**Numerical Linear Algebra**
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**Lecture - 13**
# Eigen values & Eigenvectors- I

Hello friends, I welcome you to my lecture on Eigen values and Eigen vectors. There will be two lectures on this topic. This is our first lecture. Now, Eigen values have their greater significance in dynamic problems, the solution of the equation d u by d t equal to A u.

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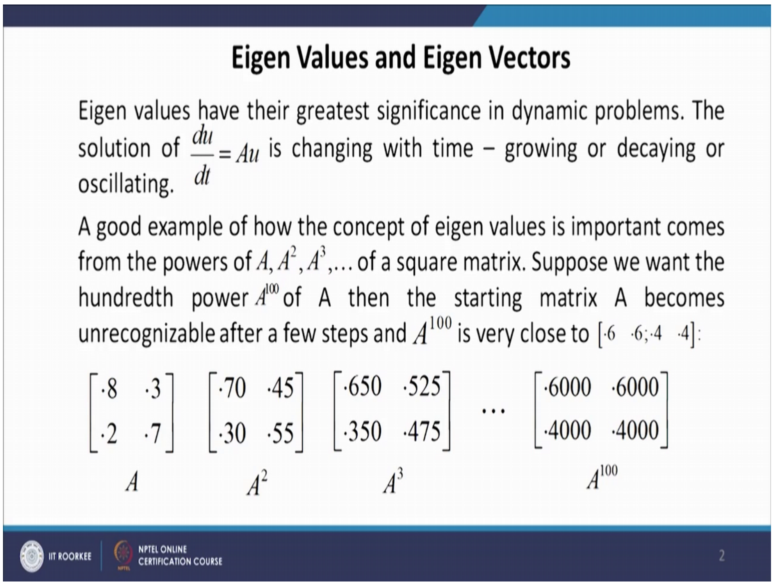

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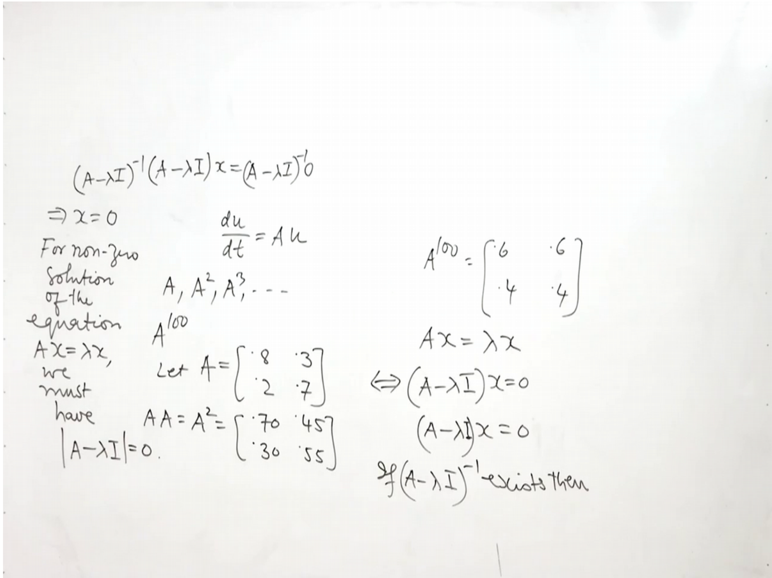


Where A is a matrix and u is a vector, changing with time. With time growing, decaying or oscillating are good examples. The concept of Eigen values comes from powers of A, A square a cube and so on. Suppose, we want 100 power of A. Then let us take the matrix A to B. Let the matrix AB be equal to 0.8, 0.3, 0.2, 0.7. Then, we can calculate A square, A cube up to a certain stage after that, it becomes very difficult to calculate the powers of A. So, if you calculate A square, A square comes out be 0.70, 0.45, 0.30 and 0.55.

So, A into A which is A square. We can calculate by matrix multiplication. We can multiply A square by A again, get A cube. 0.650, 0.525, 0.3, 0.0, 0.475 but after a few steps, the matrix will become unrecognisable and we will see that, by using the concept of Eigen values and Eigen vectors, we can calculate A 100 and it is very close to 0.6, 0.4, 0.66 and 0.44. So, we shall see how we can calculate A to the power 100. This A to the power 100 is not calculated by the matrix multiplication, which is by multiplying the matrix 100 times. We will be calculating it by using Eigen values and Eigen vectors, to explain the Eigen values, first, let us explain Eigen vectors. Almost all vectors change the direction.

But when they are multiplied by the matrix A, there are certain exceptional vectors, which keep the same direction, when they are multiplied by x. So, those vectors are called the Eigen vectors. Thus, the vector A x, the vector x 1 multiplied by A, is elder number lambda times the original vector x. So, the basic equation is A x equal to, let us again repeat, we have the vector x, when we multiply it by A, we get a number lambda times the original vector x. So, certain exceptional vector satisfies this kind of thing. So, such a vector is called Eigen vector and lambda is called its Eigen value. So, we can also write this equation as A minus lambda I into x equal to 0. Remember, we cannot write it as A minus lambda into x equal to 0. This is not right because, A is a matrix and lambda is a scalar quantity, we cannot subtract the lambda from A.

So, we should multiply this by identity matrix of the same order as A. So, the correct way of writing A x equal lambda x is, A minus lambda I into x equal to 0. So, this is the alternate form of A x equal to lambda x. Now, suppose so happens that the matrix A minus lambda I is invertible, then, we can see from here if A minus lambda I is invertible, that is A minus lambda I inverse adjust then, 2 multiply this equation 3 multiply this equation by A minus lambda inverse, what we will get is A minus lambda I inverse into 0 vector. So, A minus lambda I inverse into A minus lambda I is identity matrix. Identity matrix into 2 x is x. So, this gives us x equal to and A minus lambda inverse multiplied by 0 vector, will give 0. So, x equal to 0.

