Numerical Linear Algebra
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Lecture – 12
Row Space, Column Space and Null Space

Hello friends. Welcome to my lecture on row space, column space and null space. Let us say, we have a matrix m by n in size let a b equal a i j we an arbitrary m by n matrix where the entries of the matrix belong to a field K, then the rows of A.

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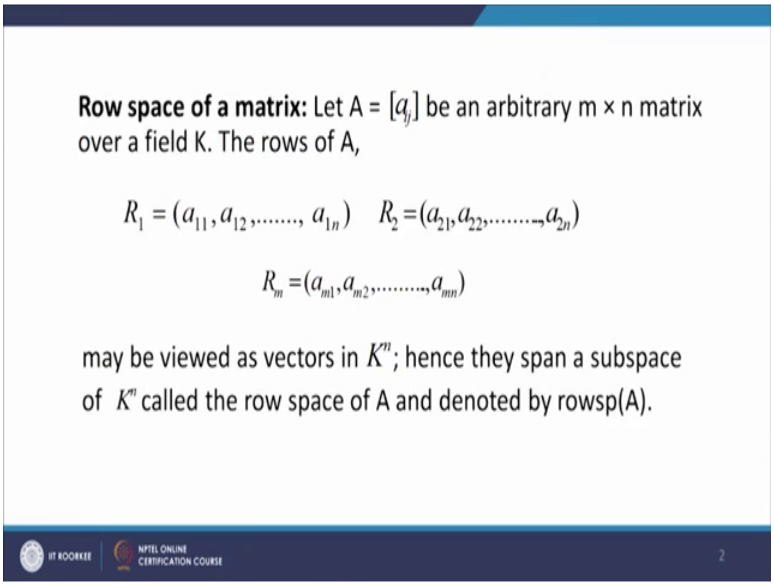

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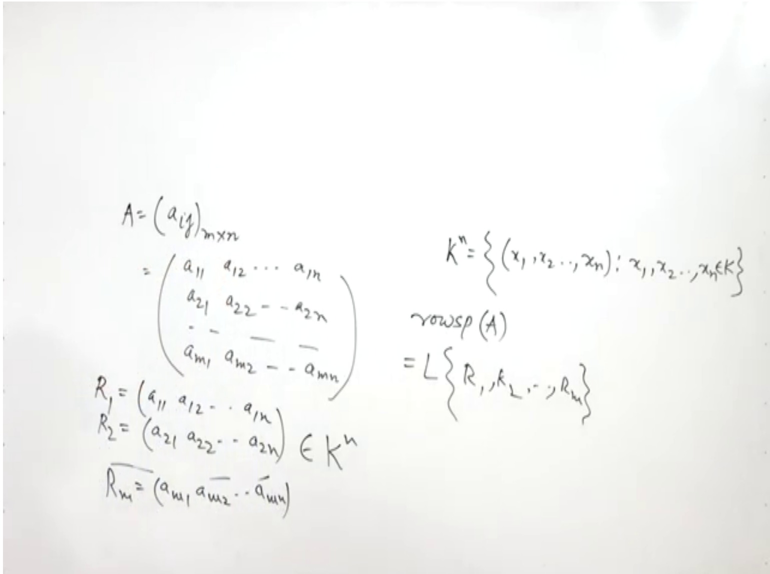

Let us say A is this one.

So, there are m rows and n columns; let us denote by R 1 the first row, R 2 denotes the second row and R m denotes the mth row. So, this can be viewed as vectors belonging to K to the power n because K to the power n is this set of n tuples where x 1, x 2, x n belong to K. So, this can be viewed as vectors belonging to K to the power n.

Now, these a span A subspace of therefore, they will a span A subspace of K to the power n which means which is called as the row space of A. So, these m vectors when we take linear combination of those m vectors, they span A subspace of K to the power n which is called the row space of A and so, row space of a row space of A, we write as it is the linear span of these R 1, R 2 and R m. So, these linear span of the m vectors R 1, R 2, R n is nothing, but the row space of A suppose A and B are row canonical matrices ok.

