Numerical Linear Algebra
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Lecture – 17
Gram Schmidt Orthogonalization and Orthonormal Bases

Hello friends. Welcome to this lecture. In this lecture, we will discuss the method known as Gram Schmidt orthogonalization method. Because in next lecture, if we recall, we have discussed the concept called orthogonal set and we have seen certain properties of orthogonal set. So, whenever we have the possibility, we always try to use orthogonal basis rather than standard rather than any given basis.

So, in this lecture, what we try to do here? We try to find out orthogonal basis from a given basis. So, it means that given a linearly independent vectors, we try to find out a new set of vector having the same number of element and with the additional property that; now the additional set of factors is orthogonal rather than only linear independent. So, that is the content of this gram Schmidt organization process.

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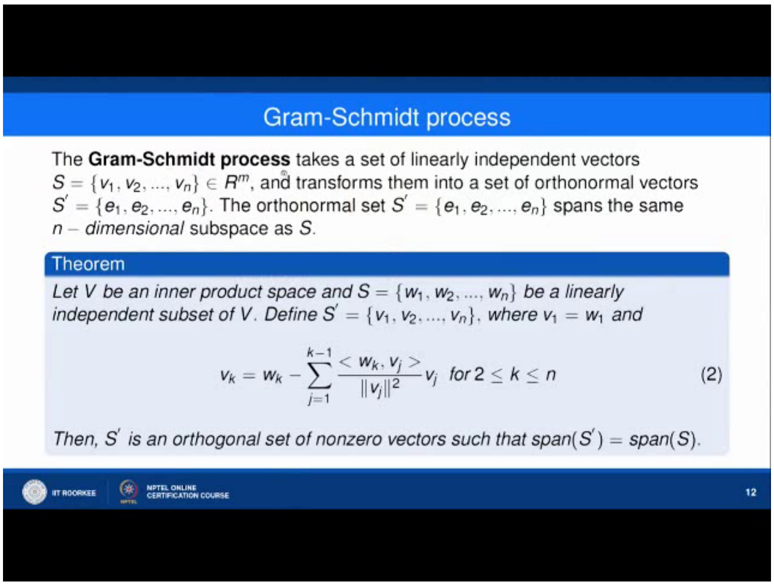

So, the Gram Schmidt orthogonalization process takes a set of linearly independent vectors S. So, let us say that we have a set S be consisting these n element v 1 to v n. Now these S is basically some linearly independent set of R m and transform this set S into a new set of vectors as S e 1 to e n where e 1 to e n's are orthogonal vectors and norm of each vector is 1. So, it means that S dash is going to be orthonormal vectors.

So, that is the use of gram Schmidt organization process and the property it retained that if S is span certain subspace then S dash will span the same subspace. So, it means that the orthonormal set S dash span the same n dimensional subspace as S. So, S generate n dimension vector subspace of R m.

Similarly S dash will also generate the same n dimensional subspace as S. So, let us consider the statement of this theorem. So, statement of theorem goes like this that. Let V be an inner product space and S is given as w 1 to w n be a linearly independent subset of V. So, it means that S is a linearly independent subset given here. So, it means that S generate a n dimension vector subspace of V.

Now, we want to define S dash as v 1 to v n where v 1 is nothing, but w 1 and v k's are given as w k minus summation j equal to oh; 1 to k minus 1 inner product of w k with v j divided by norm of v j square into v j for all k from 2 to n. So, it means that if we define your v i's like this, then this S dash is an orthogonal set of nonzero vectors such that span of S dash is same as span of S and once we have a set of orthogonal set of orthogonal vectors, then we can divide by norm of each norm of respective quantity and we can make this S dash S set of or set of orthonormal vectors.

So, the idea behind this theorem is this that.

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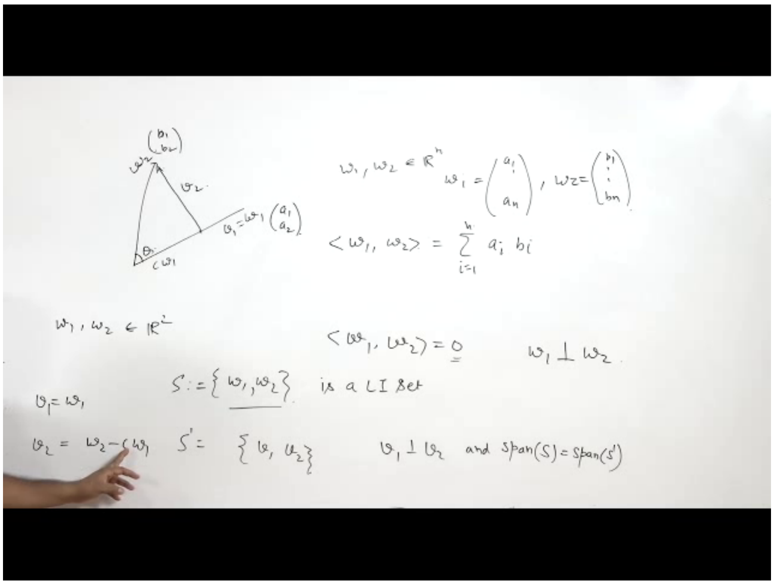

Suppose we have 2 vector say call it w 1 and say w 2 here, right. So, here w 1 and w 2 both are vectors int say R n or R m whatever the space is then we have define what is inner product of w 1 w 2 that we know how to define it depending on the coordinates here. So, if w i suppose we can write it w 1 w one as say a 1 to say a n and w 2 as b 1 to say b n then inner product of w 1 w 2 is given as a i b i; i is equal to 1 to n.

