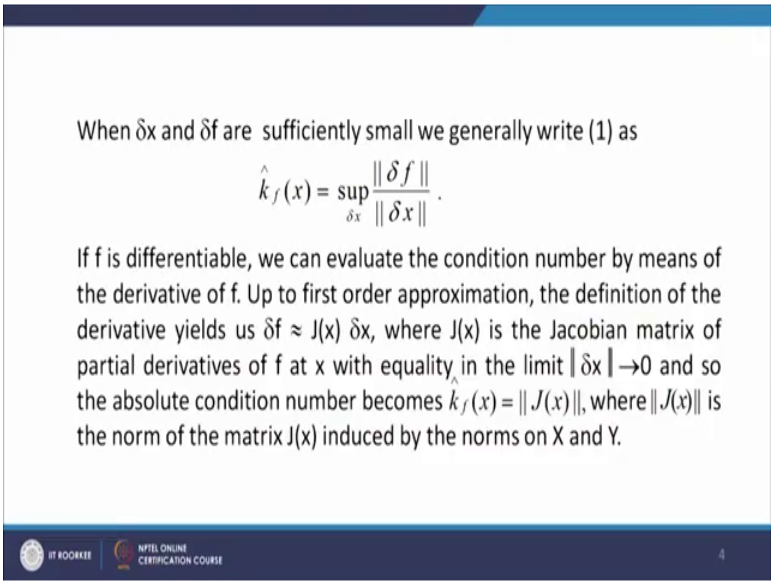

Now, if f is differentiable and then we can evaluate the condition number by means of the derivative of f up to a first order approximation up to a first order approximation means the second and higher order powers of delta x may be neglected. So, then up to the first order approximation the definition of derivative yields delta x equal to J x delta x if f is differentiable, we can evaluate the condition number by means of the derivative of f up to a first order approximation means these second order and higher order terms containing delta x are neglected.

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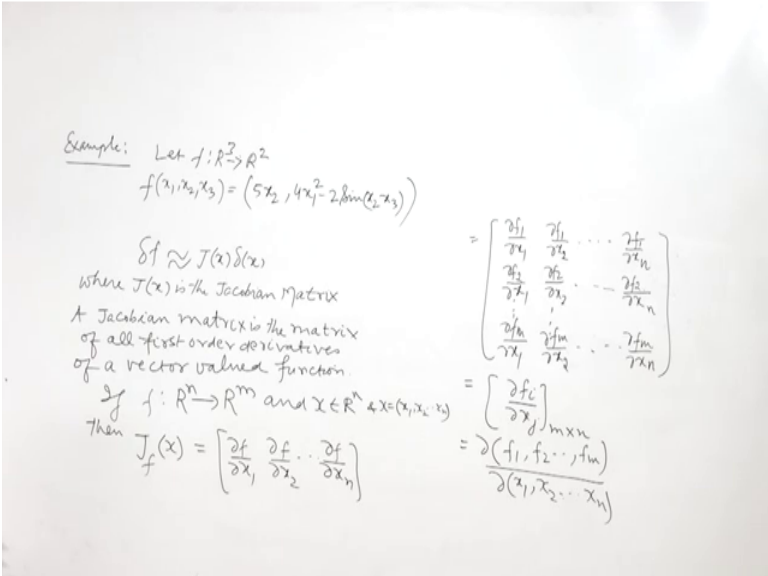

So, up to first order approximation, the definition of derivative yields us delta f is approximately equal to J x into delta x where J x is the Jacobean matrix and as you know the Jacobean matrix is the matrix of all first order derivatives Jacobean matrix a Jacobean matrix, it is the matrix of all first order derivatives of a vector valued function.

So, if f is the function from R n to R m, R n to R m, then the Jacobean of f, let us take any x belonging to R n, if f is a mapping from R n to R m and x belongs to R n, then the Jacobean matrix of f is defined as that is this is equal to oh this is delta f 1 over delta x n here we have delta f 2 over delta x m and so on delta formula over delta x n. So, we get m by n matrix, this row has got m this matrix has got m rows and n columns.

So, if f is mapping from R n to R m that is it is a vector valued function which takes as input the vector x belonging to R n and produces an output the vector f x belonging to R m then the Jacobean matrix J of x is an m by n matrix as we have seen here this can also be written as where i denotes the row and z denotes the column. For example, this can also be written as here the vector x belongs to R n this x we have taken as R x 1, x 2, x n and f 1, f 2, fm are the m components of the vector valued function f.