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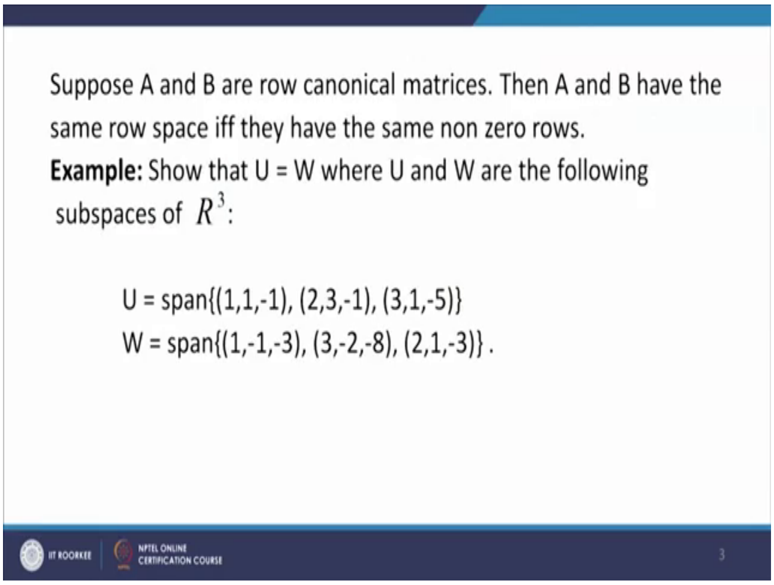

Then A and B have the same row space if and only if they have the same number of nonzero rows, so, let us say for example, you take u to be the linear span of the 3 vectors 1 1 minus 1 2 3 minus 1 3 1 minus 5 and W be the span of 1 minus 1 minus 3 3 minus 2 minus 8 2 1 minus 3, then let us show that they span these vector span the same space U and W. So, row space of the 3 vectors 1 1 minus 1 2 3 minus 1 3 1 minus 5 is the same as the row space of the other 3 vectors.

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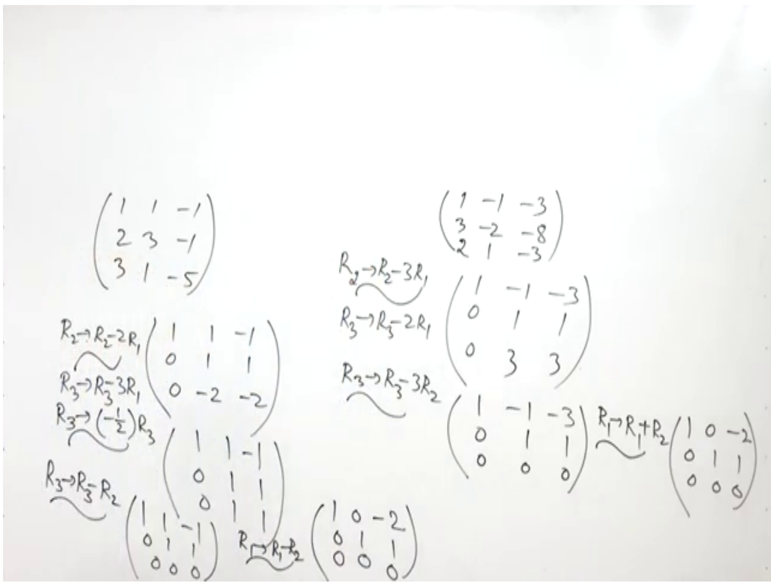

So, what we do is in order to show this let us form the matrix. So, 1 1 minus 1, then we have 2 3 minus 1 and then we have 3 1 minus 5. So, will row reduce it to row canonical form, then we will have this is R 2 goes 2, R 2 minus 2 R 1, R 3 goes to R 3 minus 3 R 1 and we have 1 1 minus 1, this is 0, we have 1 here, 2 minus 1 is 1, then we will get 0 here 1 minus 3 is minus 2 and then we have here we are sub multiplying by we are multiplying by 3. So, 1 minus 3 is minus 2 and here minus 3 means we minus 3 into minus 1 is 3. So, 3 minus 5 is minus 2 ok.

Now, this leading coefficient is 1, here also we have 1, but here we have minus 2. So, divided by minus 2 the third row; so, R 3 goes to minus half R 3 and we get 1 0 1, 1 1 1, 1 1 1 minus 1 0 1 then 0 1 1 and then we reduce further. So, this is what we do? We will multiply it by 1 subtract it here. So, R 3 goes to R 3 minus R 2 and we get 1 1 minus 1 0 1 1 0 0 0. So, we have this is row canonical form of these 3 vectors and then the other 3 vectors let us take. So, we have the other 3 vectors are 1 minus 1 minus 3 1 minus 1 minus 3 and then we have 3 minus 2 minus 8 and the third vector is 2 1 minus 3.

