Numerical Linear Algebra
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Lecture – 02
Determinant of a Matrix

Hello friends, I welcome you to my lecture on determinants. So, whenever we take a, a square matrix of order see n, then it can be associated to an expression or a number which we call as the determinant.

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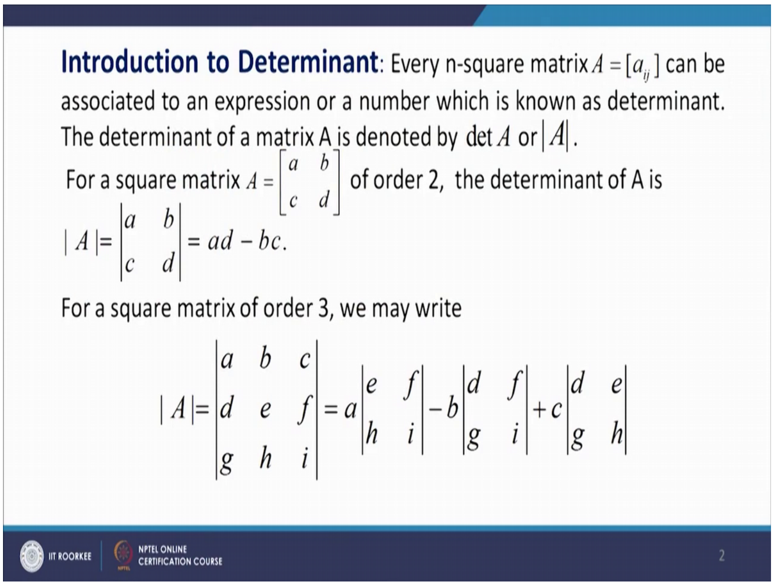

The determinant of a matrix A is denoted by det A, either we write it det A or we denote it by mode of A.

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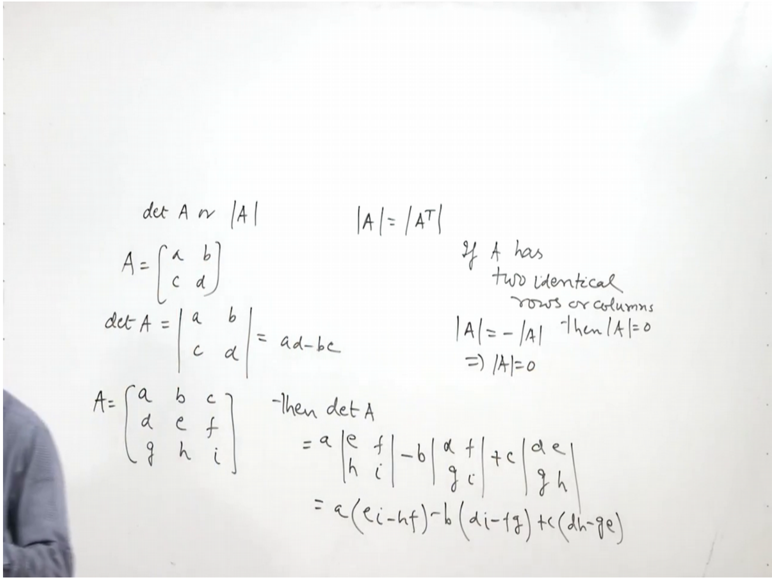

So, the determinant of a matrix is denoted by determinant of A or mode of A, for a square matrix a b c d. So, let us see A equal to a b c d determinant of A is equal to we write like this is equal to a d minus b c. When we take a square matrix of order 3 see A equal to let us say it, a b c d, a b c d e f and then g h and then i ok.

So, then determinant of A is equal to if we expand it by the first row then a times e f h i that is its co factor, a times h i, then b times its co factors that will be minus b times d f g i, and then c times it co factor so we have d e g h. So, when then the elements of a row are multiplied by its co factors and we and added what we get is the determinant value of the determinant. So, this will be a times e i minus h f minus b times d i minus f g plus c times d h minus g.

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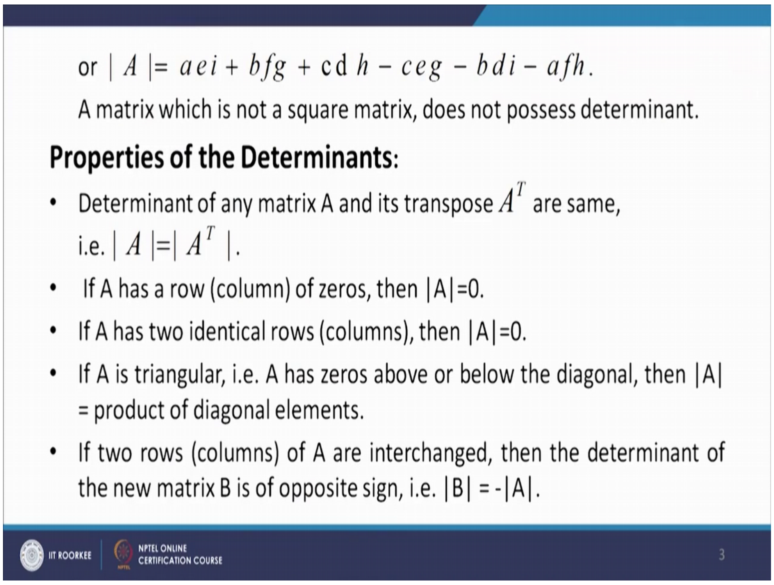

Now, a matrix which is not a square matrix, if you do not take a square matrix then it does possess determinant. Properties of the determinants, the determinant of the matrix A and its transpose are same. So, determinant of A is equal to determinant of a transpose this is the first property, the second one if A is a row of 0s, A has a 0 row or 0 column then the value of the determinant is always 0, because you can expand that determinant by that 0 row or by that 0 column.