For example, let us consider this would be f 1 and this will be x 1, x 2, x 2. So, this is m by n matrix. So, when we have a vector valued function f from R n to R m, then the Jacobean matrix is of size m by n, we can also express it as delta f 1 f 2 fm divided by delta x 1 x 2 x n.

Now, let us take an example on this to make it clear suppose we take a function f from R cube to R x square where it f is defined as x 1, x 2, x 3, a vector in R cube as 5 x 2 4 x 1 square minus 5 2 sin x 2, x 3,

So, let us take a function from R cube to R square which is defined as f x 1, x 2, x 3 equal to 5 x 2 4 x 1 square minus 2 sin x 2, x 3, then let us find the Jacobean of x with respect to the vector x that is x 1, x 2, x 3.

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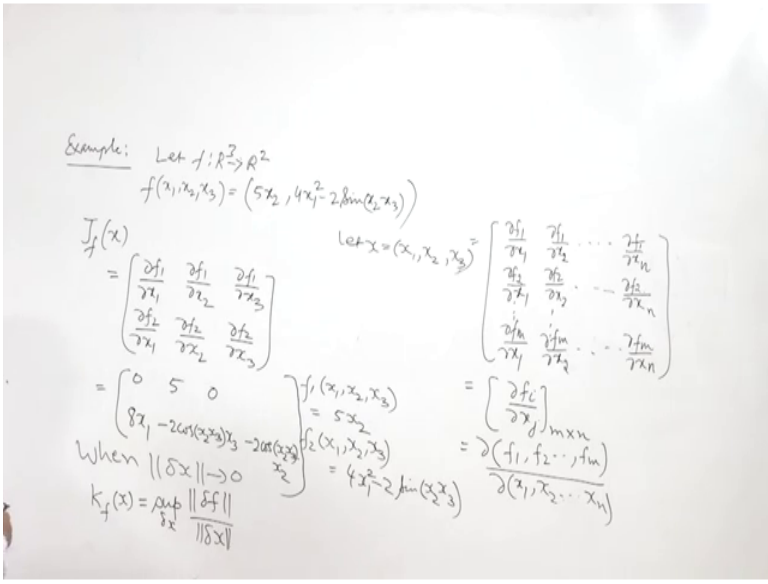

So, what we will have then J of x where x is let x be equal to x 1, x 2, x 3, then J of x is equal to. So, f has got 2 components. So, this f 1; f 1, x 1, x 2, x 3, this is f 2 x 1, x 2, x 3. So, f has got 2 components.

So, delta f 1 by delta x 1 and then delta f 2 by delta x 1 we will have, then delta f 1 by delta x 2 delta f 2 by delta x 2 and then we have delta f 1 by delta x 3 delta f 2 by delta x 3 n is equal to 3, here m is equal to 2. So, we have 2 by 3 matrix and this is equal to now when you differentiate f 1 x 1 x 2; f 1 x 1 f 1 is a function of x c 1 x 2 f 1 is equal to 5 x 2 and f 2 x 1; x 1 x 2 x 3 and f 2 x 1 x 2 x 3 is equal to 4 x 1 square minus 2 sin x 2 x 3.

So, we can find the Jacobean matrix easily when you differentiate f 1 with respect to x 1 you get 0 when you differentiate f 1 with respect to x 2, you get 5 when you differentiate f 1 with respect to x 3, you get 0 when you differentiate partially f 2 with respect to x 1 you get 8 x 1 and then you differentiate f 2 with respect to x to. So, you get minus 2 cos x

2 x 3 into x 3 and then the derivative partial derivative of f 2 with respect to x 3 we will get again as minus 2 cos x 2 x 3 into x 2 . So, we will get a 2 by 3 matrix

So, Jacobean of f which is defined from R cube to R square gives us a matrix of size 2 by 3 this, how we can obtain the Jacobean matrix. So, let us go back to a our discussion of the condition number we see that delta f is approximately equal to J x into delta x where J x is the Jacobean matrix of the partial derivatives of f at the point x. Now here when norm of delta x goes to 0 that is norm of delta x sufficiently small the condition number becomes this like this.