So, when A minus lambda I inverse exists, what we get is x equal to 0 and we can see from here that x equal to 0 clearly satisfies this equation. So, we are interested in those vectors x which are nonzero vectors and satisfy the equation A x equal to lambda x. So, where we want the vector to be a nonzero vector, for the nonzero vector A minus lambda I inverse must not exist, that is, determinant of A minus lambda I inverse B equal to 0. So, for nonzero solution of the equation A x equal to lambda x, we must have determinant of A minus lambda I equal to 0, because we know that if determinant of A minus lambda is nonzero, then x inverse exists.

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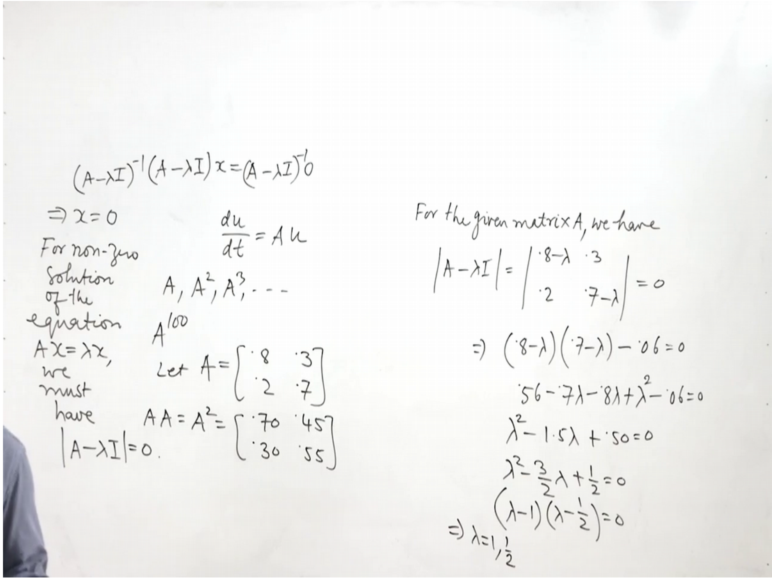


So, now let us find determinant of A minus lambda I for the given matrix A. For the given matrix A, we have determinant of A minus lambda I equal to determinant of A minus lambda I. The matrix A minus lambda I is the matrix where, the diagonal elements of A are subtracted by lambda. So, we shall have 0.8 minus lambda 0.3 and then 0.2, 0.7 minus lambda. So, let us put this equal to 0. So, this will give us 0.8 minus lambda into 0.7 minus lambda minus 0.06 equal to 0.03 into 0.2, determinant of this is AB CD AD minus BC and when we solve this is 0.56 minus 0.7 lambda minus 0.8 lambda plus lambda square minus 0.06 equal to 0.

So, I can write it as lambda square. This is minus 1, 0.5 lambda and here we get 0.50. So, this is lambda minus square minus 3 by 2 lambdas plus 1 by 2 equal to 0. I can write it as lambda minus 1 factors are, lambda minus 1 into lambda minus half equal to 0. So, the Eigen values are lambda equal to 1 and half we get two values of lambda 1 and half for which determinant of A minus lambda I is equal to 0. Now, we shall calculate the Eigen vector for each of these two Eigen values. So, let us first calculate the Eigen vector for lambda I equal to 0.

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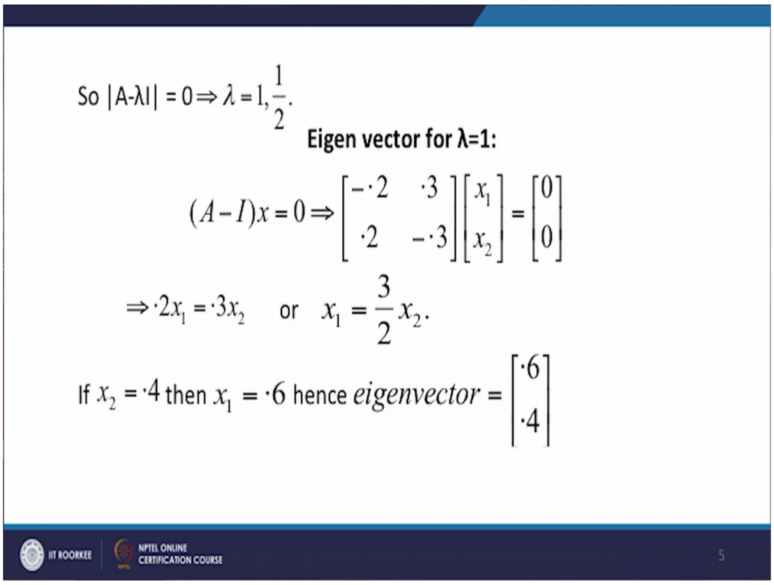


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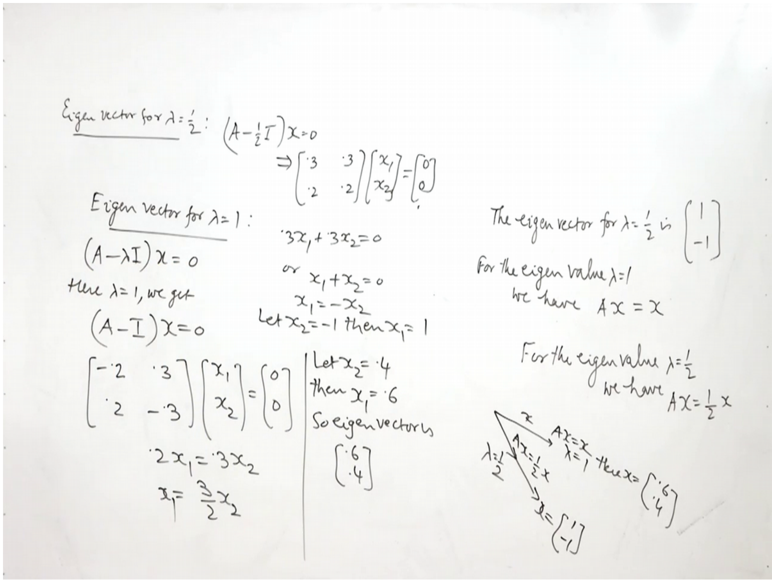


So, the equation from which we will determine the Eigen vector x is A minus lambda I x equal to 0. So, let us put here lambda I is equal 1. So, we get A minus I into x equal to 0. A minus I, means subtract the unit matrix from the matrix A. So, A minus I becomes 0.7, 0.3 and 0.2 and 8 minus 0.28 minus lambda is equal to 11 minus 0.8 is 0.0 minus 0.23, then 0.27 minus 1 is minus 0.3, and let us say the vector x has components x1, x2. 0 is the 0 vector. So then, we will have these two equations minus 0.2 x1 plus 3 x2. 3x2 equal to 0 and then 0.2 x1 minus 0.3 x2 equal to 0, they are both same equations.