So, this also be reduced to row canonical form. So, R 2 goes to R 2 minus 3 R 1, R 3 goes to R 3 minus 2 R 1, we get 1 minus 1 minus 3 0 here. So, we multiply by minus 3 under here. So, 3 minus 2 is 1, then we get here minus 3 into minus 3 nine minus 8 is 1 and then 2 minus 2 0, then here 2 minus 2 we are multiplying. So, 2 plus 1 is 3 and here we are multiplying by minus 2. So, we get 6 minus 3 is 3, then we multiplied by 3 subtract from here. So, R 3 goes to R 3 minus 3 R 2, then we will get 1 minus 1 minus 3 and then we will get 0 1 1, then we get 0 0 0.

So, we will have the first set is 1 0 minus 2, 0 1 1, 0 0 0, wait, we have to row reduce it to row canonical form. So, this means we have to this we have this is further we have to do it. So, this has to be subtracted from here. So, we write R 1 goes to R 1 minus R 2. So, 1 0, alright and this will subtracted. So, minus 2 here and then 0 1 1, 0 0 0. Now this is row canonical form these row canonical form because this is the only nonzero entry in its row column and here, we add it further second row to first row. So, R 1 goes to R 1 plus R 2. So, we get 1 0 minus 2, 0 1 1, 0 0 0. Now this is row canonical form these also row canonical form and we have they have same number of 0s and the nonzero same number of 0 rows and nonzero rows are identical. So, they span the same subspace. So, u s d equal to W.

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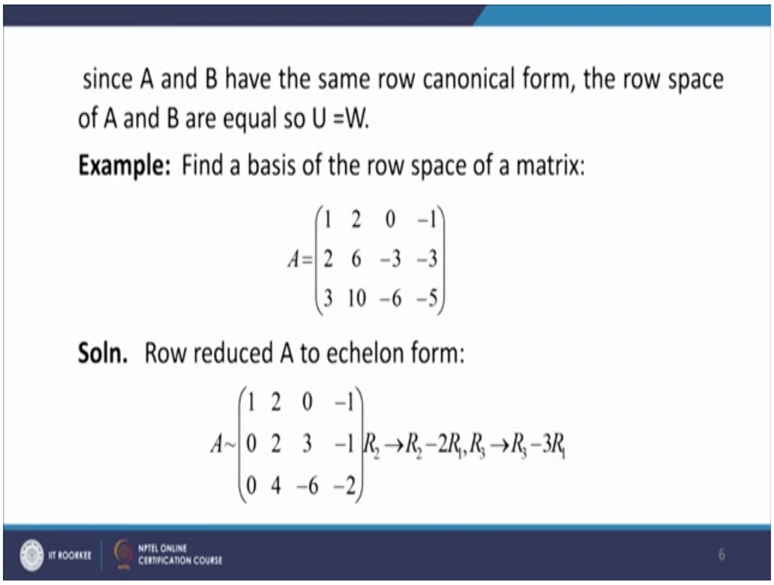

So, this is what we have; now let us go to example where we find the basis of rows space of a matrix this matrix is 1 2 0 minus 1 2 6 minus 3 minus 3 3 10 minus 6 minus 5. So, let us row reduced echelon form; that means, we subtract from the second row 2 times the first row, and 3 times the first row be subtract from the third row. So, we will get the matrix A to be equivalent to 1 0 minus 1 2 0 minus 1 0 2 3 minus 1 0 4 minus 6 minus 2 and then with the help of the second row be made the third row in the third, we reduce 4 element into 4 to 0.

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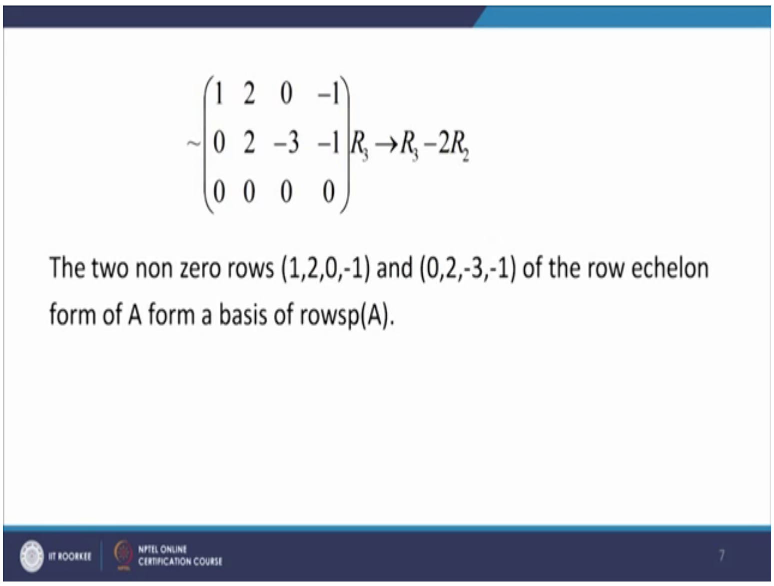

So, R 3 minus 2 R 2 we do. So, when we do R 3 minus R 2 we get the matrix 1 2 0 minus 1 0 2 minus 3 minus 1 and 0 0 0 0.

