Numerical Linear Algebra
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Lecture - 16
Orthogonal Sets

Hello friends; are welcome to this lecture. In this lecture, will discuss the concept of orthonormal sets and then, after this is in this orthonormal set will discuss the method known as a gram Schmidt orthogonalization process to obtain orthonormal set from a given set. So, first let us discuss the concept of inner product by which we can define the term orthogonal. So here, we discuss orthogonal and then, orthonormal be set of vectors.

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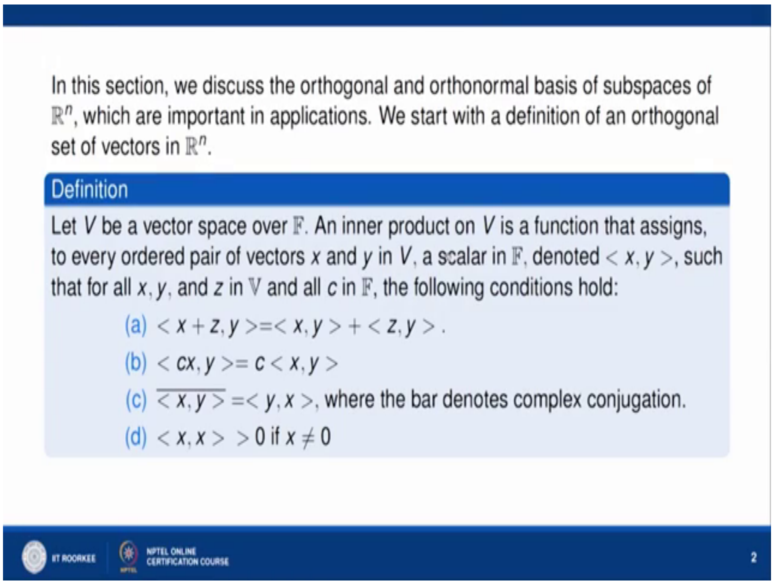

So, first we define inner product. So, let V be a vector space over F F is a scalar field here, it may R or c and inner product on V is a basically a function from V cross V to R plus union 0, that assigned to every order period of vectors x and y in V a scalar in F, which is denoted as in inner product x with y such that, for all x, y and z in V and all the scalars in F the following few conditions hold.

So, first condition says that, inner a product of x plus z with y is equal to inner a product of x with y plus inner a product z with y here. So, this condition shows that this inner a product function is linear with respect to first variable and second condition is that, c x comma y is equal to c inner product of x comma y. So, here these 2 to a and b is simply represent that, our inner product is linear with respect to the first variable, that and then c which sees that conjugated bar or conjugate of inner product of x comma y is equal to inner a product of y comma x where, this bar represent the complex conjugate conjugation and last one is that, inner a product of x with x is greater than 0 if x is nonzero.

So, this is the definition of inner product, which is denoted by this symbol symbol and it satisfies the following properties now, let us consider set an example.

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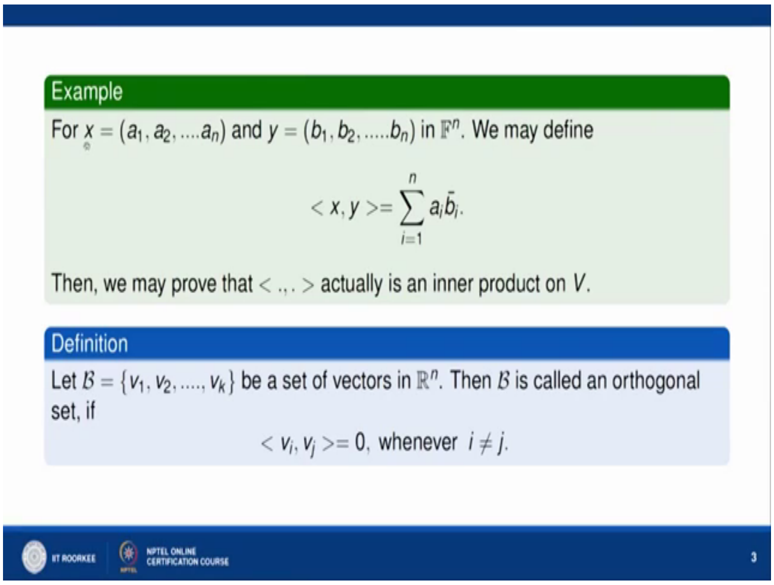

So, let us says x is equal to a1 to a n and y equal to b1 to b n or 2 vectors in F n and then, we may define our inner product x with y a summation i equal to 1 to n a i b i is star. So, here we can verify that, this satisfy the properties listed here listed as ab cd I am not proving it and I am ask you people to verify with that, this is an inner product on V and here if this F is scalar field are then, here we can reduces as this inner product is reduce to i equal to 1 to n a i b i. So, in case of real vector space we have i equal to 1 1 to n a i b i. Now, now with the help of this inner product we try to define what is known as orthogonal set ? So, let be is a set having v1 to v k as element in Rn.

So, here v is equal to v 1 to v k be a set of vectors in R n. So, all these members v 1 to v k vectors in Rn then, B is called an orthogonal set if inner product of v i with v j is equal to 0 whenever, i is not equal to j. So, it means that if we take inner product of v 1 with any vector remaining any vector reaming here then, it is inner product which is defined earlier must be 0. So, if a set having this property we call this set as orthogonal set and vectors are known as orthogonal vectors. So, let us consider certain one example.