Now, we have define the concept orthogonality is that w 1 is orthogonal to w 2, if is this inner product is coming out to be 0. So, what is exactly meaning of this orthogonal orthogonality of w 1 and w 2 that with the help of this inner product, we can define the notion of angle.

So, it means that whenever we say that w 1 is orthogonal to w 2, it means that the angle between w 1 and w 2 is 90 degree or we can say that w 1 is perpendicular to w 2. So, if we can represent w 1 on. So, here since our physical constraint allow us to represent only elements of R 2. So, here let us consider this that w 1 and w 2 element of R 2. So, this represents some a 1 a 2 and this represents some b 1 and b 2. So, these 2 represent 2 vectors here and if this angle theta which we denote if this angle is a 90 degree, then we say that w 1 is orthogonal to w 2.

So, now that is what is what we mean by orthogonal orthogonalily of w 1 and w 2. So, now, we how to obtain a new set of vectors from w 1 and w 2 as v 1 and v 2 that is the content of theorem that we need to find out; out of these w 1 w 2 it is already given that it is a LI set what we want to know is we want to find out a new set of vectors say v 1 and v 2 such that v 1 is orthogonal to v 2 v 1 is orthogonal to v 2 and a span of your this call it this S and call this as S dash. So, span of S is same as span of S dash.

So, for this, we always take v 1 as w 1. So, the first factor is written as it is, we want to find out v 2 as linear combination of w 1 and w 2 in a way that this v 2 is orthogonal to v 1. Now how to find out this thing, then if you look at here v 1 is nothing, but w 1. So, it means that if you look at w 2 here, then w 1 is same as v 1, then we want to look at this w 2, then w 2 can be written as some same c of u c of w 1, right some vector here and a vector which is orthogonal to this and we say that whatever we obtained is nothing, but your v 2. So, v 2.

So, w 2 can be written as c w 1 plus w 2 and v 2 is something having angle 90 degree with w 1 or we can say 90 degree with vector v 1 here. So, we want to find out the c w 1 that if we know this c w 1 and we did we subtract this c w 1 from w 2, we can get our v two. So, it means that our v 2 is going to be w 2 minus component along this vector w one. So, it means that in w 2 if we reduce the component which is along w 1 then we will have only component which is perpendicular to w 1. So, that is the concept of this orthogonalization process.

So, we want to find out this c such that this v 2 is orthogonal to this v one. So, how to find out this c let us see. So, here what we have shown here that w 2 can be written as c union sum of c w 1 and v 2 and you want to find out this v 2 which is perpendicular to this v 1 or we can say perpendicular to w one. So, we want to find out contribution of w 2 along w one. So, we want to find out this c says that v 2 and v 1 are orthogonal to each other.

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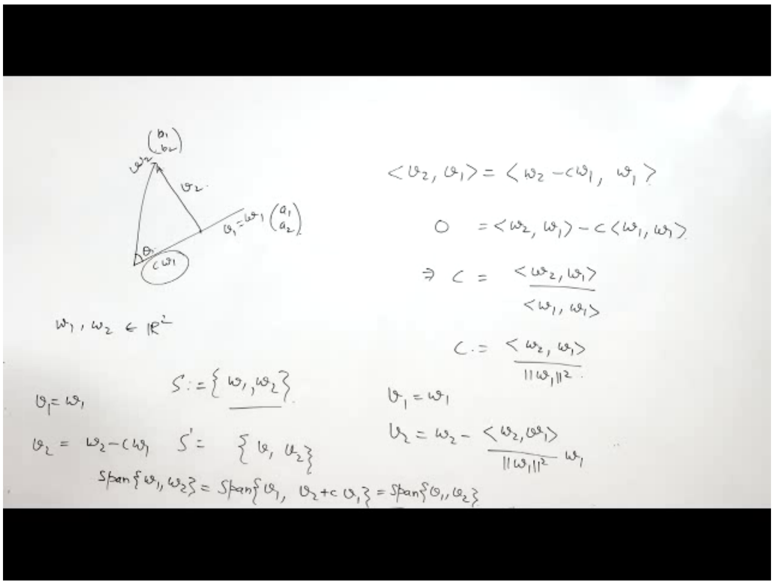

So, consider v 2 with v 1 which is nothing, but w 2 minus c w 1 with v 1 which is nothing, but w 1. So, this is written as inner product w 2 with w 1 minus c inner product of w 1 with w 1 here and this is going to be 0 because we want that v 2 is orthogonal to v 1. So, this is going to be 0. So, this implies that here c is going to be inner product of w 2 with w 1 divided by inner product of w 1 with w 1. So, this is nothing, but norm of w 1 square. So, we can write it c as inner product of w 2 with w 1 divided by norm of w 1 square. So, this is your c. So, it means that if v 1 is equal to w 1, then v 2, you can define as w 2 minus c means w 2 inner product with w 1 divided by norm of w 1 square with w 1.