So, then we will its value will be always be 0, if there are two identical rows then the value of the determinant is also 0 or two identical columns. So, if A has two identical rows or columns, then its determinant is equal to 0 this is because when you inter change any two rows of the determinant, there is a change of sign.

So, when you have two identical rows the determinant of A will be equal to minus determinant of A, because when you inter change two rows you get the same determinant, but by the property that when we inter change two rows there is a change of sign. So, here what will happen mode of A will be equal to minus mode of A and this implies that determinant of A equal to 0.

So, if A is a triangular matrix, then determinant of A will be product of its diagonal elements, if whether it is lower triangular matrix or A it is a upper triangular matrix. Now, if two rows of A are inter change that two columns are inter change then the determinant of A is multiplied by minus 1.

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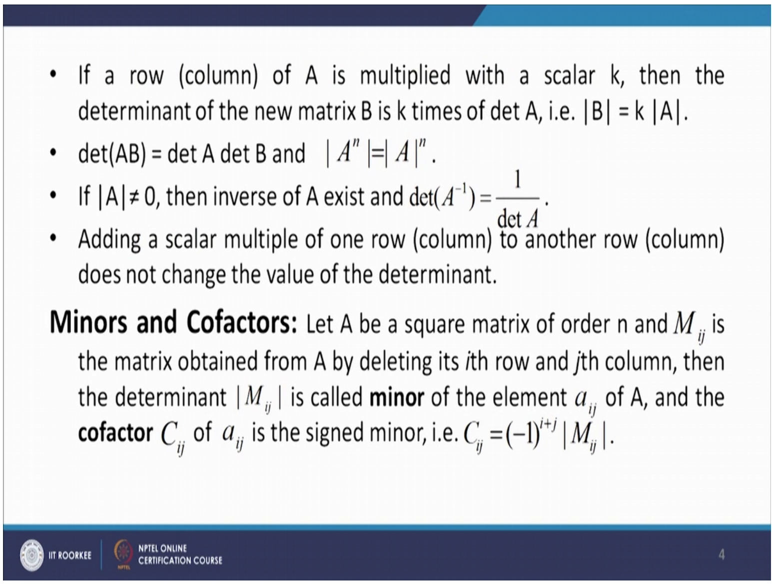

So, if a row are, a column of A matrix is multiplied by k then all the elements all the entries of that row or that column are multiplied by k. Determinant of a product of two matrices suppose we have two matrices, a and b of size n by n then the determinant of a b is equal to determinant of a into determinant of b.

So, if we take see a to the power n that is a is multiplied with itself and times then determinant of a to the power n is equal to determinant of a raise to the power n. Now, if A is a matrix whose determinant is non-zero, such a matrix is called as a non-similar matrix.

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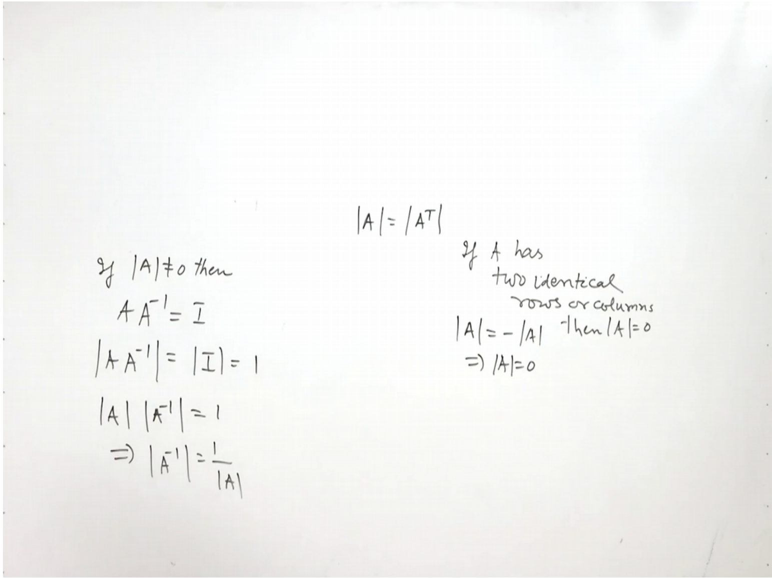

So, if determinant of a is non-zero, then the inverse of A, determinant of inverse of A we can find, so if determinant of A is not equal to 0, then A A inverse we know that A A inverse is equal to identity matrix. So, determinant of A A inverse will be equal to determinant of i, i is a unit matrix, determinant of i is equal to product of its diagonal elements.

So, it is 1 and here we have determinant of A into A inverse equal to determinant of A into determinant of A inverse. So, this gives you determinant of A inverse equal to 1 upon determinant of A, adding a scalar multiple one row or one column to another row or to another column does not change the value of the determinant.

Now, minus and co factors let a b a square matrix of order n and m i j with the matrix obtained from a by deleting the its ith row and jth column, then the determinant of m i j is called the minor of the element of a i j of a and the co factor c i j of a i j is the signed minor, that is c i j is equal to minus 1 to the power i plus j into determinant of m i j.