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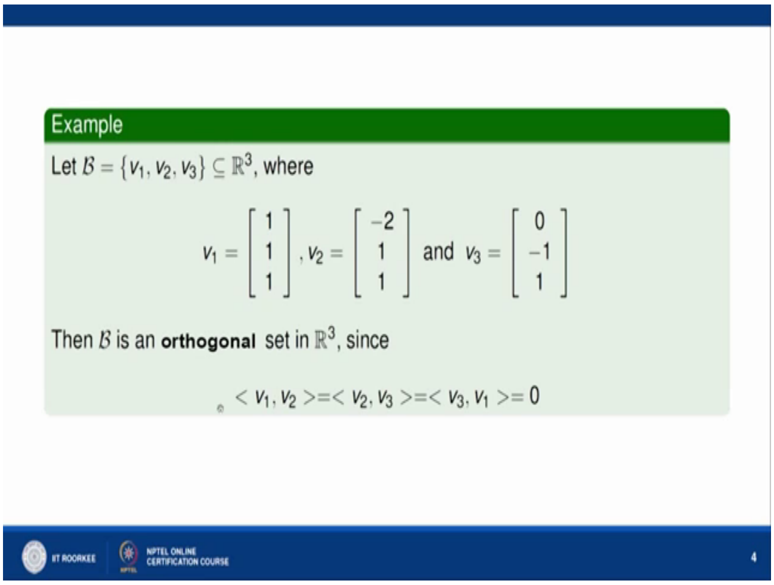

So, here we let we take a set B in a consisting these 3 element v1, v 2, v 3 were v 1 is equal to 1 1 1, v 2 is minus 2 1 1 and v 3 0 minus 1 and 1. So, here we want to show that, this B is a orthogonal set. So, to show that it is orthogonal set we need show see that, v 1 inner a product of the v 1 with v 2 and v 3 is 0 similarly, if you take inner product of v 2 with v1 and v 3 R 0 similarly the same process is true for v 3 also. So, if you look at inner a product of v1 and v 2 you look at here, just multiply component by. So, it is 1 minus 2 let be write here.

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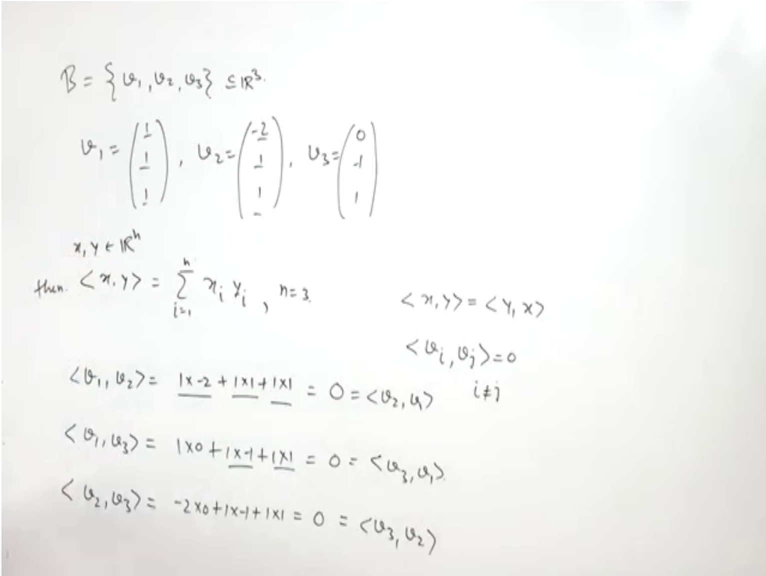

So, example here is that, we have to show that the the set B which consists of the 3 vectors v1,v 2, v 3 is a subset of 3 4 v1 is given by 1 1 1, v 2 as minus 2 1 1 and and v 3 as 0 minus 1 1 then, this set B is a orthogonal set to show that, it is orthogonal set we have to show that inner productive of v1 with v 2 and v 3 is 0 and similarly, V inner product of v 2 with the v 3 is 0. So, we recall the definition of inner product, that if we take 2 element x y in Rn and then, inner product x y is given by summation i equal to 1 to n x i y i.

So, it is basically component wise multiplication here. So, here since we are taking real vectors. So, we are not putting any bar here. So now, let us defined in inner product of v1 and v 2. So, let us see the component wise. So, 1 in to minus 2. So, that is 1 in to minus 2 plus 1 in to 1 here and then 1 into 1. So, this is see inner product if you sum them up then it is coming out to be 0 this is minus 2 n1 n1. So, it is coming out to be 0. Similarly, it is a here v1 with the v 3 here. So, it is v1, v 3 means, component wise multiplication. So, 1 into 0 plus 1 in to minus 1 plus 1 in to 1 here. So, this will be giving this will give minus 1 and this will give 1. So, it is coming out to be 0.

Now, by the property of inner product we know that, we have this property that x inner product of x y is equal to inner product of y with x bar know since, we are considering their vector space. So, there is no bar here. So, it means that inner product of x with y is same as inner product of y with x. So, taking this in mine then inner product of v1 v 2 is equal to 0 implies at it is same as v 2, v1. Similarly, inner product of v1, v 3 is same as inner product of v 3 if v1 here. Now, we want to show the inner product of v 2 with v 3 is also 0; so, for that is minus 2 into 0 plus 1 into minus 1 plus into 1 that is coming out to be 0 here.

So, this is same as v 3 with v 2 by the same property. So, here what we have shown here, we have shown that inner product of v i with v j is equal to 0 where, i is not equal to j here. So, it means that, these vectors of this set b satisfied this property it means that, this set is an orthogonal set and vectors are orthogonal vectors here. Now, moving on next theorem were we want to understand the property of orthogonal set one important property of orthogonal set this statement of the this theorem says let B ah, which set of k elements we want to v k be a set of nonzero vectors in Rn, if B is an orthogonal set then B is linearly independent. So, B is orthogonality implies the linearly independence. So, to prove this let us say that.