So, it is the process when we consider for 2 vectors. So, what we have done here we have subtracted the component from w 2 which is along the vector w 1. So, it means that now this v 2 will have no component along w 1, it means that the component left in as v 2 as perpendicular to v 1. So, angle between v 1 and v 2 is going to be 0 here. So, we are simply subtracting the part which is along the vector w 1 that is all.

So, that is how we are going to prove this theorem now regarding this span thing that the span of w 1 and w 2 is same as span of v 1 and v 2. So, you can say that span of w 1 and w 2 is basically what span of w 1 I can write at write as v 1. Now w 2; how we can write it w 2 we can write it as v 2 plus c w one. So, we can write it v 2 plus c w 1. Now w 1 is nothing, but v 1.

So, it means that span of w 1 and w 2 is nothing, but a span of v 1 comma v 2 plus c v 1. Now this is same as writing a span of v 1 and v 2. So, it means that by this process, there is no change in a span. So, it means that a span of original set is same as span of the new obtained set. So, it means that a spanning set is having no problem. So, it means span of S dash is same as span of S.

So, let us consider the proof of this theorem.

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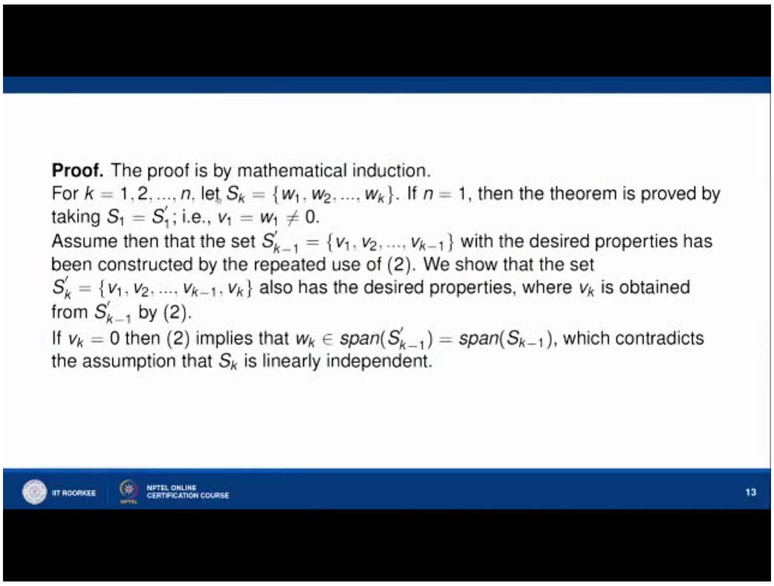

And this proof is can be done by mathematical induction. Now induction we are running over the number of vectors available on in a given set S. So, let us say for k equal to 1 to n let S k is define as w 1 to w k. So, for you take k equal to one then S 1 is defined as w 1 if you take k equal to n your S n is defined as w 1 to w n which is your set S now if n equal to one then this theorem is trivially true because for k equal to one S 1 is w one. So, S 1 is w 1. So, your S dash is again same as w one. So, here S 1 is same as S 1 dash and a span of S 1 is same as span of S 1 dash. So, it means that our theorem is done for n equal to one.

Now, we assume that this theorem is true for n equal to k minus one. So, assume then that the set S k minus 1 dash equal to v 1 to v k minus 1 with the desired property has been constructed by the repeated use of two. So, it means if you follow and the procedure given here and for k equal to one we have already seen for up to k minus 1 we have already constructed we want to show that for k plus one it is also true. So, we want to show that the set S k dash which is consisting v 1 to v k minus 1 comma v k also has a desired property. So, we have assumed that for n equal to k minus 1 theorem is true we want to show for n equal to k theorem is prove theorem is true. So, we want to show that S k dash has the desired property.

Now, let us say that this v k is 0 or not. So, if v k 0, then this S k dash is linearly dependent and this is not going to be an orthogonal set. So, first we want to avoid the possibility that v k is a never 0. So, if v k 0, then look at your equation number 2 if v k is 0, then w k can be written as linear combination of w k is written as j equal to 1 to k minus 1. So, inner product of w k with v j divided by norm of v j square and v j.

Now, so, it means that w k can be written as linear combination of v j's. So, it means that w k belongs to span of v j j is from one to k minus 1. Now we already know that for k minus 1 a span of S dash k minus 1 is same as a span of S k. So, it means that w k belongs to span of w 1 to w k minus 1 which is a contradiction because we have already assumed that S is a linearly independent subset of v. So, it means that w k cannot be written as a linear combination of w 1 to w k minus 1. So, it means that w k cannot be written as linear combination of v 1 to v k minus 1.

So, it means that this vector v k is never going to be 0. So, it means that if v k is equal to 0, then equation implies that w k belongs to a span of S dash k minus 1 which is same as a span of S k minus 1 which is the assumption we have already made that for n equal to k minus 1 where the theorem is true. So, w k belongs to a span of S k minus 1 which is a contradiction because we have w i's are linearly independent set of vectors. So, it means that v k is never 0.