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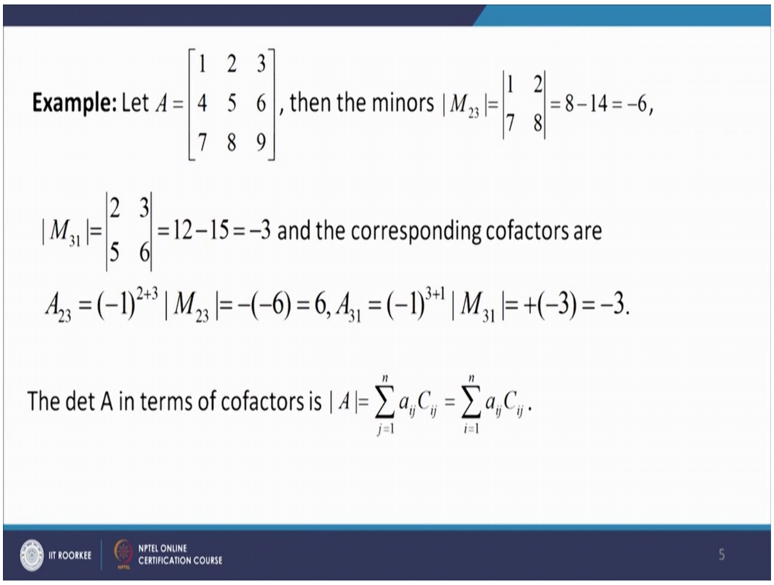

So, this is known to us and then let us see for example, we can take the matrix A equal to one this one.

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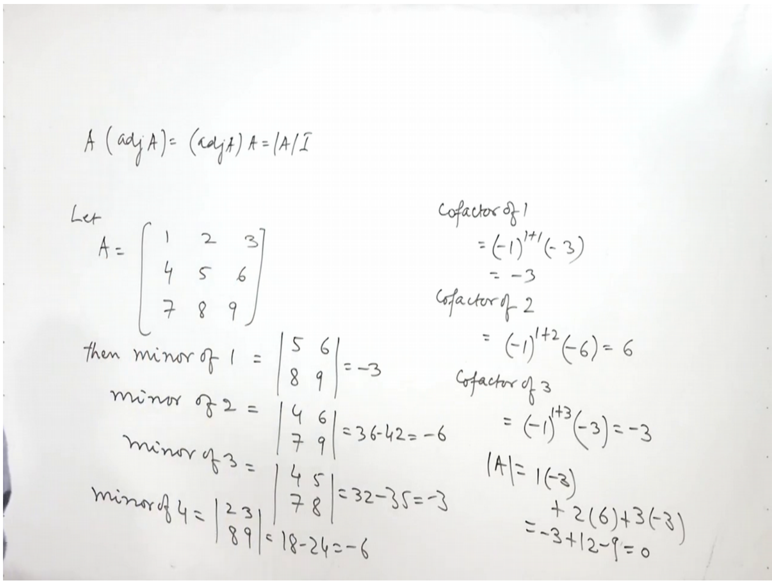

So, let a b equal to 1 2 3 or 4 5 6 7 8 9. So, then let us find the minor then the minor, minor of 1 minor of 1. So, the minor of 1 will be found by deleting the first row, first column, this will be equal to determinant 5 6 8 9, which is 9 into 5 45 45 minus 48 so minus 3. Then similarly minor of 2, so it will be found by deleting second column and the first row. So, 4 6 7 9, so this will be equal to 9 into 4 36, 36 minus 42. So, this will be equal to minus 6 ok.

So, similarly we can determine the minor of 3, minor of 3 will be delete the third column first row, so 4 5 7 8. So, this is 8 into 4 that is 32 minus 7 into 5 35. So, we get minus 3 ok. Similarly, we can find the minor of 4. So, delete the second row first column, so 2 3 8 9 and what we get is minus 6 ok. So, this way we can go on when we find the minor of 5, we delete the second row the second column and the determinant of 1 3 7 9 will give the minor of 5.

Now, then we multiply these minus by minus 1 to the power i plus j. So, here the co factor of 1 will be now the element one occurs in the first row and first column so minus 1 to the power 1 plus 1 into minus 3.

So, this is minus 3, then co factor similarly of co factor of 2. So, minus 1 to the power 2 occurs first row second column. So, 1 plus 2 into its minor that is minus 6, so this we get as 6 and co factor of 3, we can find co factor of 3 is minus 1 to the power first row third column so1 plus 3 into its co factor that is equal to minus 3.

So, we can write it minus 1 to the power 4 that is this is minus 3 and we know that the when the co factors of a certain row, I mean of any row the elements of any row are multiplied by its co factors the their sum gives the determinant of the matrix.

So, determinant of A, we can write determinant of A will be 1 into its co factor that is 1 into minus 3 plus 2 into its co factor. So, minus 3, 3 into its co factor sorry this is 6 here. So, 1 into its co factor that is minus 3 2 into its co factor that is 6, 3 into its co factor which is minus 3, what we get is minus 3 plus 12 minus 9.

So, what we get is we are getting 0 here. So, this must be a singular matrix, alright because right. So, this is how we can find its determinant and when we want to find the adjoint of the matrix, in a similar manner like we are found here minus we found the minors of we find the minors of all the entries of the matrix and their corresponding co factors write the co factor matrix and then take its transpose to get the adjoint matrix.