See when norm of delta x goes to zero; that means, delta x sufficiently small. So, that we can neglect the second and higher order terms containing delta x then k of x k of x is equal to supremum of norm of delta f divided by norm of delta x delta x. So, the maximum value of perturbation delta x in the input delta x. So, this is this when delta x tends to 0 we can take k f x to be approximately equal to norm of delta f over delta norm of delta x.

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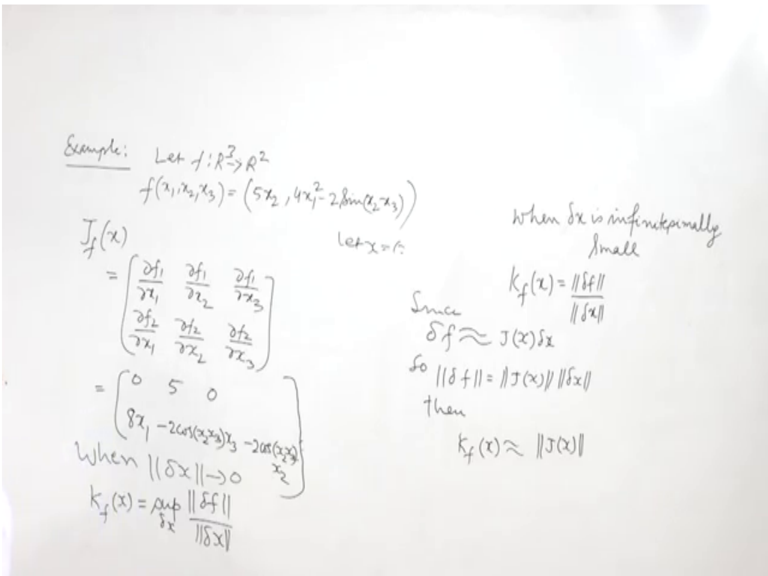

So, when delta x is infinitesimally small k fx can be approximately taken as equal to norm of delta f divided by norm of delta norm of delta x, but just now we have seen that norm of delta f is equal to delta f is equal to J x into delta x. So, since delta f is approximately J x in to delta x norm of delta f will be equal to norm of J x into norm of delta x.

So, let us put this value there. So, then k fx is approximately equal to norm of jx. So, when delta x is sufficiently small the absolute condition number is approximately the norm of the Jacobean matrix where norm of Jacobean matrix where norm of the matrix J x is the norm induced by the norms on x and y.

Now, let us go to the relative condition number.

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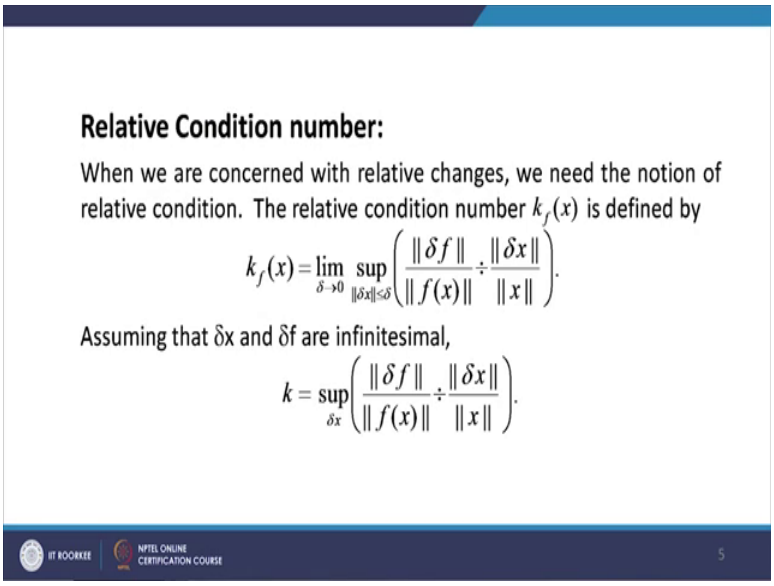

When we are concerned with relative changes we need the notion of relative condition. So, the relative condition number k fx is defined as k fx equal to limit delta tends to 0 supremum of norm of delta x less than or equal to delta.