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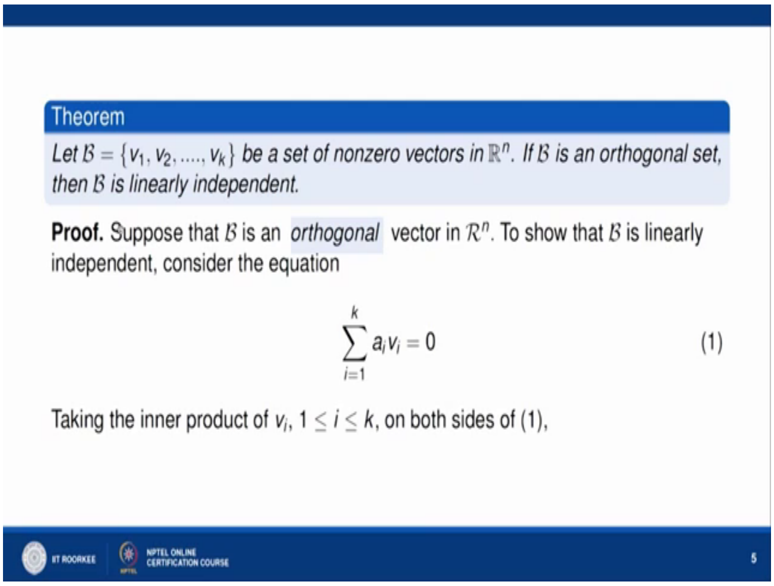

Suppose, that B is an orthogonal vectors B is orthogonal set set an Rn to show that, B is linearly independent consider this equation summation i equal to 1 to k a i v i equal to 0 what we want to show here that, all these coefficience a i are nothing but 0 or we can have only a trivial solution of this equation. So, of to show that all a i is there 0 we take the help of orthogonality, that all these v1 to v k are orthogonal vectors (Refer Time: 9:34). So, what we do here we take the inner product of v i, but i is running from 1 to k on both sides of this equation.

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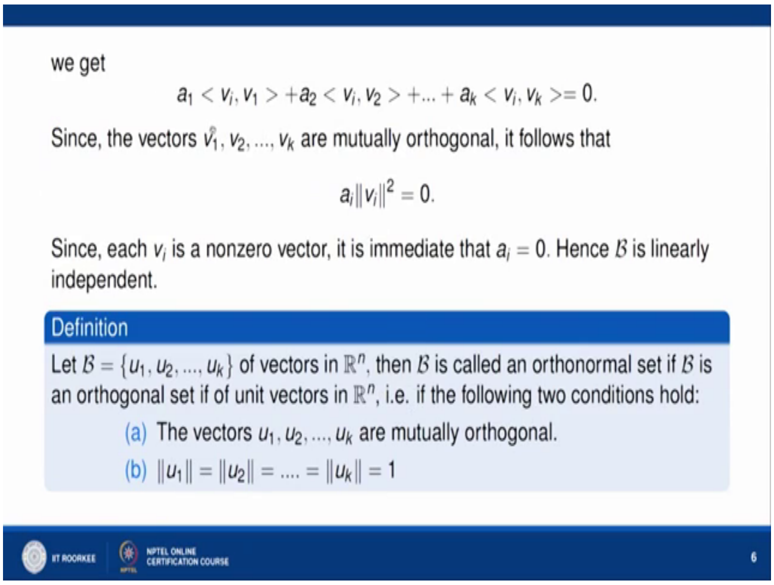

So, taking this inner product we have a1 let us see, a1 inner product v i with v 1 plus a 2-inner productive with v 2 plus. So, on a k inner product of v i with v k equal to 0. Now, here we already know that, that v i is are orthogonal vectors. So, it means that inner product of of v i with v j is equal to 0 if j is not equal i. So, it means that if i is not equal to say. So, it means that only ith factor will come out to be nonzero rest of all 0. So, since the vectors v1 to v 2 are mutually orthogonal it follows that, v i norm of v i square is equal to 0.

So, we want to show that this set is a B, which is a set of v1 to v k which is given as orthogonal set we want to show that, this is an linearly independent set. So, to show that, these vectors are linearly independent vectors we have we have to consider this linear combination i equal to 1 to k a i v i equal to 0 here, and then we want to show that all these a i is are 0 for all i equal to 1 to k. So, far that let us take the inner product of these in the equation with v i. So, here since we are taking inner product with v i. So, we are considering this dummy variable as j variable.

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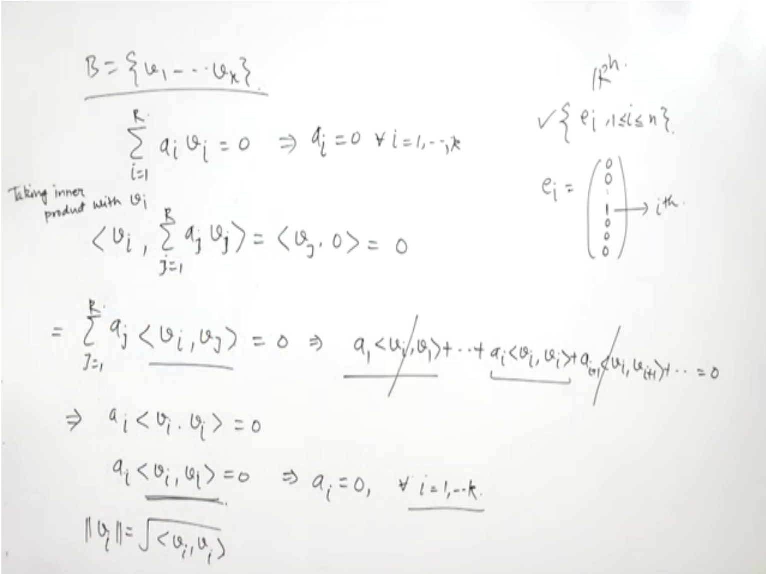