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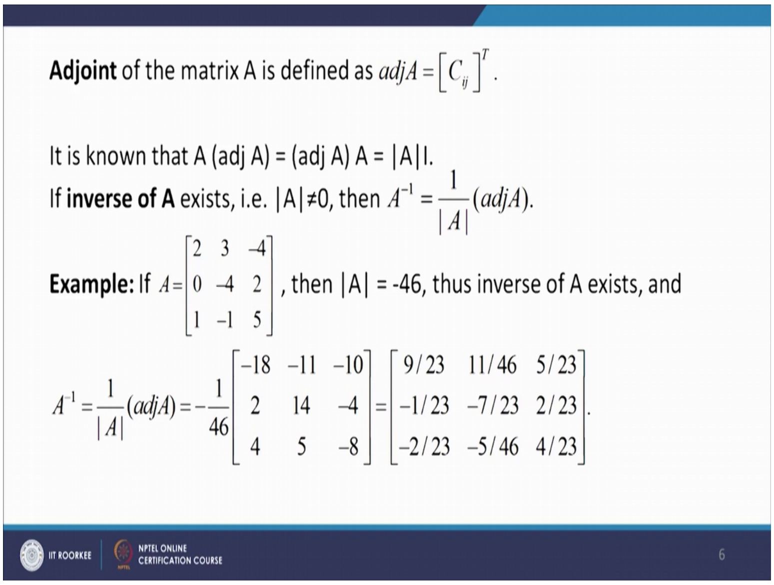

Now, we go over to adjoint of the matrix, adjoint of the matrix as I said is the transpose of the matrix of co factors. Now, we know that it is well known fact that, A times adjoint of A equal to adjoint of A into A equal to determinant of A into I, this is very simple to prove let us see how we can prove this.

Suppose A is this one let us see to prove this let a b equal to a 1 1, a 1 2, a 1 n, a 2 1, a 2 2, a 2 n and so on a n 1, a n 2, a n n ok. Let us write the matrix of co factors the matrix co factors of A. So, co factor of a 1 1, lets write c 1 1, c 1 2, co factor of a 1 2 and so on, c 1 n and then c 2 1, c 2 2, c 2 n and lastly c n 1, c n 2, c n n ok.

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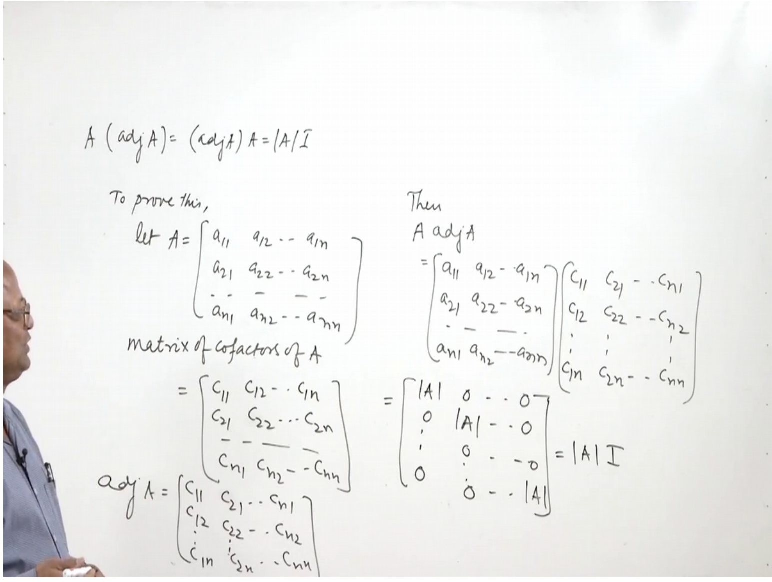

So, multiply then let us find adjointment matrix. So, adjoint of A is the transpose of this matrix. So, this is c 1 1, c 1 2, c 1 n, then c 2 1, c 2 2, c 2 n and lastly c n 1, c n 2, c n n. Now, let us find a times adjoint of A. So, then a times adjoint of A, will be equal to this one so c 1 1, c 1 2, c 1 n, c 2 1, c 2 2, c 2 n, c n 1, c n 2, c n n ok. So, let us take first column of this matrix the adjoint matrix and multiply to all the rows of the matrix A. So, then what will get a 1 1 into c 1 1, a 1 2 into c 1 2, a 1 n into c 1 n 1 so; that means, the row of the row the first row of A is multiplied by its co factors.

So, it will give you determinant of A, then a 2 1 into c 1 1 a 2 2 into c 1 2, a 2 n into c 1 n, that is second row is multiplied by the co factors of the first row. So, and then we are adding. So, that will give you 0, and similarly when you multiply the remaining rows by the first column you are multiplying the rows of the matrix A, other than the first one by the co factors of the first row.

So, they will all give you 0s and then we go to the second column, second column when we multiply to the first row will give you 0, because second column consist of the co factors of the second row of A. So, then it is when multiplied to the first row of a gives you 0, and second row of A is multiplied by the co factors of the second row will give you determinant of A and so on, 0 0 0 and similarly, you can multiply other columns what you get is this.

So, this is a scalar matrix because all the diagonal entries are same determinant of A. So, I can write it as determinant of a times identity matrix ok.

Similarly, if you write adjoint of A times A you will get determinant of A into I. So, adjoint of a when multiplied to A, it gives you determinant of A into I. So, this is a well-known result so if we know that a determinant of a is non-zero, then what we will get, let us divide by determinant of A in this equation.