So, what we get is 0.2 x1 equal 0.3x2 or we can say, x is equal to 3 by 2x 2x1 equal to 3 by 2 x2. So, if I take x2 equal to 0.4. Let us say, I take x2 equal to 0.4 then x1 is equal to 0.6, and so, Eigen vector maybe taken as 0.6, 0.4. So, there is an Eigen vector corresponding to lambda equal to 1. Similarly, eigenvector for lambda is equal to half. So here, in this equation, let us put lambda equal to half, then we shall have A minus half I x equal to 0. So, we subtract half I from the matrix A. So, this will give you 0.5 minus 0.8. So, 0.3 and then 0.2 and then 0.2, x is again a, let us say x1 x2.

So, we get 0.3 x1 plus 0.3 x2 equal to 0. Net equation is 0.2 x1 plus 0.2 x2 equal to 0. So, we can say x1 x2 equal 0 or x1 equal to minus x2. So, taking x2 equal minus 1, we will get x1 x1 as 1. So, let x2 be minus 1, and then x1 is equal to 1. So, we have Eigen vectors as 1 minus 1. So, we got the Eigen vector forwarding vectors for both the Eigen values. So, the Eigen vector for lambda equal to half is 1 and minus 1 for the Eigen value lambda equal to 1. We notice that we have x1 Ax equal to x, again the Eigen vector x after we multiplied by A remains the same, it does not change, and for the Eigen value lambda equal to half, what we have? We have Ax equal to half x.

So, suppose we have this vector for lambda equal to 1, this vector is x and A lambda is equal to half. So, Ax equal to x, after we operate by A, it remains the same. Now, in the case this is for lambda equal to 1, for lambda equal to half, let us say this is the vector, this is the Eigen vector. Here, x is equal to your 0.6, 0.4 for the case lambda equal to 1 x is equal 0.6, 0.4 0.4. So, let us say this represents x, when you operate on it by A, what you get is, this same vector, but here, suppose this is your vector x in the case lambda equal to half.

So, x let us say, this one 1 minus 1. So, this is 1 minus 1 vector, when we operate by A it becomes half. So, let us say, this is half x. So, this is Ax equal to half x. So, if we keep the same direction, but its magnitude becomes half of that. So, length becomes half. So, this is how we see that in the case of lambda equal to 1, director does not change, it remains the same. When we take lambda equal to half, it shrinks. So, the Eigen value lambda tells whether the vector x is stretched, for example, if you take lambda equal to 2, the Eigen vector Ax will become 2x.

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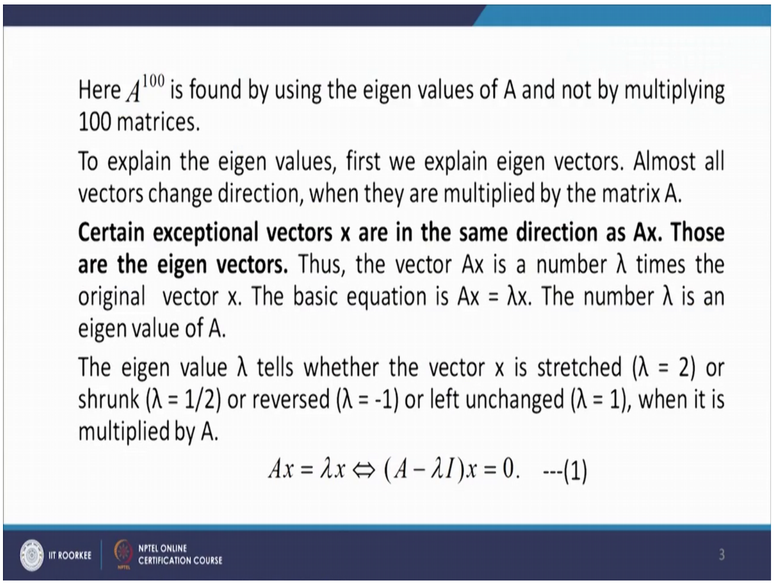


So, it will stretch Eigen vector I, vector x will stretch and if we take lambda equal to half, we will have Ax equal to half x. So, Eigen vector will shrink and if we take lambda equal to minus 1, then Ax will become equal to minus x. So, it will be reverse and if we take lambda I equal to 1, it will remain unchanged. So, how we interpret the positive and negative values of lambda and the value which are more than 1 are the value which are less than 1. So, now let us see how we found A 100 by the use of Eigen values, Eigen vectors.