Now, the 2 rows nonzero rows 1 2 0 minus 1 and 0 2 minus 3 minus 1 of the row echelon form of A, they form a basis of row space of A because these 2 vectors are linearly independent and they span the row space of A. So, there will form basis ok.

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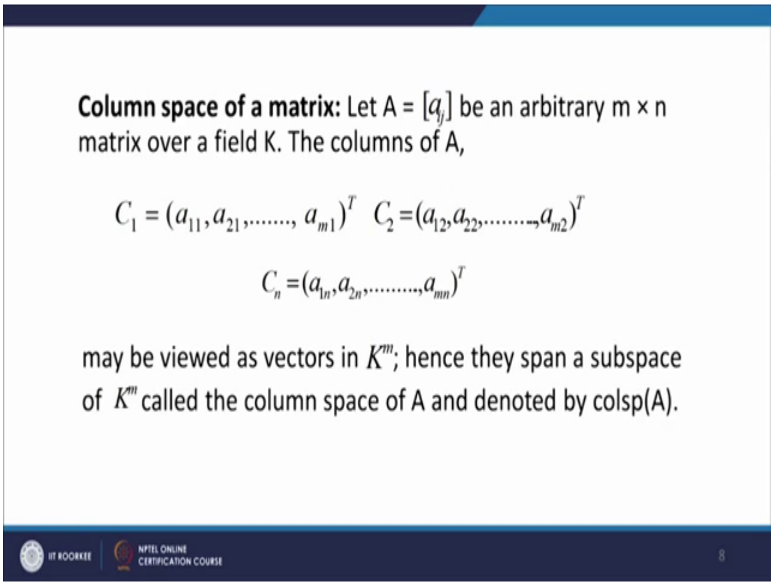

Now, next we discuss the column space of a matrix. So, let A equal to a i j be an arbitrary m by n matrix over A field K.

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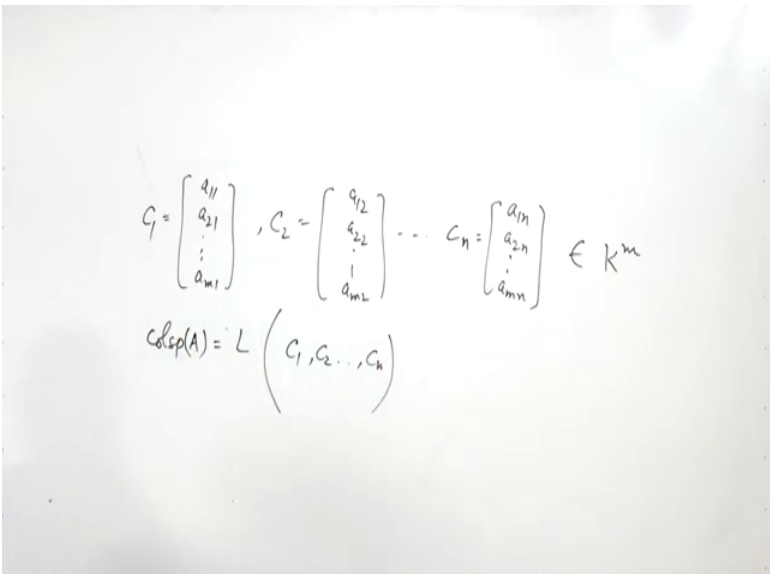

The columns of A R C 1 equal to a 1 1, a 2 1 and then a m 1, C 2 equal to a 1 2, a 2 2, a m 2 and then we have C m C n we have C n, there are n column n columns. So, C n equal to a 1 n, a 2 n, a m n. So, these n columns can be viewed as vectors in K to the power m because each has m components.

So, they be belong to K to the power m and. So, therefore, they span a subspace of K to the power m which we call as a column space of the matrix a and be denote that column space by this notation column space column space of a means a span of these n vectors. So, C 1, C 2, C n.