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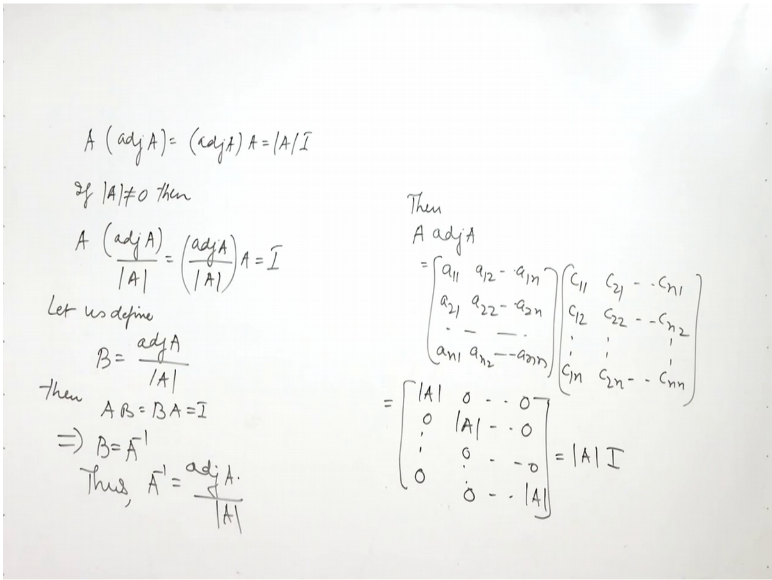

So, if determinant of a is non-zero, I can divide this equation by determinant of A. So, A times adjoint of a will have this ok. Now, let us define, B equal to adjoint of A divided by determinant of A, then we have A B equal to B A equal to I, and hence by the definition of inverse of a matrix ok.

So, B is equal to A inverse, thus we have the formula A inverse is equal to adjoint of A, divided by determinant of a provided determinant of a is non-zero. So, here we are giving an example A is a 3 by 3 matrix you can find the determinant of A it is minus 46 and therefore, its inverse adjust.

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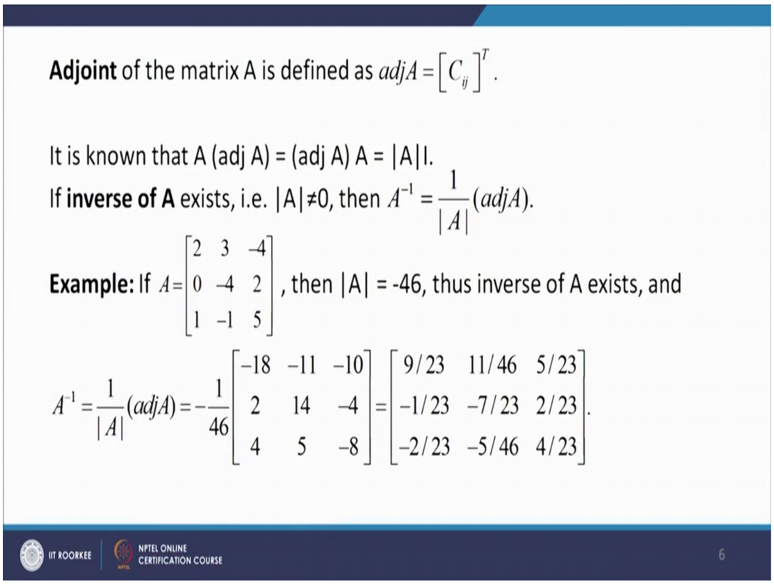

And then we have found the adjoint of A, which is the transpose of the matrix of co factors and then divided it by its determinant which is minus 46 and. So, one what we have is minus 1 over 46 times this, 3 by 3 matrix minus 18, minus 11, minus 10, 2, 14 minus 4, 4, 5 minus 8 and when we multiply minus 1 by 46, to all the elements of the matrix this 3 by 3 matrix what we get is the matrix this one. 9, 9 by 23, 11 by 46, 5 by 23, which is the first row and then we have similarly the second and third rows.

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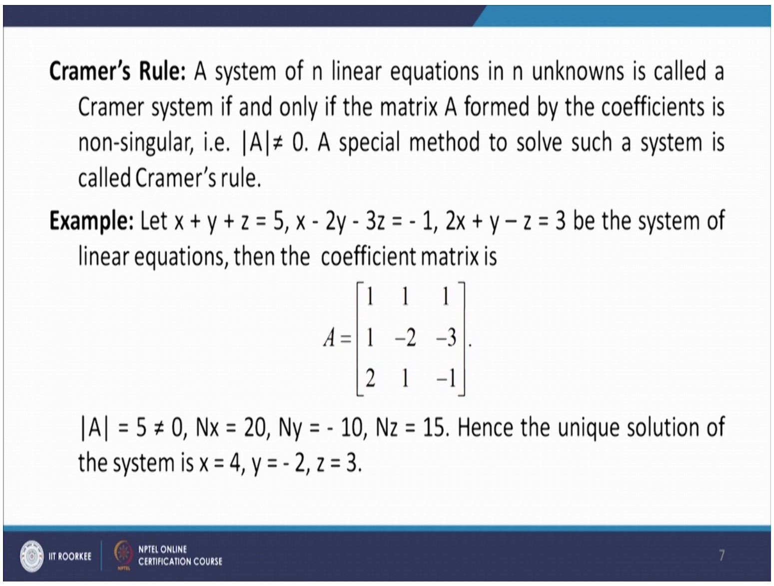

So, we get the inverse of the matrix like that and then we have the Cramers rule, if we have a system of n linear equations in n unknowns then it is called a Cramers system if and only the matrix of A found the matrix A formed by the co efficient is non-singular so let us look at this; so suppose we have this matrix we have.