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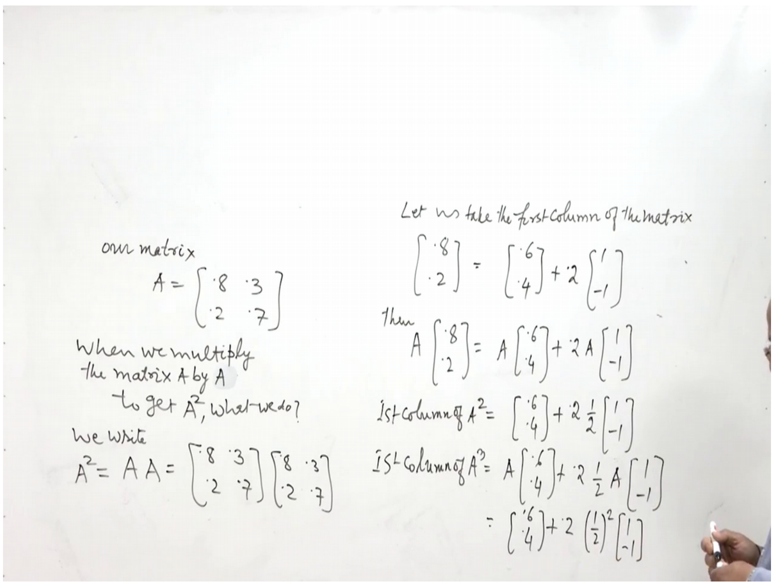

Our matrix A is 0.8, 0.3, 0.2, 0.7. Its first column is 0.8, 0.6 for lambda equal to 1, the Eigen vector is 0.6, 0.4 and for lambda, equal to half the eigenvector is 1 minus 1. When you multiply the matrix A by A, what we do is, when we multiply the matrix A by A to get A square. What we do? We write A into A, A square is, A into A, then the matrix is multiplied by the column of, by the columns of the matrix A, that is A matrix. So, 0.8, 0.3, 0.2, 0.7, we multiply it by the first column of this matrix, 0.8, 0.3, 0.2, 0.7. We multiply the first column to this matrix, and then when you multiply first column to both - first row and second row, we get the first column of the matrix A square, and then we multiply second column to the rows of the matrix A, we get the second column.

So, what we do is? So let us take the first column of the matrix A and then take the first column. Lets take the first column of the matrix A, which is 0.8 and 0.2 and write it as a linear combination of the Eigen vectors for lambda equal to 1 and lambda equal to half. So, the Eigen vector for lambda equal to 1 is 0.6 0.4 and for the Eigen vector for the Eigen value lambda equal to half, it is 1 minus 1. So, what we will do, we will have to multiply it by half, 0.6, no 0.2 we have to multiply. So, 0.2, so the first column of the matrix which is 0.8, 0.2 can be written as 0.6, 0.4 plus 0.2 1 minus 1, then will get this.

Now, what we do is, we pre multiplied this y A. So, then A, when you pre multiplied 0.8, 0.2 by A, what you get is the first column of A square. First column of A square is obtained when we multiply A matrix by the first column of the matrix A. So, first column of A square, we are getting. Now here, what happens, this is a Eigen vector corresponding to lambda equal to 1. So, Ax equal to X. So, we get 0.6, 0.4 and here 0.2 times, this Eigen vectors, corresponding to lambda I equal to half. So, Ax equal to half x. So, we get a half times then minus 1. So, this is how we get the first column of A square.

Now, the first column of A square which we get, is again multiplied by A and will get the first column of A cube. So, first column of A cube, this first column of A square is pre multiplied by A to get the first column of A cube. So, again A times 0.6, 0.4 plus 0.2 times half, but A minus 1. Let us again see, this is Eigen vector for lambda equal to 1. So, Ax equal to x. So, we get 0.6, 0.4 and here 0.2 times this Eigen vector for lambda I equal to half. So, Ax equal to half x. So, 0.2 then by 2 square.

So, first column of A cube is, when we find what we get is the Eigen vector for lambda equal to 1 plus 0.2 times half square then, the Eigen vector for lambda equal to half, we continue this process and then the first column of A 100 will be equal to continuing this process, first column of A 100 will be equal to 0.6,0.4,0.21 by 2 raise to the power 99, 1 and minus 1. Now, 1 by 2 raised to the power 99 very small value, it does not show up to the first 30 places.

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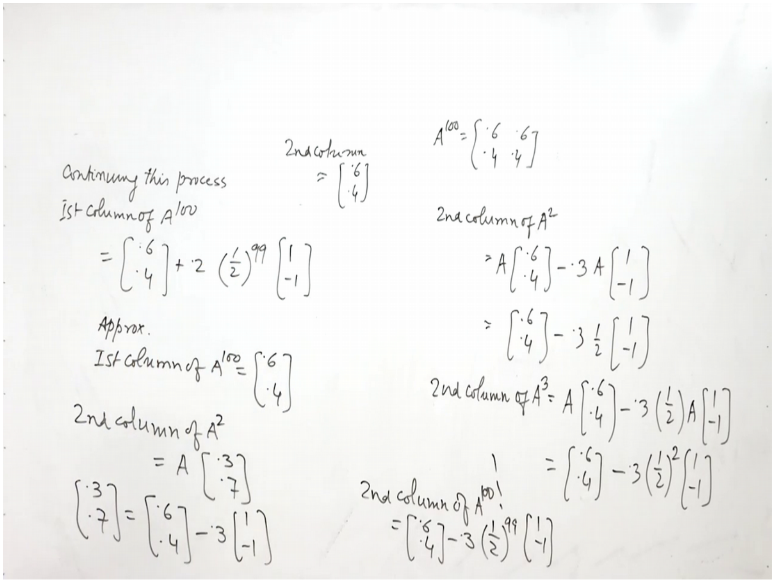


So, approximately we can say that the first column of A to the power 100 is nothing, but 0.6,0.4. So, approximately, first column of 0.6, 0.4. Similarly, when we want to find the second column of A to the power 100, we note that, the second column of A square is obtained when we multiply the matrix A by the second column of A.

So, second column of A is equal to A into second column of A and second column of A was equal to 0.3, 0.7. This is second column of A. So, second column of A1 multiplied to A gives you second column of A square. Now, we will again repeat the same thing, but the second column of A matrix, shall be written as a linear combination of the vectors A corresponding to the Eigen values 1 and half. So, 0.3, 0.7. Let us write as a linear combination of A, those 2 values of 0.6, 0.4 and that what I want is 0.3. So, minus 0.3, so, minus 0.3 times 1 minus 1. So, I can write 0.3, 0.7 as 0.6, 0.4 point minus 0.3 times 1 minus 1, then I pre multiply by A.