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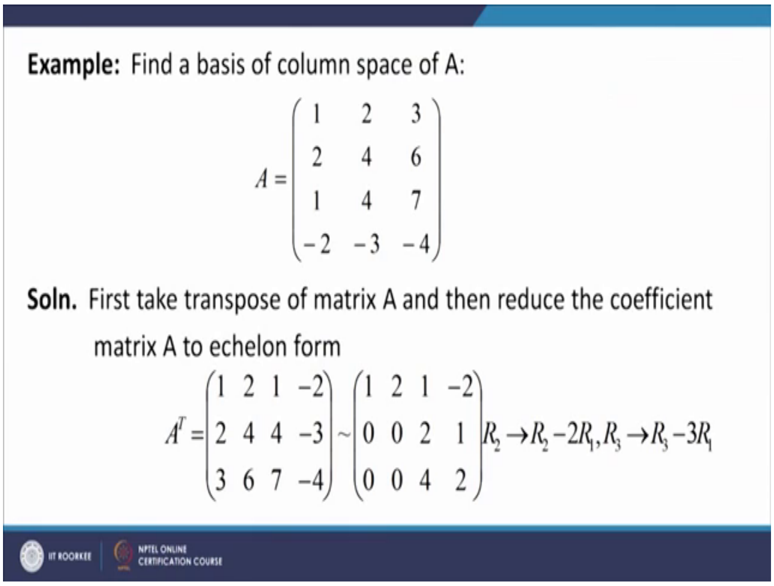

So, let us this example in this example we find the basis of the column space of the matrix A. So, here what we do is the columns of the matrix A R, the first column is 1 2 1 minus 2 second column is 2 4 4 minus 3, 3 6 7 minus 4. So, what we will do is that will write will take the first the transpose of the matrix A and when you take the transport of the matrix A. These columns of A become the rows of a transport. So, then be reduced this matrix A transpose to the echelon form. So, a transpose will have the columns of a as rows. So, 1 2 minus 1 minus 2, 2 4 4 minus 3, 3 6 7 minus 4; so, we row reduced to echelon form. So, R 2 goes to R 2 minus 2, R 1 R 3 goes to R 3 minus 3 R 1 will reduce the matrix to 1 2 1 minus 2, 0 0 2 1, 0 0 4 2 and then we subtract 2 times the second row from the third row.

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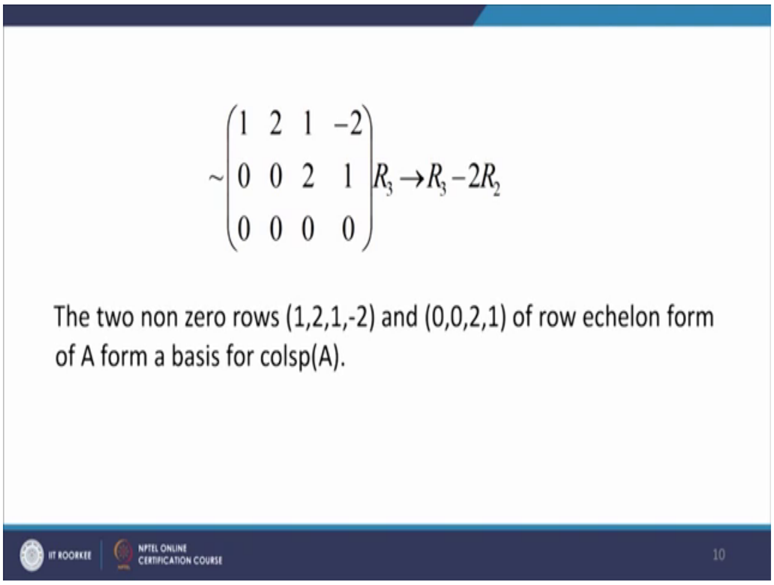

So, we get the matrix 1 2 1 minus 2, 0 0 2 1, 0 0 0 0.

So, the 2 rows first 2 rows are you can say the first 2 columns of the matrix A are linearly independent and there a span is the subspace of a column space of A. So, the 2 non zero columns 1 2 1 minus 0 0 2 1 of the row reduced echelon form a basis of the column space of A. So, now, we discuss the null space of a matrix.