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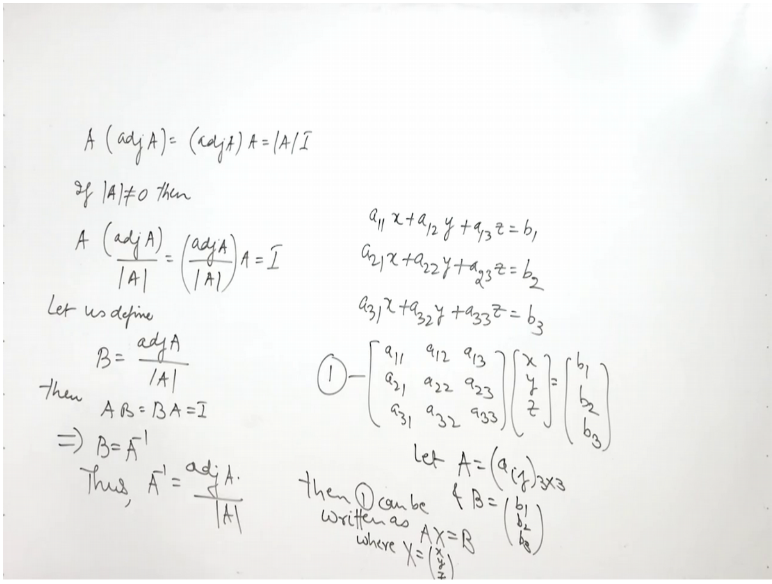

Suppose 3 equation A, I can write them as a 1 1 x plus a 1 2 y plus a 1 3 z equal to see b 1 a 2 one x a 2 2 y a 3 3 z a 2 3 a 2 3 z equal to b 2, a 3 1 x a 3 2 y, a 3 3 z equal to b 3. So, I am writing 3 equations in 3 unknowns, the unknowns are x by z the co efficient of x by z, I have written as a 1 1 a 2 2 and a 3 3. So, that we can identify the position of any element ok so this is nothing, but I can write it in the form of a matrix equation a 1 1, a 1 2, a 1 3, a 2 1, a 2 2, a 2 3, a 3 1, a 3 2, a 3 3 into the column matrix, x y z equal to you can see if you multiply the this column this is a column matrix it is consisting of only one column.

When you multiply this column to this first row you get a 1 1 x plus a 1 2 y plus a 1 3 z and then second row will give you a 2 1 x plus a 2 2 y plus a 2 3 z third row will give you a 3 1 x plus a 3 2 y plus a 3 3 z which is equal to b 1 b 2 b 3 the 2 matrix are equal if their corresponding elements are equal. So, when you multiply first row by first column here, you get a 1 1 x plus a 1 2 y plus a 3 1 z which will be equal to a 1 b 1.

So, we will get first equation, similarly we will get second and third equations. So, this system of 3 linear equations in 3 unknowns can be expressed in the form of a matrix equation this is matrix equation this I if this matrix I denote by A ok. So, let A B equal to a i j 3 by 3 and b equal to the column matrix b 1 b 2 b 3, then I can write this equation let me call it as equation 1.

So, then A 1 equation, equation 1 can be written as A X is equal to B, where X is the unknown column vector x y and z ok. This is X this is A this is B, this a is called as the co efficient matrix of the system of equations, now this system is called as the Cramer system, if determinant of the co efficient matrix is non-zero.

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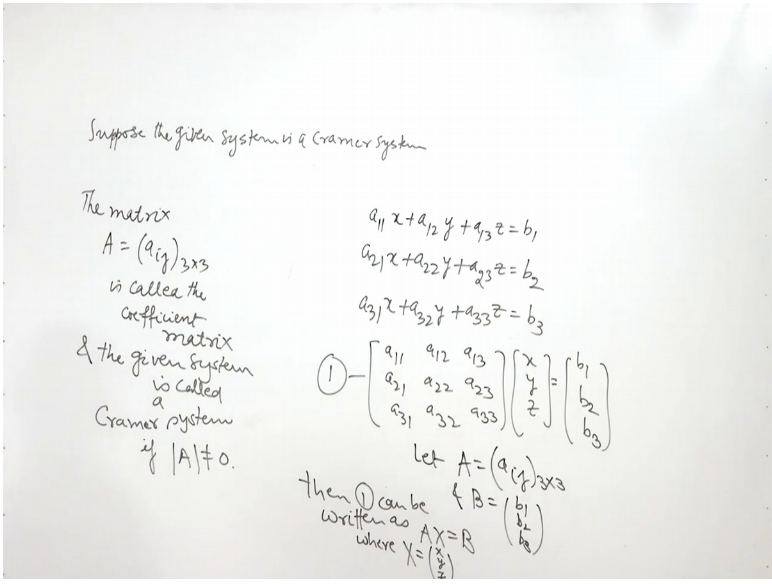

So, the matrix a is called the co efficient matrix, because it is found from the co efficient of the unknowns and the given system is called a Cramer system. If determinant of a is not equal to 0. So, when a is the co-efficient matrix in veritable will call it as a Cramer system. Now and when it when this system is a Cramer system, it can be solved by the Cramer's rule. So, let us see how we solve this so suppose the given system is a Cramer system.