So, when we pre multiplied by A. So, second column of A square will be a 0.6, 0.4 minus 0.3 A 1minus 1. So, this is point 0.6, 0.4 minus 0.3 into half, 1 minus 1, because this Eigen vector for lambda equal half is equal to 1. Similarly, second column of A cube will be equal to A, let us apply A of matrix on this pre multiply. So, this will be equal to A here, and then 1 minus 1. So, this will be 31 by 2 square, 1 minus 1 and again continuing this process the second column of A 100, this will be equal 0.6, 0.4 minus 0,3, then by 2 raise to the power 99 and then 1 minus 1, again this is a very small value, so it does not show up to the first 30 places.

So, approximately second column is also 0.6, 0.4. So, second column approximately is 0.6, 0.4. So, the matrix A to the power 100, has first column as 0.6, 0.4 and second column is 0.6, 0.4. So, this how, we get the matrix A to the power 100 by using Eigen values and Eigen vectors. So, this is what we have discussed here. Now, the Eigen vector x equal to 0.6, 0.4 is a steady state, because it does not change,

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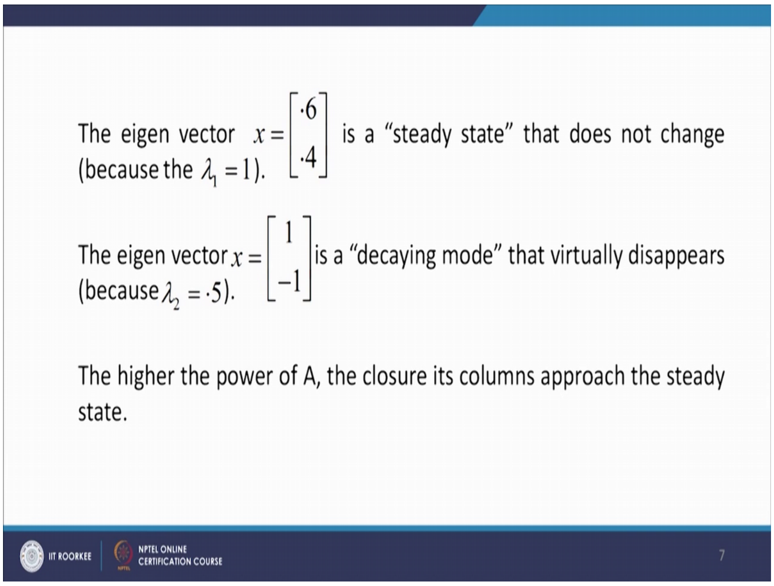


It remains the same due to the fact that Eigen value is lambda 1 equal to the eigenvector x equal to 1 minus 1, is an decaying mode because it does virtually disappear, because lambda 2 is, half the higher the power of A, the closure its columns approach the steady state in this case.

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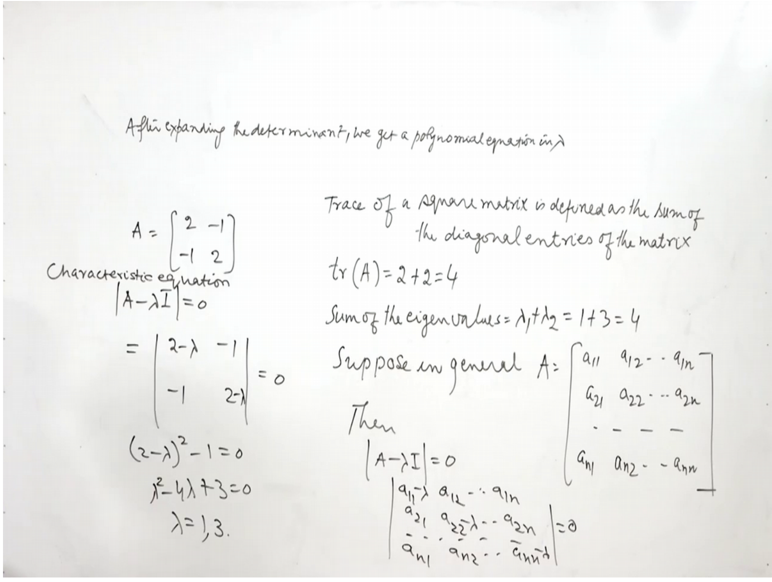


So, let us now take away another example. Suppose, we take the matrix a equal to t 2 minus 1 and then 1 minus 2. We want to find the Eigen values and Eigen vectors of this matrix. So, what we will do is, we will write the characteristic equation determinant A minus lambda I equal to O is called the characteristic equation, and its roots are called the Eigen values or characteristic values. So, Eigen values are also called characteristic equation roots. So, here what we will have is 2 minus lambda minus 2 minus 1 2. So, this will give you here, 2 minus lambda. So, 2 minus lambda whole square when we write that, the value of the determinant then we get, minus 1 equal to 0. So, what we will get is lambda square minus 4 lambdas plus 3 equal to 0. So, the roots of this are lambda equal to 1and 3. So, the Eigen values of the matrix A are: - 1 and 3. Now, let us note the following sum of the Eigen values, you can see is equal to sum of the diagonal elements.

So, it is the property of the Eigen values. So, the trace of a matrix is equal to trace of a square matrix, defined as the sum of the diagonal entries of the matrix. So, here trace of a which we write as trace of a is equal to the Eigen diagonal entries, to which is a 1 plus a 22. So, 4 and you can see some of the Eigen values, let us say, the sum of the Eigen values is equal to lambda 1 plus say, 1 Eigen values. Lambda 1, at the other 1 is lambda 2. So 1 plus 3 equal to 4. So, there is a check on the values of lambda that you get you can see, whether the sum of the Eigen values equals the sum the trace of the matrix a or not, that we can easily find trace of the matrix.