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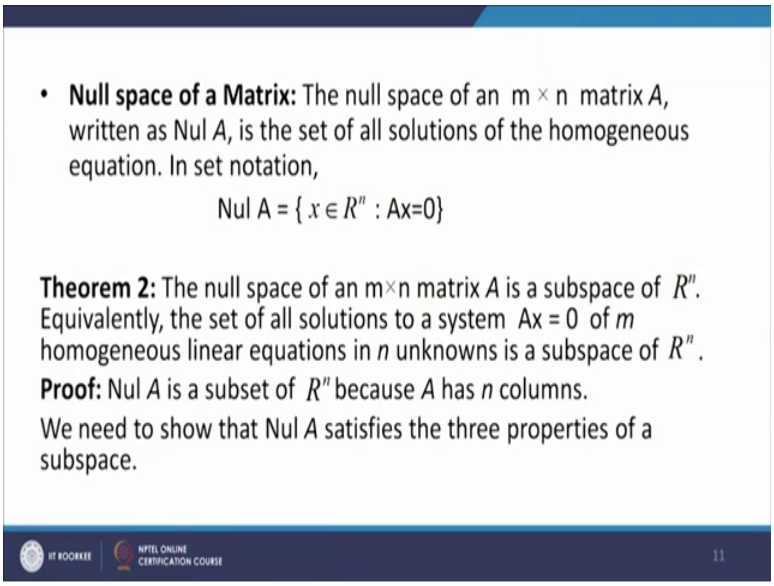

The null space of a m by n matrix A written as null space of a null A is the set of all solutions of the homogeneous equation x equal to 0.

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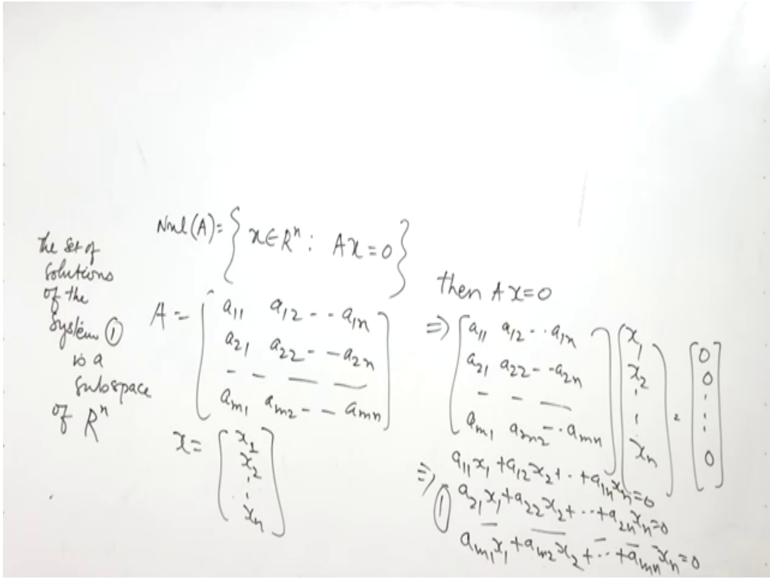

That is null space of a is a set of all x belonging to a R n such that x equal to 0. So, null space of A. So, the null space of a m by n matrix a is of the space of R n equality in the set of all solutions to a system x equal to 0 of m homogeneous questions.

In an unknowns is a subspace of R n, see we have suppose A is this one m by n matrix and x is an element of R n. So, let us say it has component x 1 x 2 x n then A x equal to 0 gives us an questions in an unknown.

0 is an element of R m 0 vectors. So, these equal these gives you a 1 1 x 1m a 1 2 x 2 and so on a 1 and x n equal to 0 then a 2 1 x 1, a 2 2 x 2, a 2 and x n equal to 0 and so, on a m 1 x 1, a m 2 x 2, a m n x n equal to 0. So, we get m linear equations in an unknown the unknowns are x 1 x 2 x n the components of the vector x. So, this sort of solutions of this m homogeneous linear equations in an unknown is a subspace of R n this we can easily prove. So, let us see how we prove this. So, null space of A the set of solutions of the system one let me say this is system one of the system one is a subspace of R n. So, let us see we can easily prove that that the null space of A satisfies the 3 properties of a subspace.

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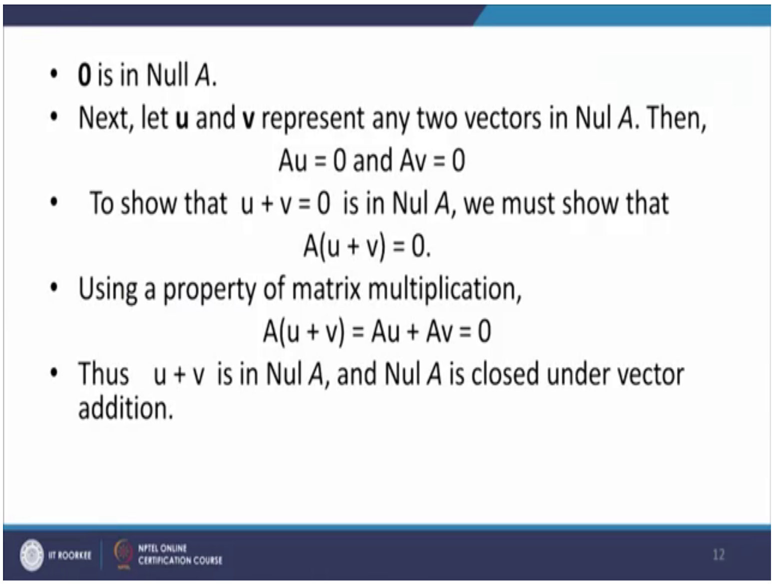