So, inner product of v i with respect with j equal to 1 to k a j, v j equal to inner product of v j with 0 here. Now, inner product of with v j with 0 is nothing but 0 here. Now, he here use in the property of inner product we can write this as summation j equal to 1 to k a j inner product of v i with v j is equal to 0. Now, we already know that, these are orthogonal set of vectors here. So, it means that v i is equal to v j inner product of v i with v j is equal to 0 for all i not equal to j. So, it means that it means that, for i equal j only this inner product is nonzero for all other it is simply 0. So, we have only one time left that is a j inner product with v j with v j is equal to 0. So, this is nothing but, this implies that a j this is v j v j is equal to 0 now, this v j is a nonzero vectors. So, this implies that this a j is equal to 0 here.

Now, this can be done for any j here. So, it means that this j is we can write it that, j for all j equal to 1 to k here. Now, here one thing we try to note it down that, here we can define norm of v j as under root of inner product with v j with v j. So, basically this quantity is nothing but, norm of v j square and non-if v j is nonzero than norm of v j square is nonzero and hence a j has to be 0 and this we can repeat for any j here. So, j is equal to 1 to k means, all a j is are simply 0. So, here we are considering the inner product of v i with this linear this linear combination. So, here if we take we use a property of inner product then, this can be written as a equal to 1 to k a j inner product of v i with v j equal to 0. Now, since we have that these vectors are orthogonal vectors. So, it means that this inner product of v i with v j is equal to 0 until i is not equal to j.

So, it means that, only for j equal to i we have nonzero by quantity for rest it is simply a 0 value. So, it means that if we write it here, this nothing but a1 v i with v 1 plus. So, on a i inner product with v i with v i plus a i plus 1 inner product of v i with v i plus 1 and so on equal to 0. Now, we know this is going to be 0 because i is not equal to 1.

So, this is 0 same this is 0 and all other terms are 0 only non0 term left is this. So, it means that a i inner product of v i with v i is equal to 0 now, this is what this is the norm of v i square because, we know this relation than norm v i is given as as under root of inner product of v i with v i. So, we can say that this norm is square is norm 0 because, v i is taken as nonzero vectors. So, it means that a i has to be 0. So, it means at this we can say for all i equal to 1 to k. So, it means that this summation i equal to k a i v i equal to 0 implies that all a j 0.

So, this implies that if B is orthogonal set then, B is a n linearly independent set. So, moving on next definition of orthogonal set. So, let B set of k vectors in Rn then, B is called and orthogonal set if B is an orthogonal set and B is an orthogonal set of unit vectors in Rn it means that, set B satisfy the following 2 condition first the vectors u1 to u k are mutually orthogonal and second that, norm of each vector is equal to 1 here. So, it means that this set B is an orthonormal set if these vectors are orthogonal and in up and norm of each vectors is equal to 1.

So, let us consider some example of orthonormal set. So, the standard ordered basis for F n is an orthonormal basis for Fn for example, if we consider Rn then, standard orthogonal basis Rn is basically e i, e i is I think we can write it like this that if we have Rn here then, standard basis are e i i is from 1 to n here, what is e i here? E i is basically vectors were all other it is simply 0 only the ith place it is nonzero and it is equal to 1.

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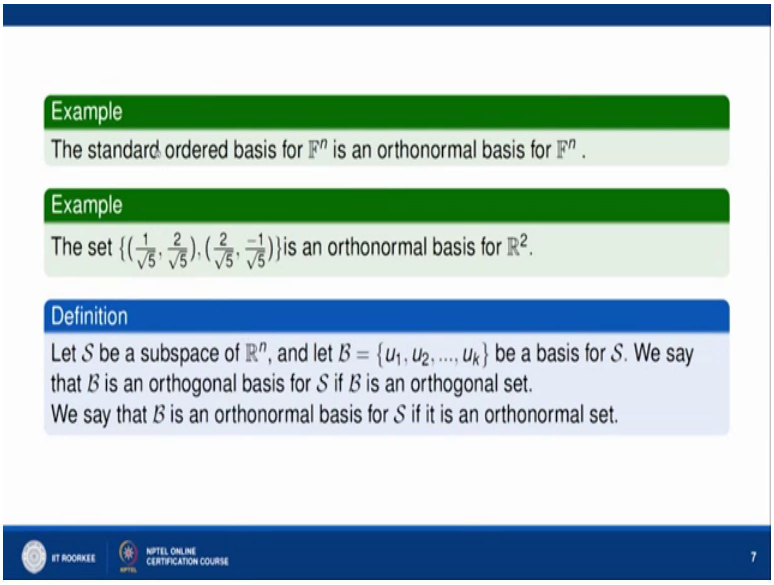

So, this e i is basically this set is an orthonormal basis for Rn here. So, that you can easily verify that, if he take the inner product of e i with e j, j is not equal to y you can see that, inner product is going to be 0 and if you consider the inner product of e i with e i itself, which is nothing but norm of e i square it is coming out to be 1 only. Now, looking at the second example we have another example we says that, this set of 2 vectors 1 vectors is 1 by root 5 comma 2 by root 5 and 2 by root 5 and minus 1 upon root 5 is an orthonormal basis for R2.