So, suppose in general here, the matrix is 2 by 2. Suppose in general, the matrix is n by n. Suppose in general, a is the matrix. So, a by a 1, a 2, a 2 a, a 2 1 a to 2 a 2 n and then a n 1, a n 2, a n n; then determinant of A minus lambda I equal to 0 will give you a 1 minus lambda a 1 2 and so on. a 1 and a 2, 1 a to 2 minus lambda, a 2 n and so on. A n 1, a n 2, a n n minus lambda equal to 0, when we solve this, it will give you a polynomial equation in lambda of degree n.

So, the polynomial in lambda is called the characteristic polynomial. So, what we get expanding the determinant is, we get a polynomial equation in lambda. So suppose, this will be a polynomial equation lambda of degree n. So, it will have in general n roots.

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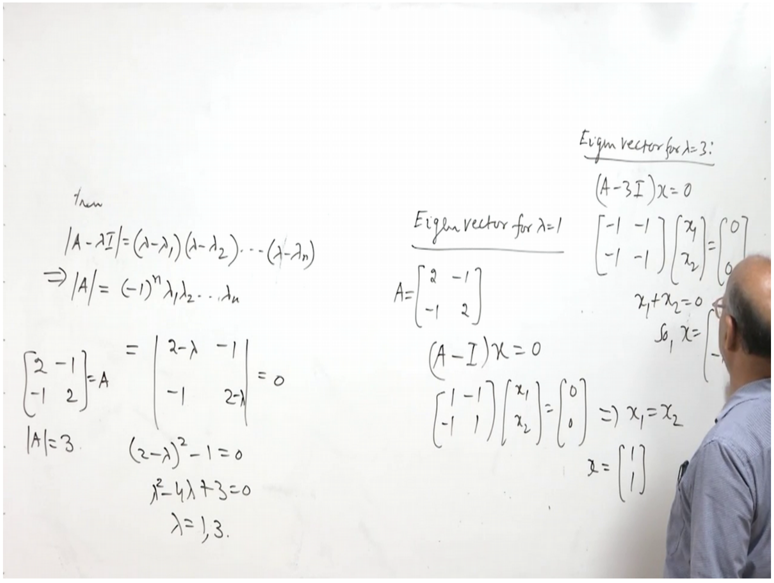

Let the roots of the equation be lambda 1, lambda 2, lambda n, then determinant A minus lambda I, we can write equal to lambda minus lambda 1, lambda minus lambda 2 and so on. Lambda minus lambda n, suppose roots are lambda 1, lambda 2, lambda n, we can notice of one more property form here, taking lambda equal to 0. You can see that the determinant of A from here turns out to this valid for any lambda. So, when we put lambda equal to 0 here, what we get is determinant of A equal to minus 1 to the power n into lambda 1, lambda 2 and so on lambda n.

So, determinant of the matrix A must be equal to minus 1 to the power n into product of the n Eigen values. You can see here, our matrixes 2 minus 1 minus 1 and 2, if you find the determinant of this matrix, what it is, is 4 minus 1. So, this is 3 and the Eigen values are o1 and 3. So, product of the Eigen values is 3 and determinant is 3 and here, we have 2 by 2 matrix, 2 minus 1 to the power two is one. So, determinant of A is equal to the product of the Eigen values. So, this is another check on the calculation of the Eigen values of the given matrix. You can multiply all the Eigen values and then multiply by minus 1 to the power 30 n, where n is the order of the matrix. It should be equal to determinant of A and when we expand here, what we notice is that the coefficient of lambda to the power n minus 1, which is lambda 1 plus lambda 2 lambda n that should be equal to trace of the matrix. So, from here 1 can also prove that trace of the matrix must be equal to the sum of the Eigen values.

So, there are two checks on the calculation of the Eigen values. one is that trace of the matrix must be equal to sum of the Eigen values, and the other one is that product of the Eigen values into minus 1 to the power n must be equal to determinant A. This means that if 1 Eigen value is 0, then determinant of A is always 0; that means, if the determinant of A is equal to 0, then you must always get 1 Eigen value equal to 0. Now, we can find the Eigen vectors for the Eigen values A lambda equal to 1 and 3. So, Eigen vector for lambda equal to 1, the matrix is 2 minus 1 minus 1 and 2 and this is A equal to this.

So, we have the equation A minus lambda is 1. So, A minus I into x equal to 0. So, we subtract unit matrix from here. So, what we get is 1 minus 1 minus 1 - 1and then we have x 1 x 2 equal to 0. So, we get two equations, one is x 1 minus x 2 equal to 0 the other one is minus x 1 plus x 2 equal to 0. So, what we get is x 1 equal to x 2 and therefore, the vector x can be taken as 1 1. The linearly independent Eigen vector A corresponding to lambda equal to 1, can be taken as 1 1 and the Eigen vector, similarly for lambda equals to 3. So, this time we subtract 3 I from a a minus 3 1 into x equal to 0. So, this will be a, if we subtract 3 I. So, we get minus 1 minus 1 and then minus 1 and then we get minus 1.

So, this will give you x 1 plus x 2 equal to 0. So, we can take x 2 equal to minus 1 then, x 1 is equal to 1. So, x is equal to 1 minus 1. So, we can find the Eigen vectors.

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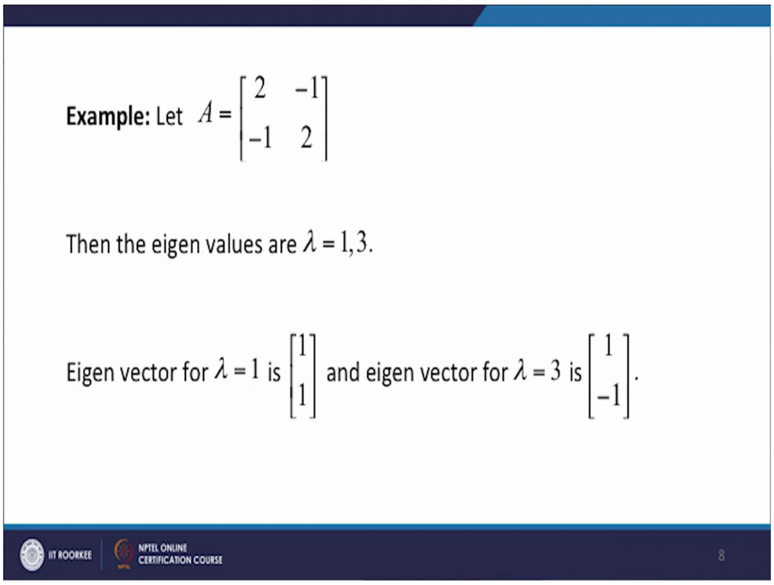


Corresponding to the two Eigen values 1 and 3, for 1 we have 1 - 1 and for 3 we have 1 minus 1.