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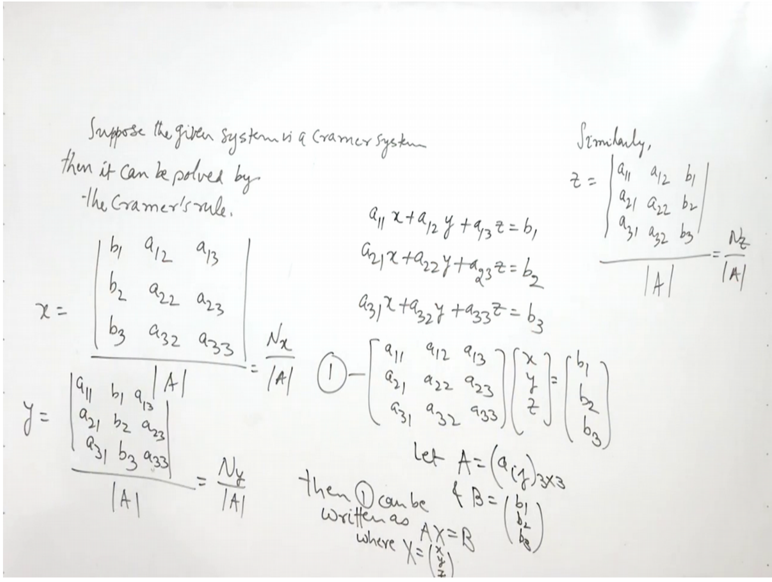

Then it can be solved by the Cramer's rule ok. The Cramer's rule says that the value of x, x is given by so in this matrix A, replace the first column by the column of constants ok. So, and then take the determinant. So, we have first row first column of the a matrix a is replaced by the column of constants b 1 b 2 b 3, remaining columns remain the same then take the determinant ok. So, replace the first column of the matrix A by the column of constants and then take the determinant divided by determinant of A it will give you the value of a x

Similarly, y is given by replace the second column of the matrix A by the column of constants. So, and take the determinant, so a 1 1 a 2 1 a 3 1, then b 1 b 2 b 3 and then we have a 1 3 a 2 3 a 3 3 determinant of this divided by determinant of a this will give you value of y and z ok, z similarly replace the third column of the matrix a by the constants. So, a 1 1 a 2 1 a 3 1, then we have a 1 2 a 2 2 a 3 2 and then we have b 1 b 2 b 3 divided by determinant of a this will give you the value of z.

So, in this so we have given an example where the questions are x plus y plus z equal to five x minus 2 y minus 3 z is equal to minus 1 2 x plus y minus z is equal to 3 and the co efficient matrix will there then be the first row will be the co efficient of x by z in the first equation they are 1 1 1. So, we have first row of a as 1 1 1, and then the second equation then the second equation the co efficient of x y z are five minus 1 minus 2 minus 3. So, they form the second row of A, and then the third row of A is the co-efficient of x y z in the third equation so 2 1 minus 1.

So, you can find the determinant of a here, it comes out to the 5 which is non-zero and when you find the n x n x I mean here, N x like this, this is N x this is N y divided by mode of A and this is N z divided by mode of A.

So, we can find Nx the numerator of x N x comes to be 20 the numerator of by that is N y comes out to be minus 10, and the numerator of z that is N z comes out to be 15 and. So, the values of x y z are N x divided by mode of a which will give you x equal to 4 N y divided by mode of A, which will give you the value of by as minus 2 and z divided by mode of a will give you the value of z as 3 and the solution of the matrix equation are the given system is unique and that solution is the 1 given by these values of x y z.

So, in the case of a Cramer's system we get a unique solution, now let us look at another example.

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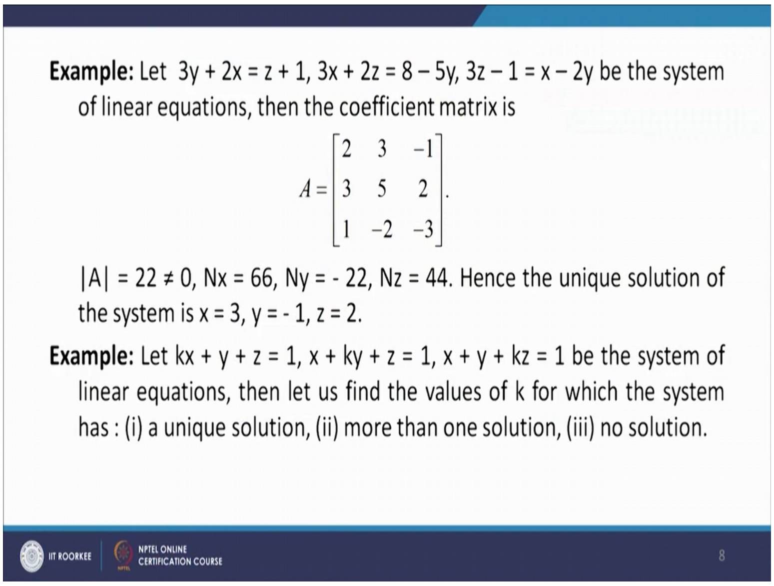

Where the questions are 3 i plus 2 x plus z equal to z plus 1 so, we have 3 y plus 2 x equal to z plus 1.

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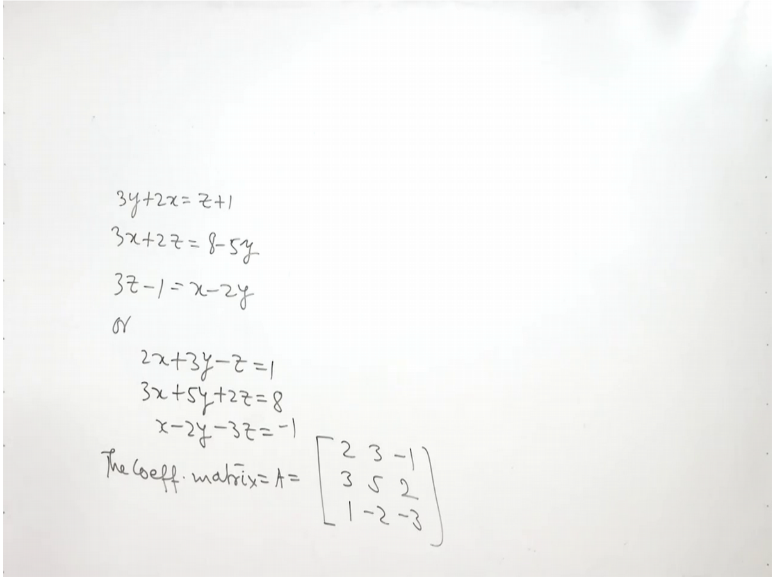

And then we have three x plus 2 z equal to 8 minus 5 y, third equation is 3 z minus 1 equal to x minus 2 y. So, first of all in order to solve the system of equation we have to write it in the standard form ok.