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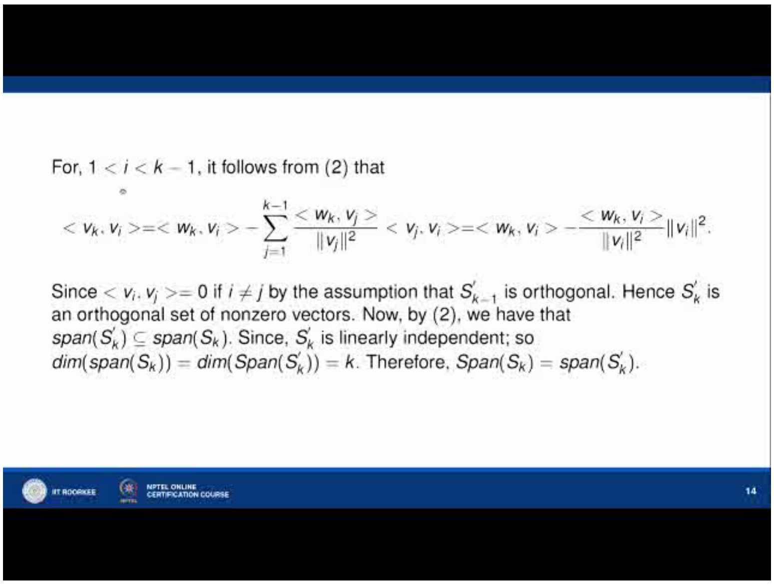

So, if v k is non zero, then take the inner product of this v k with v i where i is running from 1 to k minus 1, what we want to show here that this v k is orthogonal to v i’s i is from 1 to k minus 1.

So, to show that let us find out say inner product this and v k is written as w k minus j equal to one to k minus 1 w k v j divided by norm of v j square v j. So, when you perform the inner product it is what inner product of w k v i minus here we have taken the help of properties of inner product and we can write it like this. So, let us look at here.

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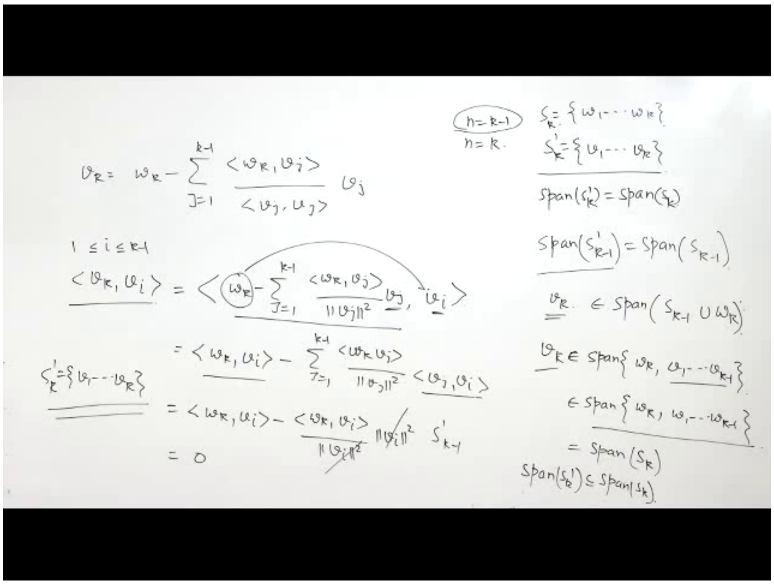

So, we have assumed that theorem is true for n equal to k minus 1 and we want to prove for theorem for n equal to k. So, it means that we want to show that S dash which is v 1 to v k is an orthogonal set of vectors and span of S dash is same as span of S here where S is given by say w 1 to say w k and we already know that for theorem is true for n equal to k minus one. So, here let us use the notation here which we have defined here this k rep represent the number of element and the corresponding set.

So, here we have defined v k as this v k as w k minus j equal to one to k minus 1 inner product of w k with v j divided by inner product of v j with v j in to v j. So, this is nothing, but norm of v j square. So, now, take the inner product of v k with v i where i is from one to k minus 1 with the help of this we want to show that this set S k dash is a an orthogonal set. So, for that taking the inner product here this v k you can utilize this expression here.

So, it is nothing, but an v k, we are writing this. So, w k minus j equal to one to k minus 1 v w k with inner product of w k with v j divided by norm of v j square into v j comma v i. Now here we use the property of inner product. So, this can be written as inner product of w k with v i. So, here we are taking inner product with this and this and minus this remaining element summation j equal to one to k minus 1 w k v j divided by norm of v j square that we can take out because this is nothing, but constant term. So, this we can take out take out and we have left is inner product of v j with v i.

So, here now j is from one to k minus 1 and i is also from one to k minus one. So, it means that this both v j and v i are element of S dash k minus one. So, hey here these are element of S dash k minus 1 and there we already know that S dash k minus 1 is an orthogonal set of vector. So, it means that this v j's and v i's where i and j is running from one to k minus 1 are orthogonal to each other.

So, it means that they will be this will be nonzero only when j is equal to y for all other thing it is going to be 0. So, this implies that this is nothing, but w k with the v i minus only j equal to i is left. So, it means that for j equal to y it is what w k v i divided by norm of v i square and here what is left here inner product of v i with v i which is nothing, but norm of v i square. So, that these 2 things are cancelled out because we already know that v i's are nonzero. So, we can cancel and if you look at this nothing, but w k v i inner product of w k v i minus inner product of w k v i which is going to be 0.