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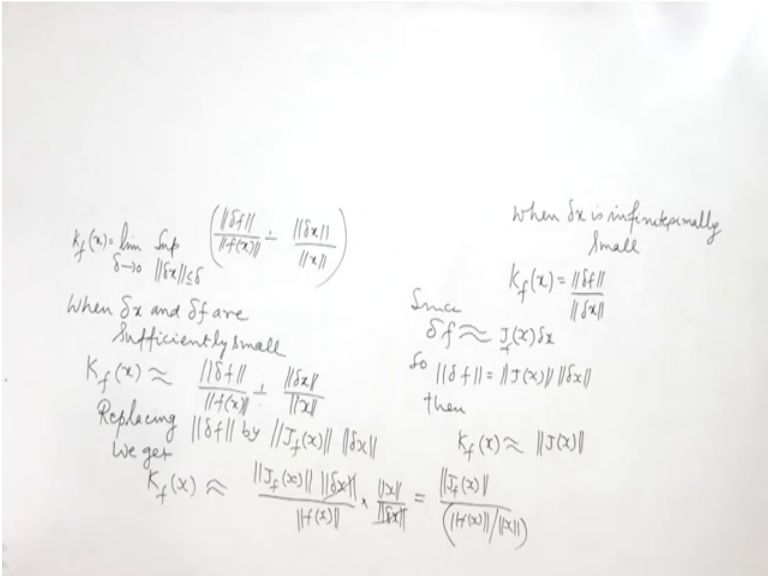

And then norm of delta f divided by norm of fx divided by norm of delta x divided by norm of. So, it is the quotient of the relative change in f divided by the relative change in x, you can see norm of delta f over norm of delta norm of fx gives us the relative change in f and norm of delta x divided by norm of x is gives the relative change in x.

So, it is the quotient of the relative change in f and the relative change in x and we take delta to be sufficiently small it is go going to 0. So, when delta x is sufficiently small we can say that when delta x and delta f are sufficiently small k fx is approximately equal to norm of delta f divided by norm of fx divided by norm of delta x divided by norm of x.

Now, when delta x is sufficiently small we have seen delta f is approximately equal to J x into delta x where J x is the Jacobean of f. So, you can write it also as fx J fx. So, then what will happen here is that again ah. So, replacing J norm of delta f Y this one norm of J fx into norm of delta x. So, replacing norm of delta f by norm of J fx into norm of delta x we get k fx equal to norm of. So, this is norm of J fx into norm of delta x divided by norm of fx into norm of x divided by norm of delta x. So, this will cancel and we will get this as same as norm of J fx divided by we write it like this norm of fx divided by norm of x.

So, k fx is given by the norm of the Jacobean matrix of f J fx divided by the norm of fx over norm of f. So, this is the case when f is differentiable we can express the Jacobean number the condition number k fx which we also write as k in terms of the Jacobean of norm of the Jacobean matrix of f. So, now this is the formula we have.

Now, let us remark here that the absolute and relative condition numbers.

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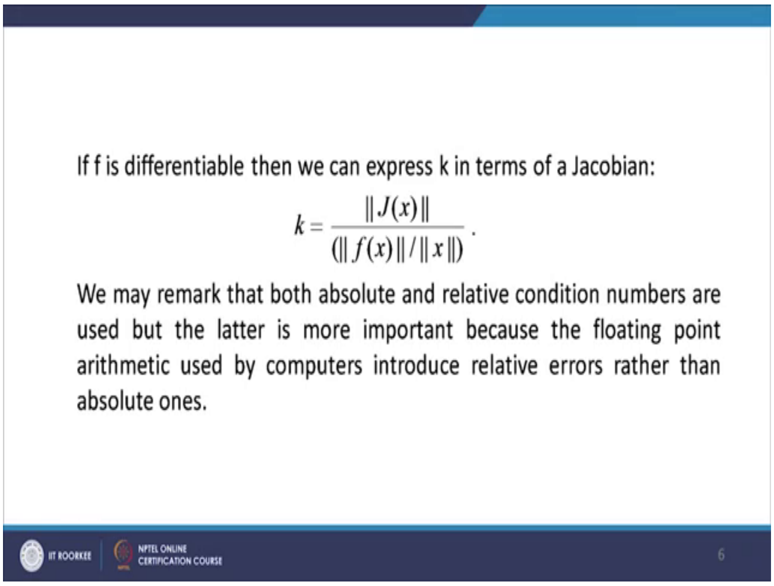

Both are used in a literature, but the condition relative condition number is more important because the floating point arithmetic used by computers introduce relative errors rather than absolute ones we have seen that by example we have seen the information that we do not get about the accuracy of the numbers from the absolute error the absolute error is same in both the cases, but when we found out the relative error it turned out that one approximation is better than the other. So, relative numbers are used in the floating point arithmetic relative errors are used.