So, how we can look at here first of all we have to show that, this is orthonormal set and then, we can show that it is orthonormal basis. So, to show that it is a basis here since, dimension of R2 is to and if you can show that this set is an orthogonal then, we know that it is a linear independent also and hence, the dam dimension of the vector subspace, which is span by this set going to be 2 and hence, than we can prove that this is going to be in orthogonal basis for R2 let us verify that, this is orthogonal this for R2. So, we want to show that this set B, which is given as set of 2 2 vectors 1 vectors is 1 by root 5 comma 2 by root 5 and another vector given as 2 by root 5 comma minus 1 upon root 5 this set is going to B an orthonormal basis for R2.

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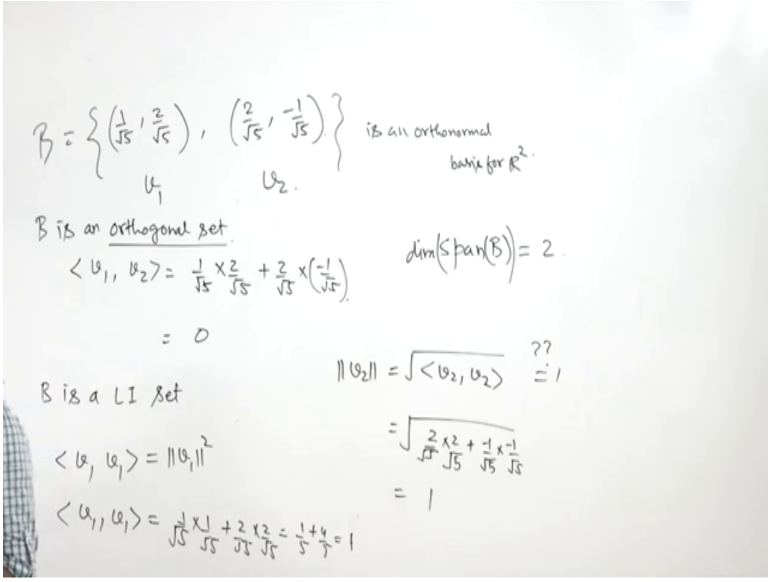

So, what we try to prove first is that, this, this B is an orthogonal set. So, first we show that, B is a orthogonal orthogonal set. So, to show that we need we show that inner product of v1 with v 2 is going to be 0. So, here we have 1 upon root 5 in to 2 by root 5 plus 2 by root 5 in to minus 1 upon root 5. So, if you look at nothing but 0 here. So, it means that, B is consist only 2 element and those 2 elements r orthogonal to each other.

So, we can say that B is an orthogonal set. So, just by orthogonality we can show that by pe previous result B is an a linearly independent set here. So, that is all, all clear, next thing we want to show that, that each factor has a norm 1. So, to show that it has norm 1 let us take the inner product of v1 with v1, which is nothing but norm v1 square. So, if you calculate this, this is nothing but v1 with v1 is equal to 1 by root 5 with 1 by root 5 plus 2 by root 5 into 2 by root 5.

So, which is nothing but 1 by 5 plus 4 by 5 it is coming out to be 1 here. Similarly, we can verify that norm of v 2 is going to be in another root of inner product of v 2 with v 2 is going to be 1, this you want to show here let us calculate the inner product of this. So, inner product of v 2 with v 2 is going to be 2 by root 5 into 2 by root 5 plus minus 1 upon root 5 into minus 1 upon root 5. So, if you calculate this is coming out to be 1 also. So, it means that B is an orthogonal set and every vector in this set has a norm 1.

So, it is mean it means that B is an orthonormal set here set of R2. Now, we already know that, this being a orthogonal set it is a l i set. So, it means that span of B is going to be vector space having dimension 2. So, dimension of span of B is going to B 2 and it is going to be a subset of R2. So, it means that this is going to be a basis for R2 here. So, we already know that either we do it like this or we can show that since, the dimension of R2 is 2 and the we have a set having 2 linearly independent vectors than, this is going to be a basis for R2 here.

So now, let us defined the terms of orthonomal basis and orthogonal basis. So, for that let us take subspace of Rn and let B is a set is a subset S and which is a which represent basis for S is we say that, B is an orthogonal basis if this set is an orthogonal set and we say that, B is an orthonormal basis if the set is an orthonormal set. So, we have show that , the set is going be an orthonormal basis is for R2 here because, we have shown that this a basis and it is a orthonormal set.

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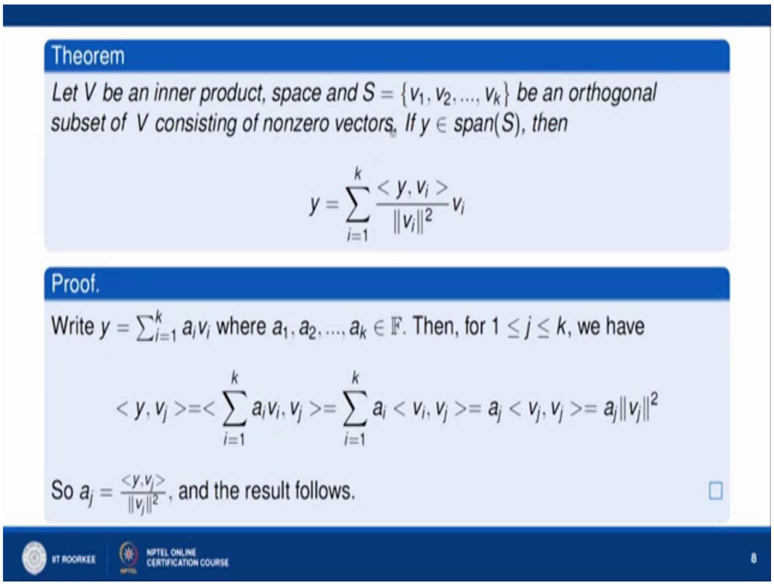

Now, let us move on to next example next theorem we say that, let me be an inner product space and S is equal to be v1 to v k, B an orthogonal subset of v consisting of nonzero vectors only.