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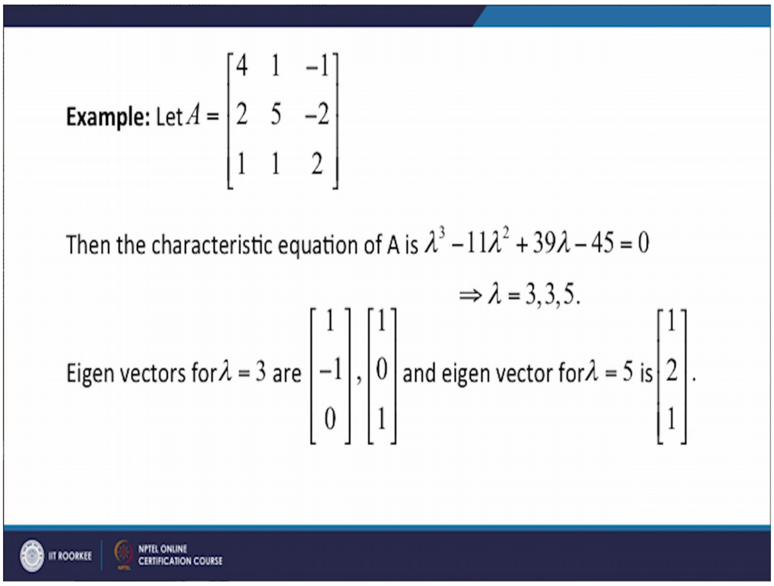


We can take one more example. Here, we are taking a 3 by 3 matrix. You can find the characteristic equation determinant of A minus lambda I equal to 0, expanding that we will get a cubic equation in lambda the equation, will come out to be lambda A cube minus 11 lambda square plus 39 lambda minus 45 equal to 0 up to a cubic equation, which is not difficult to solve. So, we can solve this equation. The values of lambda will come out to be 3, 3, and 5. Now, you can see A, if you take the sum of the 3 Eigen values, what we get is 3 plus 3, 6, 6 plus 5, 11 and the trace of the matrix is 4 plus 5 plus 2, which is also 11. So, some of the Eigen values equals the trace of the matrix. You can see the determinant of A must be equal to minus 1 to the power N into product of the Eigen values, the product of the Eigen values is 5 into 3 into 3.

So, 45 multiplied by minus 1 to the power 3. So, we will get minus 45. So, you can find the determinant. Now, Eigen vectors: - let us find corresponding to the Eigen values here, we can notice that then Eigen value occurs twice. So, A - 33. So, corresponding to Eigen value, lambda equals to 3. Let us see how many Eigen vectors we get; sometimes what we get is corresponding to a repeated Eigen value. We do not have the same number of Eigen vectors linearly independent on Eigen vectors and sometimes, we have the same number of Eigen vectors. So, here the Eigen value lambda equal to 3 occurs twice. So, its algebraic multiplicity is 2 and we shall see the geometric multiplicity. Geometric multiplicity is the number of Eigen vectors corresponding to the repeated Eigen value lambda equal to 3.

So, let us find the Eigen vector for lambda equal to 3. So, we have the matrix 4 1 1, and then we 41 minus 1.

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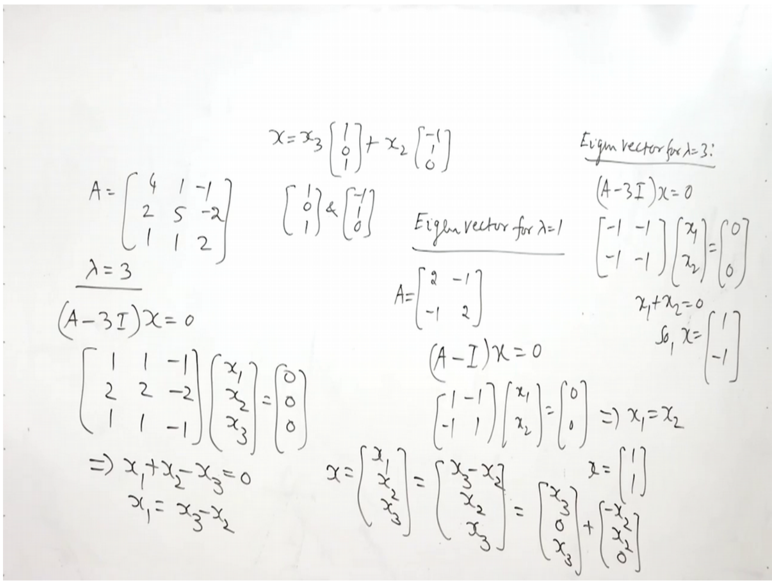


Then, we have 25 minus 2 and we have 1 1 2 lambda is equal to 3. Let us take lambda equal to 3. So the Eigen vector will be given by A minus 3 I x equal to 0. So, we subtract the 3 times identity matrix of order 3 from A. So, what we will get is 1 1 minus 1 1 2, minus 2 1 1 minus 1 and then, x is x 1 x 2 x 3.

Now, first equation will be x 1 plus x 2 minus x 3 equal to 0. Second equation is twice times the first equation. So, they are same equations. Third equation is also the same as the first equation. So, this gives you x 1 plus x 2 minus x 3 equal to 0. There is only one equation now, you can write x 1 equal to x 3 minus x 2, to write the corresponding linear, find the linearly independent Eigen vectors. Let us write x is equal to x 1 x 2 x 3 and then, we can put the value of x 1 x 3 minus x 2 and here, we have x 2 x 3. Then, we can express it as a linear combination of x 2 x 3 x 2 times.