So, now addition multiplication division with positive numbers are well conditioned problems because when we carried out carried out addition multiplication division with positive number we have seen that the there is no appreciable error in the relative in the relative error. So, that whatever change is there in the relative error as a result of addition multiplication division operations that is not very large when the relative errors in X and Y are the small the relative error in X plus Y or X into Y or X over Y is also small which is acceptable. So, they are well-conditioned problems, but subtraction is not well conditioned problem because we have seen that when we subtract to nearly equal numbers there may be situation where the relative error gets too large. So, subtraction cannot be taken as a well-conditioned problem.

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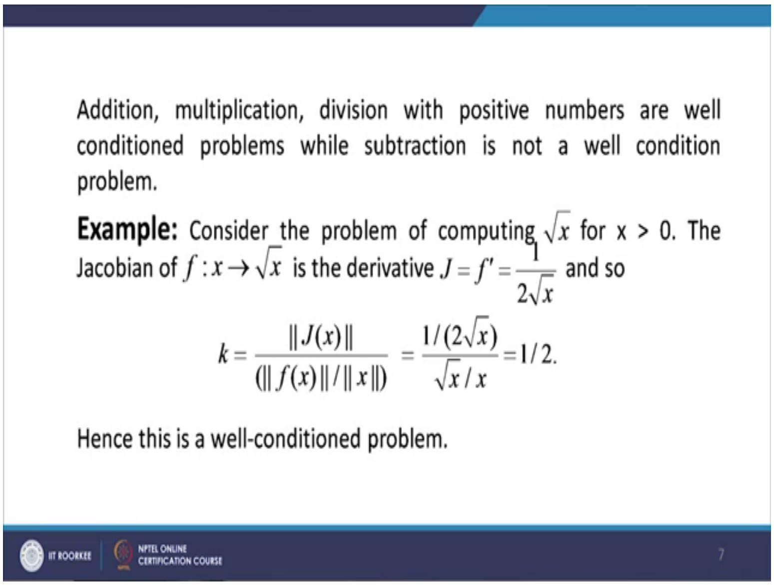

Now, let us illustrate this article are let us find out the condition number in case of some examples consider the problem of computing to root x for x greater than 0. So, we are given the function f from x to root x here which is defined as fx equal to root x.

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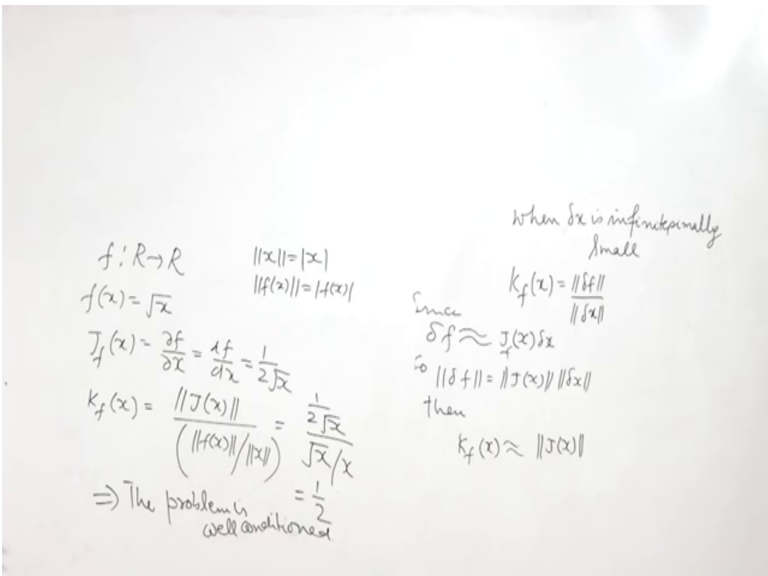

So, f is a function form from R into R f is a function from R from a normed vector space R into R defined as fx equal to root x x is given to be positive.