First of all the 0 vector; obviously, belongs to null space because we take x 1, x 2, x n to be all 0s, then this system is satisfied. So, 0 vector is there a null space of a and let u and v to the v any 2 vectors in null space of A, then by the definition of null space of A u will be equal to 0 vector A v will be equal to 0 vector and then we want to show that u plus v is a null space of A. So, we have to show that A u plus v equal to 0.

Now, A u plus v by the definite matrix multiplication property is A u plus A v, A u is 0 A v is 0. So, 0 plus 0 vector is 0 vector. So, get A u plus v equal to 0 and therefore, u plus v is in null space of A and so, null space of A is closed under vector addition next we show that null space of a is closed under a scalar multiplication. So, let us say if c is any scalar in the field of a that is field K, then A c u where u is a vector in null space of a.

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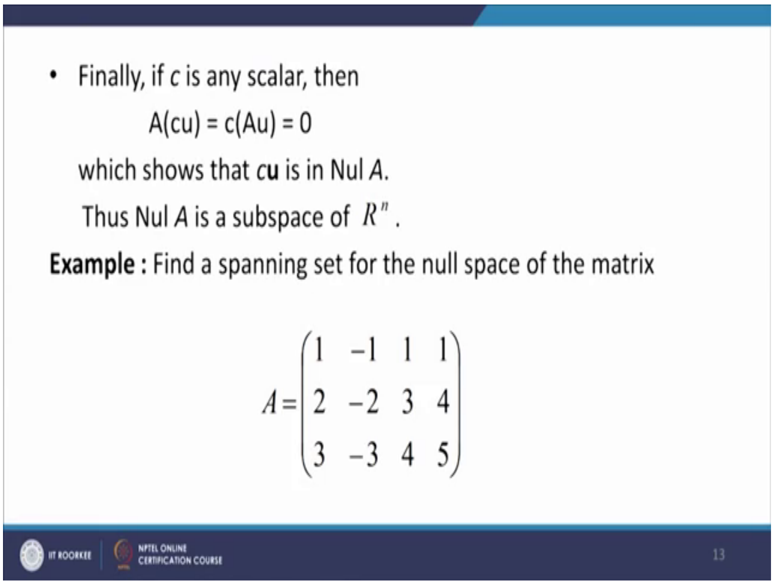

So, A c u by the matrix multiplication d property, it can be it is c times A u. So, c A u equal to 0. So, c into 0 is 0. So, we show that the vectors c u is in null space of A and therefore, null space of A is a subspace of R to the power and now let us find the spanning set for the null space of the matrix 1 minus 1 1 1, 2 minus 2 3 4, 3 minus 3 4 5. So, what will do first is that, we will find the general solution of a x equal to 0. So, did let us reduced the augmented matrix let us write the augmented matrix a 0 to the echelon form and then A matrix is equal to 1 minus 1 1 1.

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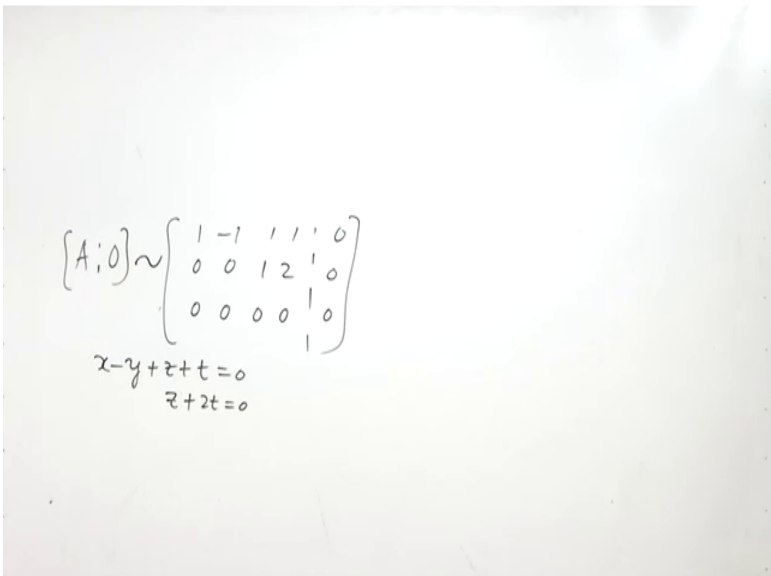

Let us write a 0 augmented matrix A 0. So, 1 minus 1 1 1; 0 and then we have 0 0 1 2; 0 and then we have 0 0 0 0. So, after we reduce the given matrix A 0 to echelon form will get this mattress. So, this matrix is it we obtained by the row reduced by reduced echelon form. So, be reduced the given matrix A 0 to echelon form and after we do that.