So, it means that v k v i is equal to 0. So, it means that this S k dash v 1 to v k is an orthogonal set. So, this we have proved that S k dash is an orthogonal set now what is left here that span of S k dash is equal to span of S k, but we already know that span of S k dash S k minus 1 dash is equal to span of S k minus 1. So, it means that only thing we want to show that v k can be written as v k belongs to span of S k minus 1 union w k which is which is trivially written here because if you look at v k v k is what v k is written as w k and linear combination of v j's.

Now, these v j's are coming from where v j's are coming from as span of S k minus 1. So, it means that v k can be written as span of w k and your v i's and span of. So, it means that here we can say that by equation number this to v k belongs to the span of w k and v 1 to v k minus 1. Now this is what this belongs to span of. So, it means that this belongs to span of w k and span of w v 1 to v k minus 1 is nothing, but a span of w 1 to w k minus 1. So, this means that this is nothing, but a span of this is nothing, but a span of S k. So, it means that v k belongs to a span of S k. So, it means that your span of S dash k minus 1 is already in span of S k minus 1. So, this is also belongs to. So, which means that span of S k dash is containing a span of S k, right. Now we already know that both has the dimension k because it is linearly independent. So, the span set has a dim vectors subspace which is generate is having dimension k. Similarly this is an orthonormal orthogonal set of vectors. So, it is also linearly independent. So, the dimension of vector subspace which is which it generate is also having dimension k.

So, it means that that dimension of span of S k is same as dimension of span of S k dash which is nothing, but k. So, it means that here not only containment is true, but equality is true. So, it means that span of S k is same as span of S k dash. So, it means that with the help of this process we are able to find out from a given linearly independent set we can find out a orthogonal set out of it. So, take the certain example of this process and let us see.

So, first example is that in R 4 let S having these 3 vectors w 1 to w 3 where w 1 is given as 1 0 1 0 w 2 is 1 1 1 1.

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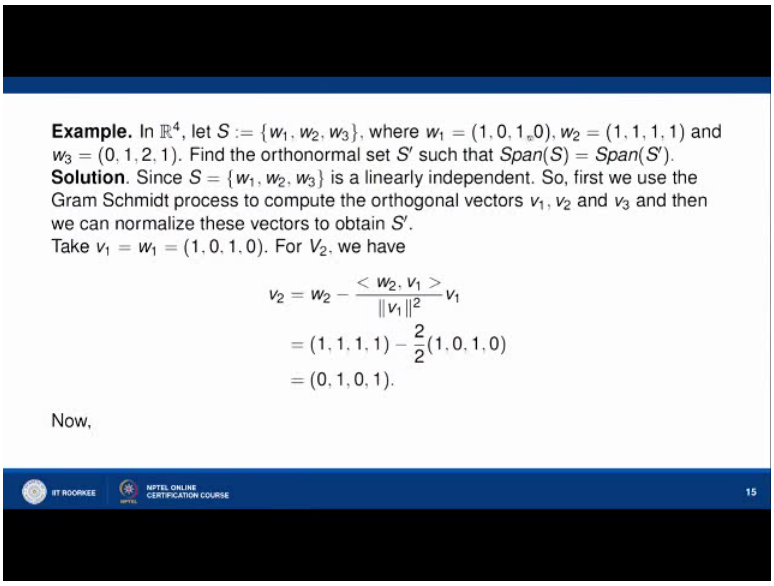

And w 3 as 0 1 2 1. So, here S is a given set we want to find out another set orthonormal set is this says that a span of S is same as the span of S dash. So, here we want to use Gram Schmidt orthogonalization process to find out this orthonormal set S dash. So, here if you look at this set S consisting these 3 element w 1 w 2 w 3 and we can easily check that these w i's are a linearly independent vectors. So, it means that S is given as a linearly independent set. So, it means that we can apply our gram Schmidt orthogonalization process to find out this S dash. So, first we try to find out orthogonal set and then we are normalize a to find out S dash.

So, let us find out v i’s means v i’s are set of orthogonal vectors. So, v 1 as it is as w 1. So, v 1 we can take as w 1. So, v 1 is nothing, but w 1 1 comma 0 comma 1 comma 0 to find out the second vector v 2, we start with w 2 and take the part of w 2 which is in the direction of w 1. So, it means that w 1 is same as v 1. So, it means that we take out the component of w 2 which is in this direction of v 1.

So, that we have shown that it is nothing, but inner product of w 2 with v 1 divided by norm of v 1 square. So, it calculate this inner product w 2 with v 1. So, if you look at w 2 is this v 1 is same as w 1. So, inner product of w 2 and w 1 is basically component wise. So, w 2 inner product w 1 is going to be 1 into 1 plus 1 into 0 plus 1 into 1 plus 1 into 0. So, it is going to be 2 here. So, inner product of w 2 with v 1 is going to be 2.

Now, looking at the norm of v 1 square; so, norm of v 1 is square basically what inner product of v 1 with v 1 itself. So, that is going to be 1 into 1 plus 0 into 0 plus 1 into 1 plus 0 into 0. So, again it is coming out to be 2 here. So, this is 2, this is 2. So, you can say that these 2 will cancel out and v 1 is given at 1 comma 0 comma 1 comma 0. So, here we can subtract this term, this component along w 1 from w 1. So, if you subtract this is nothing, but 0 comma 1 comma 0 comma 1.