Now, here Jacobean of f will be what because here m and n m and n both are equal to one. So, the Jacobean matrix will be of size one by one; that means, Jacobean matrix of f with respect to x will be the partial the derivative of f with respect to x f is a function one variables we can write df over dx which is equal to one by 2 root x here. Now condition number k fx will be equal to norm of J x relative condition number we are going to find this divided by norm of fx divided by norm of x.

In case of R here norm of x is defined as mode of x and norm of fx will be defined as mode of fx and. So, this and norm of J x norm of J x will be mode of J x which is one by 2 root x. So, one by 2 root x divided by fx is equal to root x. So, root x x divided by x, we have norm of x is equal to mode of f x and norm of fx equal to mode of fx in case of r. So, this is equal to 1 by 2 norm of J x equal to 1 by 2 root x norm of fx equal to root x and norm of x equal to x. So, this gives you 1 by 2 and therefore, we can say that the problem is well conditioned. So, the problem is well conditioned here. So, when we find when we find out root x for a given value of x the input data x then the problem of computing root x from x is a well conditioned problem.

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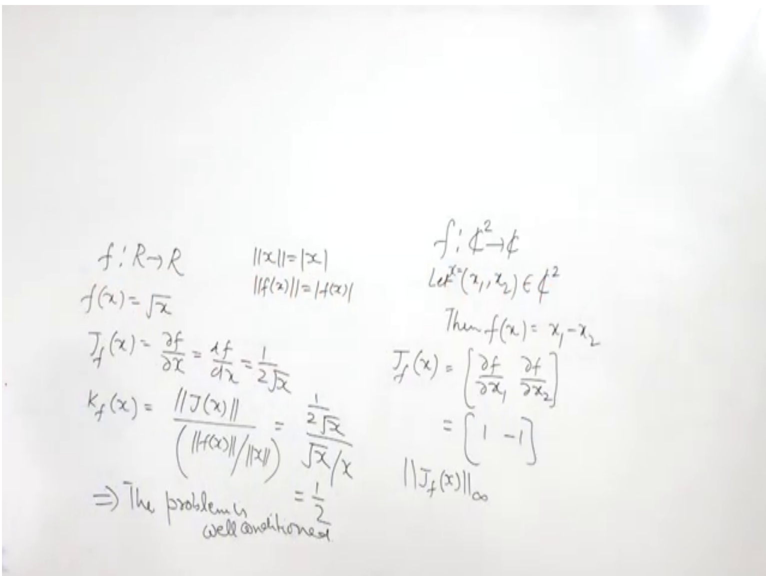

Now, we go to another case where f is a function from R c square into c f is a function from C square into C, C is the set of complex numbers and the function. So, let us say, let x 1 x 2 be an element of C square x equal to x equal to x 1 x 2 be an element of c square than fx is defined as x 1 minus x to.

Now, here we again find the condition number. So, here J f not J fx what is J fx, here x is equal to x 1 x 2 f f is a function from c square into c. So, n is equal to n is equal to 2 here and m equal to one here. So, we will get one by 2 matrix and that one by 2 matrix will be delta f over delta x 1 delta f over delta x 2 this is one by 2 matrix because the components of h there is one component of f that we can write as f. So, delta f over delta x 1 delta f over delta x 2 and this is equal to 1 and minus 1.

Now, the norm in c we are taking as infinity norm. So, norm of J x J fx infinity norm this is matrix norm the infinity norm in the case of matrix is defined as m x maximum absolute row sum.

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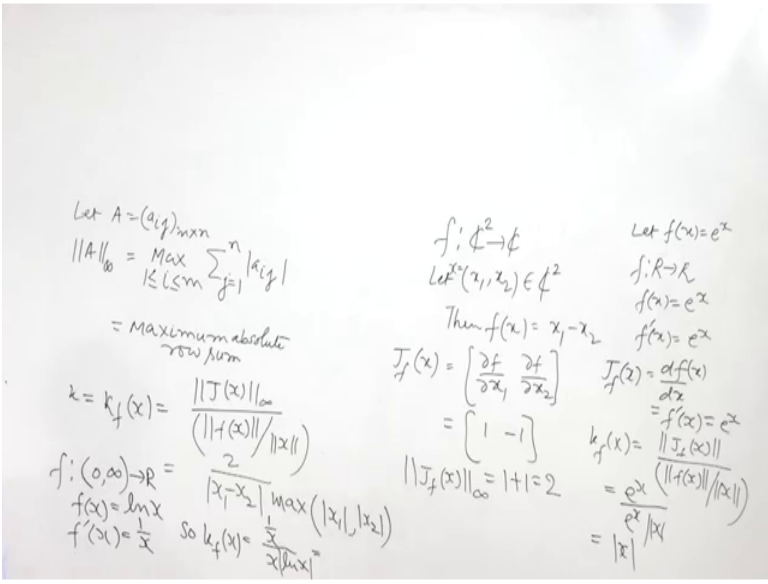