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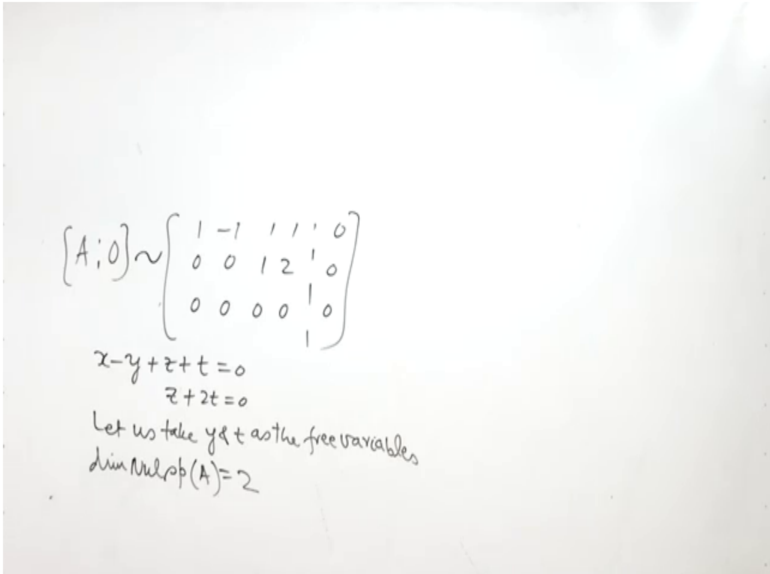

This is what we get 1 minus 1 1 1; 0, 0 0 1 2; 0, 0 0 0 0; 0 from here this is the cop, this is the first column corresponds to the x the second y, then we have z on and then we have t and then we have 0. So, we have the equations x minus y plus z plus t equal to 0 and then y plus 2 t equal to 0. So, z plus 2 t equal to 0. So, z plus 2 t equal to 0 and the last equation is 0 equal to 0.

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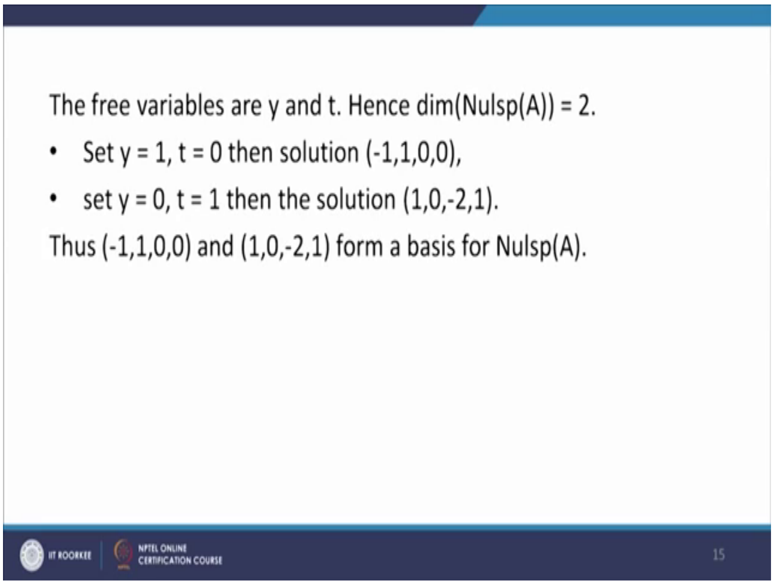

So, we from here the 2 tables are free variables we can take them as y and t y and t let us take. So, let us take y and t as the free variables then null space of a dimension of null space of A is equal to 2. So, let us put t y equal to 1 and t equal to 0, we will get the solution as minus 1 1 0 0 and then we let us take y equal to 0 t equal to 1, we will get another solution 1 0 minus 2 1, the 2 vectors are both linearly independent of each other. So, they will form a basis of null space of A with that I would like to end my lecture.

Thank you very much.