So, it means that all this v1 to v k or nonzero vectors and if y belong to span of S then y can be written as i equal to 1 to k y with v i divided by norm v i square into v i. So, this is it means that we can find out say linear coefficient of v i as inner product of y with v i divided by norm of v i square. So, in this theorem we want to show that, how we represent y in terms of linear combination of v i. So, we want to show that here, the finding out the process of finding the coefficients are not more easier than the then, the set if we this set is not orthogonal then finding the efficient are difficult, but if it is an orthogonal set then, finding the then coefficient of v i or quite easier.

So, let us consider the proof of this. So, let us a y equal to submission equal to 1 to k a i v i were these a i is are coming from a scalar field then for 1 less than j less than k we take the inner product of y with v j. So, inner product of by with v j is equal to summation i equal to k a i v i with v j is equal to now, here we take the take the property of inner product and we can write this as summation equal to k a i inner product of v i with v j. Now, we know that this inner product of v i v j is going to be nonzero only when i is equal to j. So, all other it is going to be 0. So, this means that only for i equal to j this have a nonzero value and this is nothing but, a j inner product of v j with v j. Now, we that inner product of v j with v j is going to be norm of v j is square it means, that inner product of y with v j is going to be a j into norm of v j square now, we already know that these v j are nonzero.

So, this nom is going to be norm 0 quantity. So, we can write down this a j as inner product of y v j divided by norm of v j square. So, it means that coefficient of v j can be obtained very easily, but if this v i is are not orthogonal then, this cannot a be achieved so easily. So, it means that in case of orthogonal orthogonal basis coefficient are finding coefficient are quite easy.

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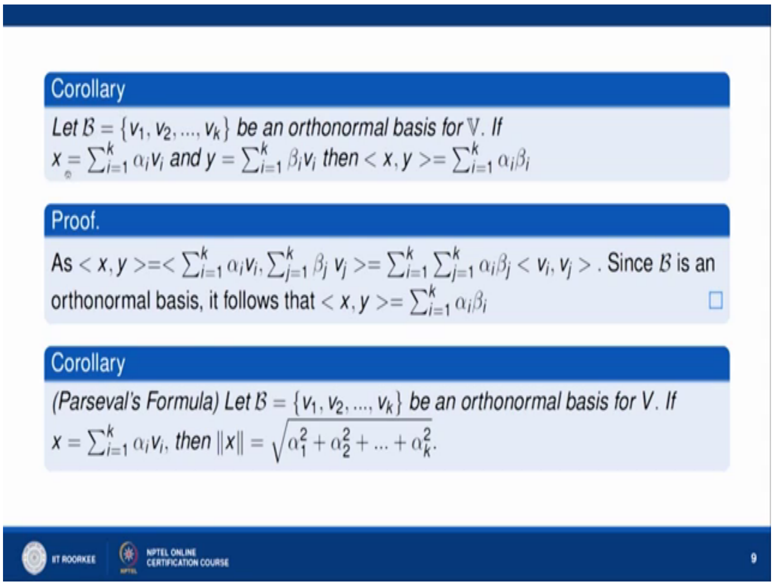

Now, moving on the corollary of this. So, we have that let B be an orthonormal basis for v and if x is written as summation equal to k alpha i v i and y is written as summation i equal to k beta y I then, inner product of inner product of x y is nothing but, inner product of coordinates of x and y that is inner product of alpha i alpha and beta i equal to 1 to k alpha i beta i.

So, inner product of x y is nothing but inner product of coordinates of x and y here. So, to prove this we simply take this as inner product of x comma y, which is equal to summation i equal to k alpha i v i comma summation j equal 1 to k beta j v j here, we are taking the inner product. So, we are taking different dummy indices. Now, this can be written as summation I equal to 1 to k summation j equal to 1 to k alpha i beta j inner product of v i with v j. Now, again using the same property that v i is are orthogonal to each other. So, it means that this will have a nonzero value only when this i is equal to j all others it is going to be 0.

So, it means that taking j has i we can say that this inner product of x y is reduces is to summation i equal to 1 to k alpha i beta i. So, this corollary says that when we have a orthonormal basis, than inner product of vectors here x and y is nothing but inner product of quadrant vectors. So, that is one use of orthonormal basis second corollary simply say that, how to find out orthonormal? How to find a find a norm of a given vectors x? So, we have let beta let B is we want to v k B an orthonormal basis for v and if x can be written as summation i equal to 1 to k a alpha i v i then, norm of x can be given as summation under under root of alpha i alpha1 is square plus alpha 2 square plus alpha k square. So, this can be proof of this corollary can be given very easily with the help of previous cor corollary. So, here we replace y by x then, it is nothing but inner product of x comma x is going to be summation equal i to 1 to k alpha i is square.

So, norm is going to be inner product is square root of inner product and the proof of this parsevals formula follows.