If you have let say, let A be equal to aij m by n, we have a m by n matrix equal to aij, then the infinity norm on the matrix is defined as maximum of this is mean that maximum absolute row sum maximum absolute row sum. You can see here J runs from 1 2 and so, we have if you take i equal to 1, then you have mode of a 1 1 mode of a 1 2 plus mode of a one n and then in the second row you have taken i equal to 2 mode of a 2 1 mode of plus mode of a 2 2 and so on plus mode of a to n.

So, you find the absolute values of all entries in the row and then take their sum and once you have done it for all rows fine take the maximum value of that. So, here what do you see here there is only one row? So, if you take the row sum absolute row sum than one plus one it is equal to 2 and there is only one row. So, this is the maximum value.

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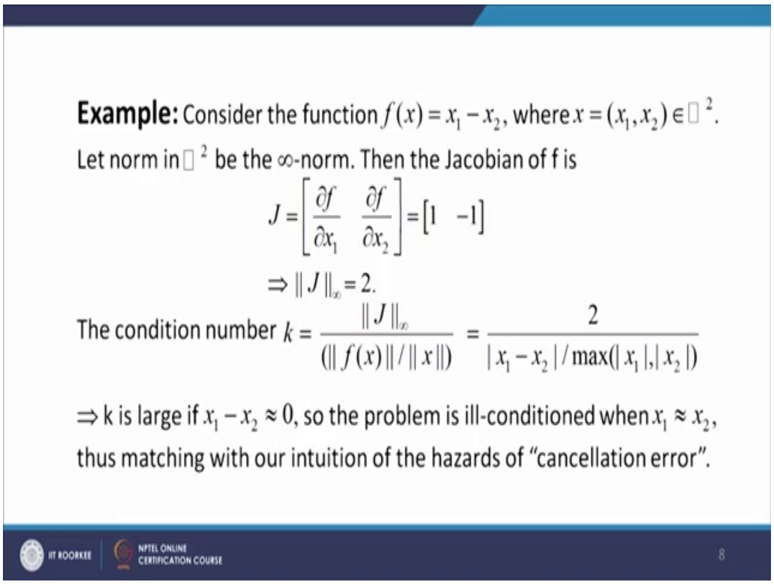

So, norm of J infinity equal to 2 and therefore, the condition number is now norm of fx in c is same as mode of f mode of fx. So, this is equal to this is this infinity norm here. So, we have 2 and then this mode of this we have mode of x 1 minus x 2 because fx is an element belonging to C which is x 1 minus x 2. So, mode of x 1 minus x 2 and norm of x norm of x, we take as maximum of f infinity norm maximum of mode of x 1 mode of x 2 which is the infinity norm in c square.

So, now we can see here this k is large if x 1 minus x 2 approximately equal to 0 and so, the problem is ill conditioned when x 1 and x 2 are nearly same this thus matching with our intuition of the hazards of cancelation error in the case of cancellation error we can we have seen that the relative error gets when becomes very large this means that the condition number becomes very large.

So, the problem is ill condition and if x 1 and x 2 are nearly equal here.