So, I will have broke bracket into like this, x 3 then 0 then x 3 and then I will write minus x 2 and x 2 and O. So, that I can write x as a linear combination. So, x is equal to x 3 times 1 0 1 and x 2 times minus 1 1 0. So, the linearly independent Eigen vectors, associated with lambda equal to 3 can be taken as 1 0 1 and minus 1 1 0. Every other Eigen vector will be a scalar linear combination of these 2 Eigen vectors and so, corresponding to the repeated Eigen value lambda equal to 3. We also have the same number of linearly independent Eigen vectors here. So, algebraic multiplicity of 3 is 2 and the geometric multiplicity is also 2. There is a result which says that the geometric multiplicity does not exceed the algebraic multiplicity

Now, we can find the Eigen vector per lambda equal to 5 in a similar manner. So, for the other Eigen value lambda equal to 5.

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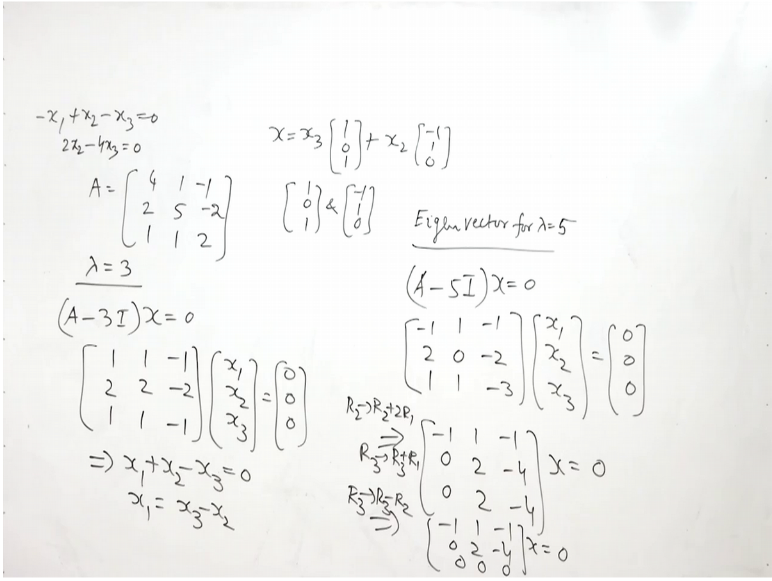

We have to subtract 5 I from A. So, when we do that, we get minus 1 1 minus 1. Then from 2 we subtract 5. So, 0 then minus 2, then we get 1 1 and we are getting minus 3. We can do elementary row operations on this to reduce it to an equivalent form. So, if I add A to the first row to the second row, this will change. We will simply do the operations on this coefficient matrix because this is 0. So, it will not change and x 1 x 2 x 3 is the column vector, these are the components of these columns.

So, that they do not of effect. So, this will be changing to minus 1 minus 1, and I multiply it in the first row by 2 and add 2 to the second row. So, we get here, 0, 2 and then minus 2 minus 2 and minus 4 and then, I add first row to the third row. So 0 then 2 and then minus 4. So, what I am doing here: 2 goes to R 2 2 R 1 and R 3 goes to R 3 plus R 1.

So, this system of equations will now change to this new system of equations. So, this is x equal to 0, now what we do in the second row, we can subtract from the third row. So, this will give R 3 goes to R 3 minus R 2 and we shall have minus 1 1 minus 1024 minus 4 and 0 0 x equal 0. So, third row becomes a 0 row. So, we are getting only two equations. So, we can write the equations now, minus x 1 plus x 2 minus x 3 equal to 0. This is the first equation and second equation is 0 into x 1 2 into x 2 minus 4 into x 3 is equal to 0. So, 2 x 2 minus 4 x 3 equal to 0 and we can solve this very easily.

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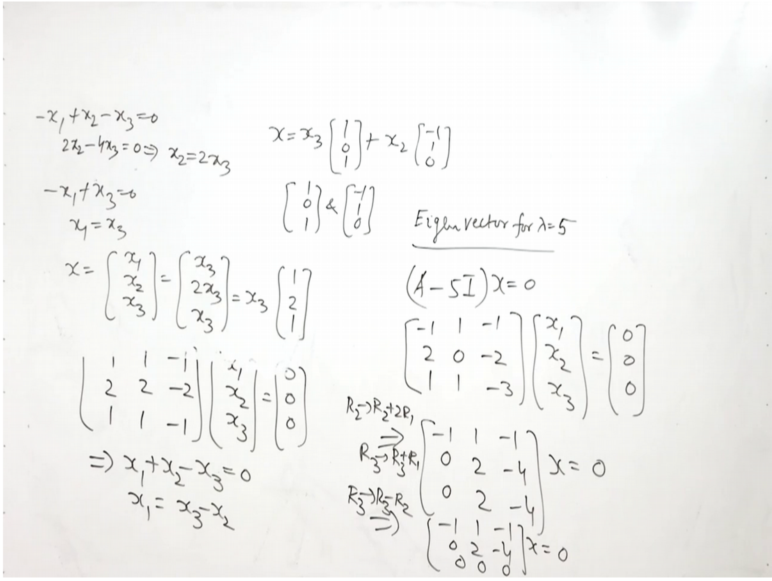

So the second equation gives x 2 equal to 2 x 3. Let us put the value there, in the first 2 equations or minus x 1, then x 2 is 2 x 3 minus x 3 is x 3. So, we get x 1 equal to x 3. So, x is equal to x 3 2 x 3 and x 3. So, I can write it as x 3 times 1 2 1. So, 1 2 1 is the linearly independent Eigen vector associated to the Eigen value lambda equal to 5.

So, this is how we find the Eigen values and the Eigen vector for this 3 by 3 matrix, with that I would like to conclude my lecture.

Thank you very much.