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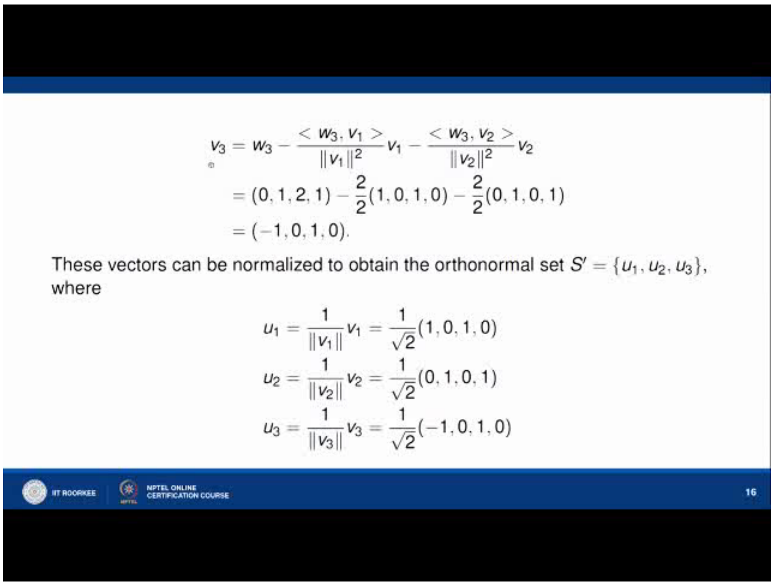

So, v 2 is coming out to be this vector 0 comma 1 comma 0 comma 1. Now to find out v 3 from v 3, v 2 find out v 3, we have to take out the component of w 3 along v 1 and v 2. So, component of w 3 along v 1 is given by inner product of w 3 with v 1 divided by norm of v 1 square and component of w 3 along v 2 is what inner product of w 3 with v 2 divided by norm of v 2 square.

So, we have to calculate these constants and then we are done. So, w 3 is given by 0 1 comma 2 comma 1 and inner product of w 3 with v 1 is basically what w 3 with v 1 is basically what one into 0 plus 0 into 1 plus 2 into 1 and 1 2. So, it is coming out to be w 1 inner product with w 3 is coming out to be 2 here. So, that is written here and norm of v 1 is square, we have already calculated is coming out to be 2 and v 1 is one comma 0 comma 1 comma 0. Similarly, we can calculate the inner product of w 3 with v 2, I am not going to calculate, you may calculate because w 3 is given here and v 2 is given by this. So, you can calculate the inner product of w 3 with v 2 and it is also coming out to be 2 and norm of v 2 is coming out to be 2.

So, if you simplify it is coming out to be v 3 as minus 1 comma 0 comma 1 comma 0. So, with this, we are done with Gram Schmidt orthogonalization process and now from in place of w 1, w 2, w 3, w 1, w 2, w 3, we have v 1, v 2 and v 3 as set of orthogonal vectors now to find out orthonormal set, we want we have to normalize these vectors. So, it means that we have u 1, u 2, u 3 and these are orthonormal set of vectors.

So, u 1 can be obtained as v 1 divided by norm of v 1 u 2 can be obtained as v 2 divided by norm of v 2 and u 3 as v 3 divided by norm of v 3. So, v 1 is given by 1 comma 0 comma 1 comma 0 and norm of v 1 is what inner product of v 1 with v 1 power half. So, inner product norm of v 1 square is coming out to be 2. So, norm of v 1 is coming out to be under root of 2.

So, u 1 is given as 1 upon root 2 1 comma 0 comma 1 comma 0. Similarly u 2 is given as v 2 divided by norm of v 2. So, norm of v 2 is again under root 2. So, this can be written as u 2. So, u 3 is norm of v 3 and you can easily calculate in norm of v 3 as nothing, but under root 2 also; so, u 1 u 2 u 3 given as these vectors. So, now, we have as S given as u one comma u 2 comma u 3 as set of a set of orthonormal vectors. So, that we have obtained from they set S. So, S w 1 w 2 w 3 are simple linearly independent set and what we have achieved here is a new set S dash where vectors u 1, u 2, u 3 are orthonormal set of vectors.

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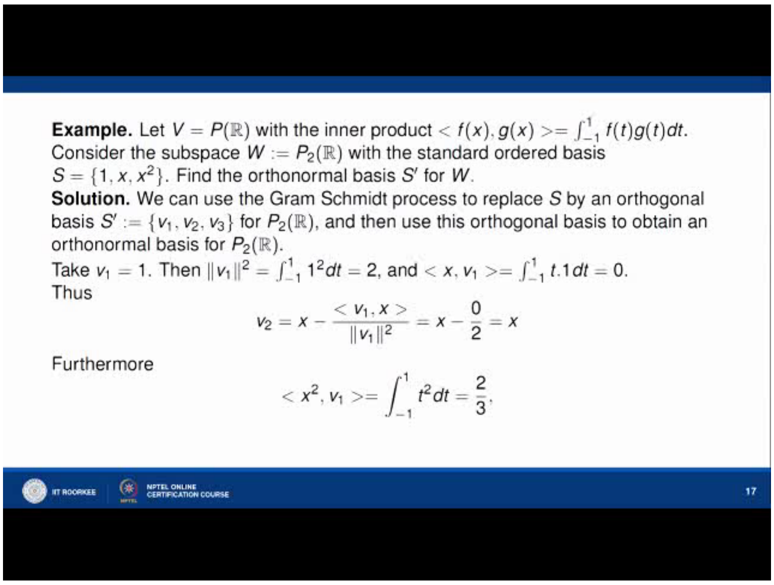