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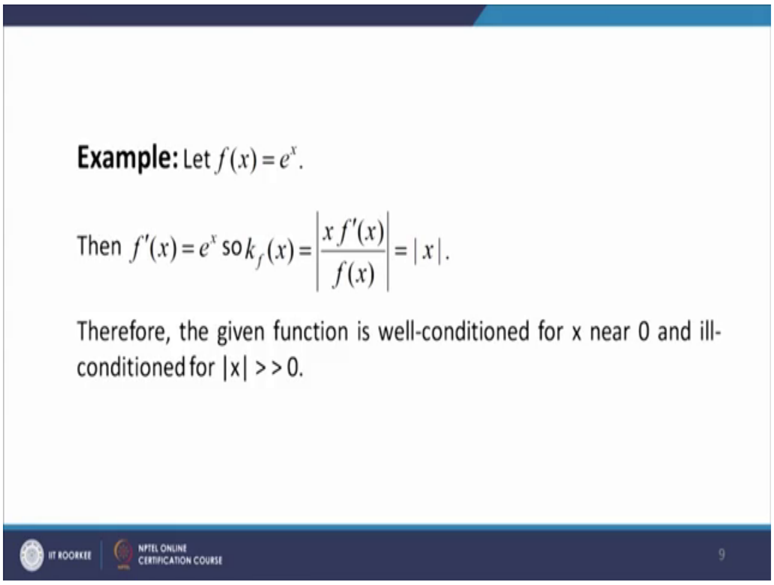

Then we can take the problem of fx equal to e to the power x here f prime x. So, f is a mapping from R into R lets say fx equal to e to the power x. So, f is a mapping from R into R defined as fx equal to e to the power x. So, f prime x we can find f prime x equal to e to the power x because in the case of n and n both equal to one here the Jacobean matrix J is 1 by 1 matrix which is the derivative of f. So, J fx equal to derivative of f with respect to x or you can say f prime x. So, here k fx will be equal to norm of J fx divided by norm of fx divided by norm of x.

So, we have e to the power x we know. So, mode this is f this is equal to f prime e to the power x. So, e to the power x we shall have norm of J fx will be the modulus of e to the power x here divided by norm of fx is again mode of e to e to the power x. So, we have e to the power x divided by we have mode of x. So, this is equal to mode of x.

So, we can say that the given function is well conditioned for x near 0 because then of the condition number will be very small and ill condition where mode of x greater the greater than zero; that means, mode of x is sufficiently greater than 0, then we can take the problem of fx equal to lnx where x is greater than 0.

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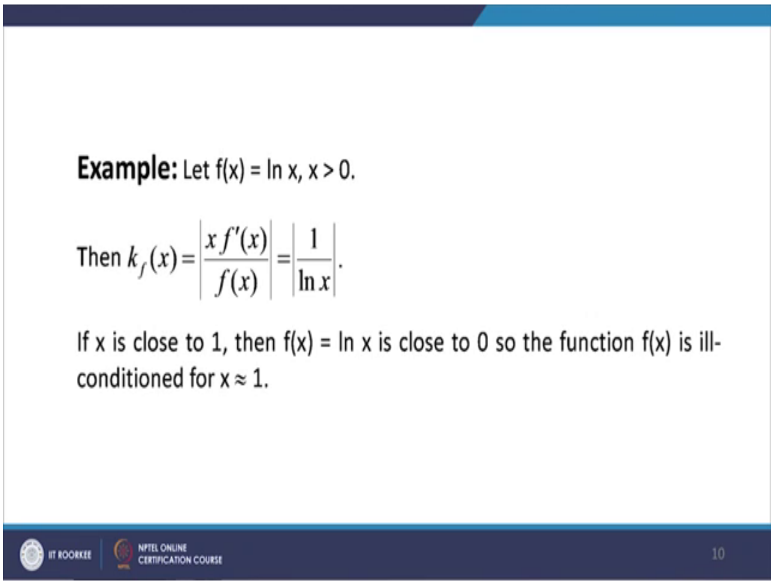

So, here again f is a mapping from R into R f is a mapping from not R f is a mapping from 0 infinity into R it is not defined at 0. So, we can say 0 infinity into R f is a mapping from 0 infinity into R defined as fx equal to lnx ok.

So, here again f prime x equal to one by x and. So, J fx equal to one by x and. So, kfx equal to J fx that is 1 by x divided by x times fx. So, that is lnx fx is equal to lnx mode of lnx mode of lnx divided by x because we have f dash x which is J fx divided by norm of fx norm of fx is mode of lnx divided by norm of x and norm of x is equal to mode of x or x because x is greater than 0. So, this equal to 1 over mode of lnx; so, when x is very close to 1, then lnx is close to 0, so, the function fx is ill conditioned for x which is nearly equal to 1 with that I would like to close this discussion.

Thank you very much for your attention.