Standard form means we first write the first equation we write as a 1 1 x plus a 1 2 y plus a 1 3 z equal to b 1 in that form. So, we write it as or we write it as first we write the term in x. So, 2 x the term in y 3 y then the term in z so minus z equal to 1, similarly first we write here term in x term in y and the term in z equal to constant, similarly here so x minus 2 y minus 3 z equal to minus 1 and we can find the determinant the co efficient matrix here is 2 3 minus 1 3 5 2 1 minus 1 minus three the determinant is 22 which is non-zero ok.

So, it can be solved by the Cramer's rule and as we discussed in the previous example N x, N y, N z can be found when we divide them by determinant of a we get these unique solution of the system, and the solution is x equal to 3 y equal to minus 1 z equal to 2. Now, let us take the last example of this lecture so we have the following. So, let us consider this let k x plus y plus z equal to 1.

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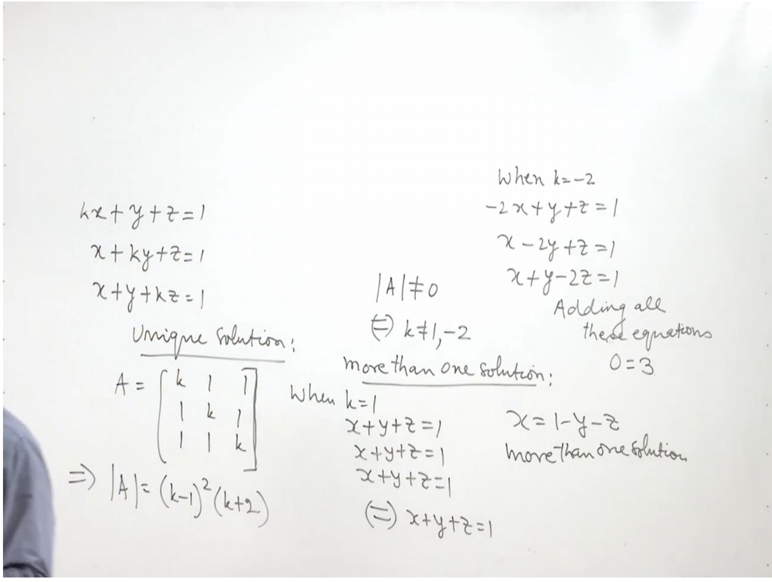

Let k x plus y plus z equal to 1 then we have x plus k y plus z equal to 1 then we have x plus y plus k z is equal to 1 ok.

So, we have 3 equations, we have a system of 3 equations they are linear equations let us find the value of k for which the system has a unique solution, when we have more than one solution and then we have no solution. So, unique solution, so we know why the Cramer's rule the unique solution exist provided the determinant of the co efficient matrix is non-zero.

So, here co efficient matrix A is k 1 1, then 1 k 1 then 1 1 k this co efficient matrix. So, let us find its determinant. So, determinant of A comes out to be k minus 1 whole square you can check it that it is k minus whole square into k plus 2 ok So, for unique solution to exist determinant of a must not be equal to 0. So, determinant of A is not equal to 0 means that k is not equal to 1 and minus 2 ok.

So, k can take any value other than 1 and minus 2 for determinant of A to be non-zero and the value of values of x y z can be found. So, the values of k for the unique solution to adjust or any value of k other than 1 and minus 2, now let us go to the second case more than one solution suppose we take.

So, there are two possibilities 1 and minus 2, let us see which gives the more than 1 solution. So, k equal to 1 let us take k equal to 1 what we have x plus y plus z equal to 1 ok. Then second equation is x plus y plus z equal to 1 third equation is x plus y plus z equal to 1. So, all the 3 equations are same ok. So, the 3 equations reduce into 1 equation, we have 1 equations and we have 3 unknowns. So, we can write it as x equal to 1 minus y minus z taking y and z arbitrarily, we can determine the corresponding value of x.

So, there are 2 independent variables y and z and there is 1 dependent variable x. So, by varying the values of y and z we can get more than 1 solution ok So, here we have more than 1 solution or we can say infinite many solutions ok. So, for k equal to 1 we get more than 1 solution let us, see what happens 1 k is equal to minus 2. So, when k equal to minus 2, this the given system becomes minus 2 x plus y plus z equal to 1, then x minus 2 y plus z equal to 1 and then we get x plus y minus 2 z equal to 1.

So, the 3 equations become like this, now when we add all the 3 equations what we get adding all these equations what we get is minus 2 x plus x plus x. So, 0 y minus 2 y plus y 0 z minus 2 z plus 2 z equal to 0. So, 0 equal to 3 which is not possible and therefore, when k is equal to minus 2 the system has no solution ok. So, with that I would like to conclude my lecture.

Thank you very much.