Now, moving on next example we have a vector space which is a set of all polynomials with inner product this f inner product of f x with g x is given as minus 1 to one f t g t d t. So, this is the inner product defined on this inner product and we want to find out say the standard order basis sorry, orthonormal basis S dash for w where w is P 2 R set of all polynomials whose degree is at most 2 and what is given here it is given that we have a standard order basis one comma x comma x square. So, it means that having this vector subspace which is given as P 2 R we have a standard order basis we want to find out a orthonormal basis for this w.

So, for this also we can use gram Schmidt orthogonalization process. So, we we want to replace this ortho standard order basis by an orthogonal basis as v 1 v 2 v 3 we want to find out what is v 1, v 2, v 3. So, for that; we take v 1 as one and if you look at the norm of v 1 square. So, norm of v 1 square is going to be inner product of v 1 with v 1 itself. So, that is in inner pro definition of inner product is given here. So, inner product of v 1 with v 1 is basically minus 1 to 1; 1 into 1. So, simply it is minus 1 to 1 1 square d t which is nothing, but 2. So, we have norm of v 1 square as 2.

Now, to find out v 2, we subtract from w 2 the component along v one. So, v 1 is one here. So, we subtract the component of x along 1. So, x minus inner component of x along one which is given as inner product of v 1 with x divided by norm of v 1 square. So, inner product v 1 with x is given by this it is what minus 1 to 1 x, now here I am using in the integration variable as t. So, this is nothing, but minus 1 to 1 t into 1 d t. Now if you look at this can be written as 0; so, value of this integral is coming out to be 0; so, it means that a x has no component along one. So, it means that x is already orthogonal to 1 with this inner product. So, it means that v 2 can be written as x itself. So, v 1 is given by one and v 2 is given by x.

Now, to find out v 3 we need to subtract the component of x square along one and x. So, for that we need to find out say inner product of x square with x and inner product of x square with one. So, let us find out say inner product of x square with v one. So, it is given by minus 1 to 1 x square we can write it t square v 1 is one here. So, t square d t and if you simplify this is nothing, but t cube by 3 limit is from minus 1 to one. So, it is given by 2 by 3. So, inner product of x square with v 1 is coming out to be 2 by 3.

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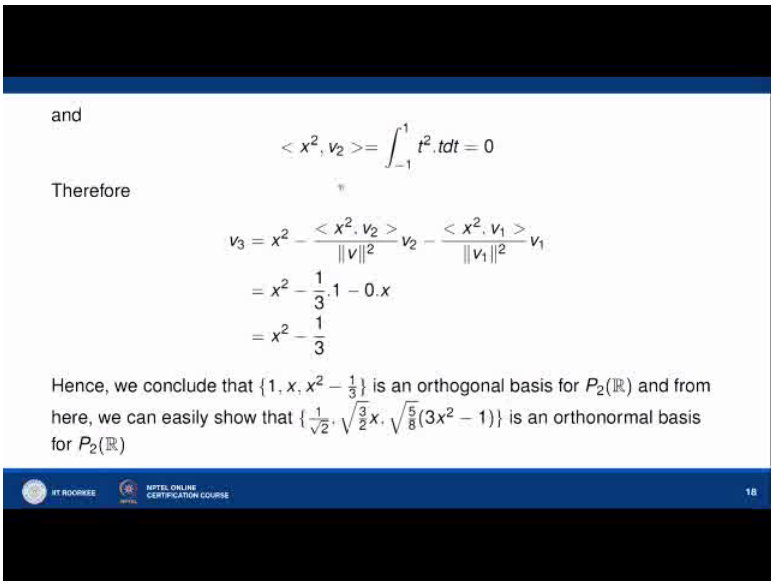

Similarly, we want to find out inner product of x square with v 2 and here v 2 is nothing, but x here. So, that is minus 1 to 1 t square into t d t if you simplify this is coming out to be 0 why because we can use simple property of integral. So, that it is odd function and limit is minus a to a. So, that can be shown that it is nothing, but 0.

So, to find out v 3 v 3 as x square minus component of x square along v 2 and component of x square along v one. So, that we have already calculated x square v 2 is nothing, but 0. So, it means that x square has no component along v 2 and then x square a component of x square along v 1 which we have already calculated 2 by 3 and norm of v 1 square we have seen as 2 here.

So, using this, this is 0. no component 0 dot x and this as 2 by 3 divided by 2. So, it is 1 by 3 into 1. So, here v 3 is coming out to be x square minus 1 upon 3. So, it means that we have this 1 comma x comma x square minus 1 by 3 as an orthogonal basis for P 2 R and from this, we can find out say orthonormal basis how we can find out the orthonormal basis we have to divide by norm of the respective vectors. So, norm of one with the help of this inner product minus 1 to 1 f t g t d t we have shown that norm of v 1 is coming out to be 2.