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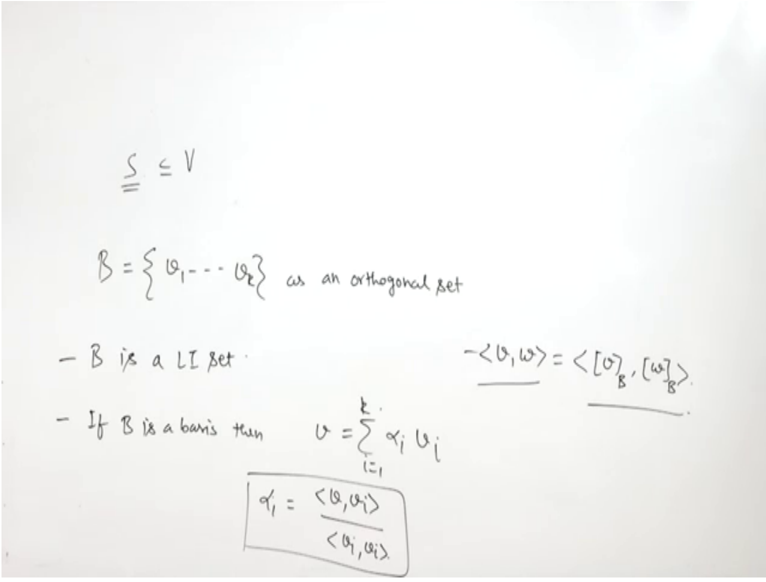

So, it means that if you look at what we have seen here, that we have a suppose a vector space v and then we have a vector subspace s c m and then we have a set B which is given as say v1 to v k as and as a an orthogonal set then we have seen that, B n an orthogonal set it satisfy 2 property first thing that B is a LI set first property, which we have proved and a second property that, if this is a basis if B is a basis then, any element we which is written as summation alpha i v i i from 1 to k then, this alpha i is can be obtained very easily as we inner product with the v i divided by inner product of v i with v i and that is easily possible because this set is an orthogonal set. So, inner product of any 2 element here inner product of v with w is nothing but, inner product of coordinate vector of v and coordinate vector of w.

So, inner product of v, w is nothing but, inner product of coordinate vectors of v and w, that is what is the meaning of this corallary here. Now, moving on next theorem we say that let cube n cross n orthogonal matrix and if x comma y are 2 vectors Rn than inner product of Qx with Qy is nothing.

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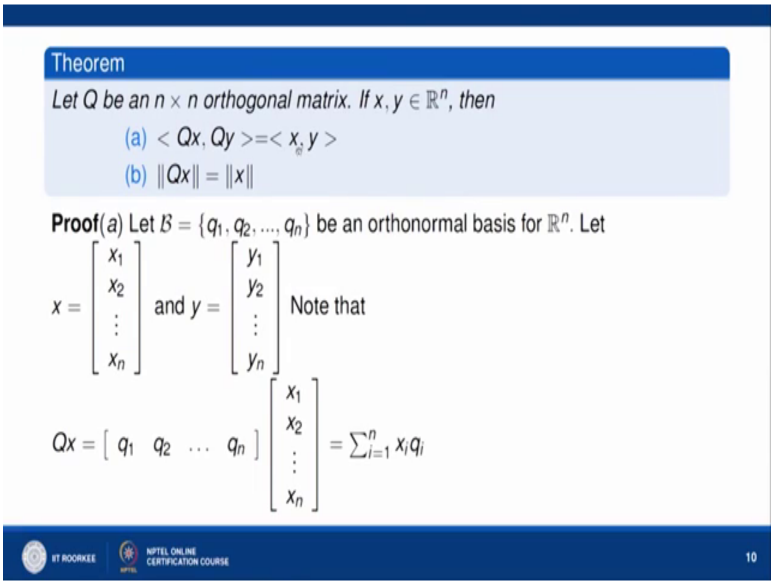

But inner product of x comma y and B part is that norm of Qx is same as norm of x. So, it means that, if we have an orthogonal say matrix ah. So, product of orthogonal matrix and a vector does not change the norm of the vector here. So, to prove this let us consider that let B is q1 to q n B and orthonormal basis for Rn and let x is a vector given as x 1 to x n and y as y 1 to y n ok. So, what is given here is at Q is an orthogonal matrix.

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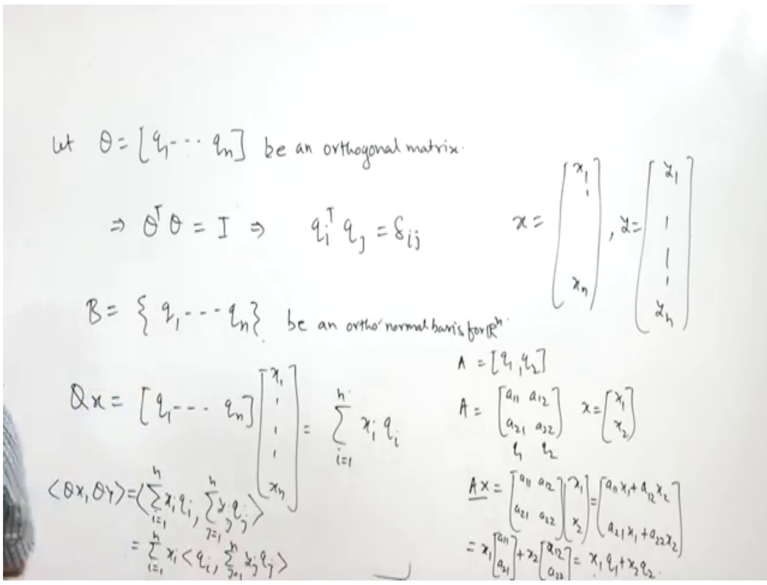

So, let Q is given as this matrix q1 to q n where, q n represent the first column of this orthogonal matrix and q1 represents the another column of this orthogonal matrix. So, since orthogonal matrix means q transpose q is i and then, we can say that this is nothing but a saying that q i transpose q j is nothing but, delta i j where delta is a Kronecker delta function. So, it means that the if i is not equal to j then, q i an inner product of q i with q j is going to be 0 and if i equal to j then q i inner product of q i with q I is going to be 1. So, it means that this q1 to q n are orthonormal vectors here. So, with the help of these n orthonormal vectors we can consider a basis for Rn and this basis is nothing but, an orthonormal basis for R n. So, then we can consider q of x where, x we can consider this as say x1 to say x n. Similarly, we can consider y as say y1 to say y n and then, consider q of x q is given as q1to q n and here, we x is given as x1 to x n and this we can write as summation say I equal to 1 to n x i q i here.