So, we divide by norm norm of v 1. So, sorry norm of v 1 is going to be 1 1 upon norm of v 1 is one a norm of v 1 is going to be under root two. So, one divided by root 2 is an orthonormal vector. Similarly norm of x we can find out. So, it is coming out to be 2 by root 3. So, we can say that x divided by under root 2 by 3 is root 3 by 2 x. Similarly, we can find out say norm of x square minus 1 by 3 and we can say that under root 5 by 8 3 x square minus 1 is the new orthonormal vector. So, it means that this set one by root 2 comma under root 3 by 2 x comma under root 5 by 8 3 x square minus 1 is an orthonormal basis for P 2 R.

So, that complete the proof that complete the solution of this just one thing since we have calculated the norm of v 1 as this, but we have not calculated the norm of v 2. So, let us just I just want to show that now how to calculate the norm of v 2 and then I will finish because if you know; how to find out the norm of v 2, you can you use a same technique to find out say norm of v 3 and that will complete that job.

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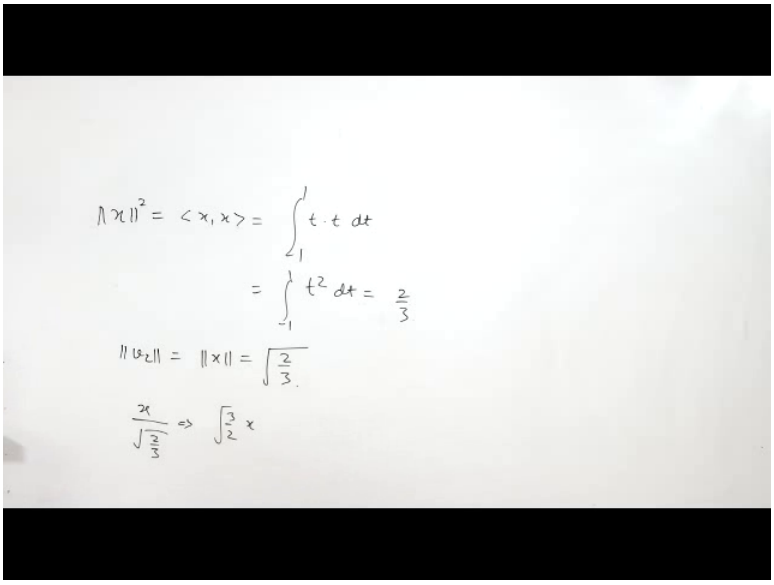

So, first we need to find out say norm of x. So, norm of x square we know that it is nothing, but inner product of x with x and it is nothing, but minus 1 to 1 f t means t, here and again t and d t. So, this is what it is t square minus 1 to 1 d t and it is coming out to be 2 by 3. So, it is t cube basically t cube and minus 1 to 1. So, this is the norm of x square. So, we can say that norm of v 2 is going to be under root norm of x here which is given as num 2 by 3. So, it means that we have x divided by under root 2 by 3.

So, it means that this is nothing, but under root 3 by 2 x is going to be an orthonormal vector here. So, that is what is written here under root 3 by 2 x is the orthonormalization of x here similarly you can find out say norm of v 3 that is x square minus 1 by 3 and this coming out to be under root this 8 by five. So, we can find out a orthonormal basis out of this P 2 R.

So, here we conclude our lecture here. So, what we have learnt from this lecture that given a LI set, we started with LI set say let us say S is you know we have v 1 to some v n it is a LI set.

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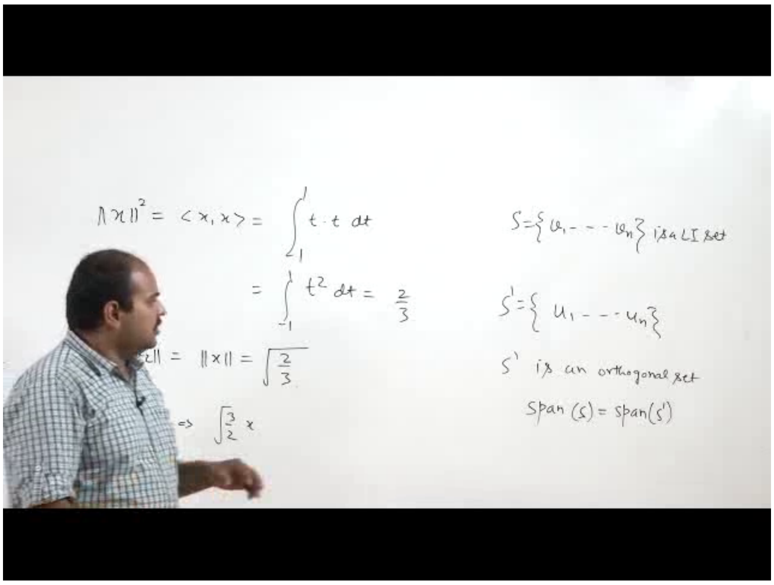

So, from this set, we are able to find out a new set S dash say u one to u 1 to u n says that this S dash is an orthogonal set right and a span of S is same as span of S dash. So, it means that by this process there is no change in span. So, it means that if this represents a basis, then this will also represent a basis. So, here if S is a basis, then S dash is an orthogonal basis for the same vector subspace. So, here we just have wind up and we will meet in next lecture.

Thank you very much.