So, So, this can be written as this product can be written as summation i equal to 1 to n x i q i I will just take very simple example of this thing let us say that, we have a matrix a which is given as a11, a12, a 21, a 2 2 and we have a vector x, which is given as x1and x 2 and you want to show that, multiplying a matrix with a vector can be written as linear combination of columns of this matrix q. So, that is what we want to write it here that say that this is q1 and this is q 2. So, we want to find out a of x a of x is going to be what a11, a12, a 21, a 2 2 multiplied by the this x1 and x 2 and this is what this can be written as a a11 x1and plus a12 x 2 and here, we can write it a 2 1 x 1 plus a 2 2 x 2 and this we can write as equal to you can take x1 out and this is nothing but, a11, a 2 1 plus x 2 this is nothing but a12 a 2 2. So, it means that in terms of column if you want to write it then, this is nothing but x q 1 plus x 2 q 2. So, it means that if you multiply if we write a as say q1 and q 2 then a x can be written as x1 q 1 plus x 2 q 2.

So, similarly we can write down Qx and if q represented as this this kind that q1 q1represent the columns of q then, this in this multiplication can be written as linear combination q i. So, i equal to one to n x i q i. So now, to find out inner product Qx with q i we can use this inner pro inner proud of x i q i with summation y i q i. So, let us use some other a dummy variable. So, let us use j here y j and q j i is from 1 to n and j is from 1 to n. So, to find out this is use the property of inner product and we can write down summation i equal to 1 to n x i then, this will be written as q i inner product summation j equal to 1 to n y of j q of j , right ? And again, we can use the same property of inner product space.

So, using again the property of inner product we can write the this as equal to summation i equal to 1 to n x i and then, we are taking these coefficient out j equal to 1 to n y of j and what is left here is inner product of q i with q j , right? And this is nothing but inner product of Qx with Qy, right? Now, this is what we already know that these qis are orthonormal vectors here.

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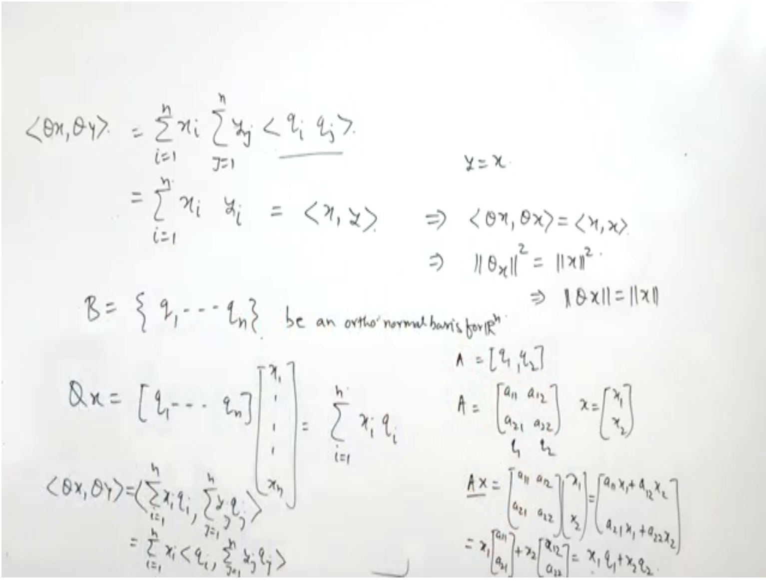

So, it means that if i is not equal to j then this is going to be 0. So, it means that we can write it i equal to 1 to n x i now, for j equal to i only we have a nonzero value. So, we can say that it is going to be y i inner product of q i with q i now, again since we have this q i's are orthonormal. So, this is going to be 1. So, this is going to be one only and now, what we can write it this, this is nothing but inner product of x with y here. So, it means that inner product of Qx with Qy is nothing but, inner product of x with y here. So, it means that inner product of Qx with q i is nothing but inner product of x with y when, q is going to be an orthogonal matrix.

So, if q is an orthogonal matrix the inner product Qx q with q i is same as inner product of x with y now, here if we replace this y as x then it is nothing but inner product of Qx with Qx is equal to inner product of x with x , which is nothing but norm of Qx square is equal to norm of x square. So, it means that it means that, norm of Qx is going to be norm of x here. So, it means that if you multiply this x by and orthogonal matrix then they are norm is not going to change and this is a very, very important property and numeric linear algebra because, as far as possible whenever we do some kind of numerical computation we always try to do that numerical computation with the help of orthogonal operators.

Now, the property of orthogonal operator's orthogonal matrix, that it will not increase any kind of error. So, if there is any any error in this is this x represent the error vector then, error vector is not going to propagate as we apply the orthogonal matrix over this. So, it means that this is very, very important property to be utilised later on. So, that is what is the proof we have considered here. So, here we are going to stop.

So, in this lecture what we have considered is we have considered the concept of inner product and with the help of inner product we have define the concept orthogonal set and we have seen certain property of orthogonal set that, even if we have a set which is orthogonal then it is automatically linearly independent then we have also seen that, if you want to write any vector in terms of these orthogonal vectors then, coefficient of these orthogonal vectors can be find out very easily and if with the help of these orthogonal vectors if we considered a orthogonal orthonormal basis of a vector space then, finding the coefficients are quite easy.

So, that we have seen here and in next lecture we want to see how to find out orthogonal set and orthonormal set from a given set of vectors. So, that is the content of next lecture. So, here we stop thank you for listening us.

Thank you.