**Numerical Linear Algebra**
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**Lecture - 11**
**Orthogonal Subspaces**

So, hello friends, I welcome you to my lecture on Orthogonal Subspaces. Let us consider a real vector space b that is a vector space where the field is the set of real numbers.

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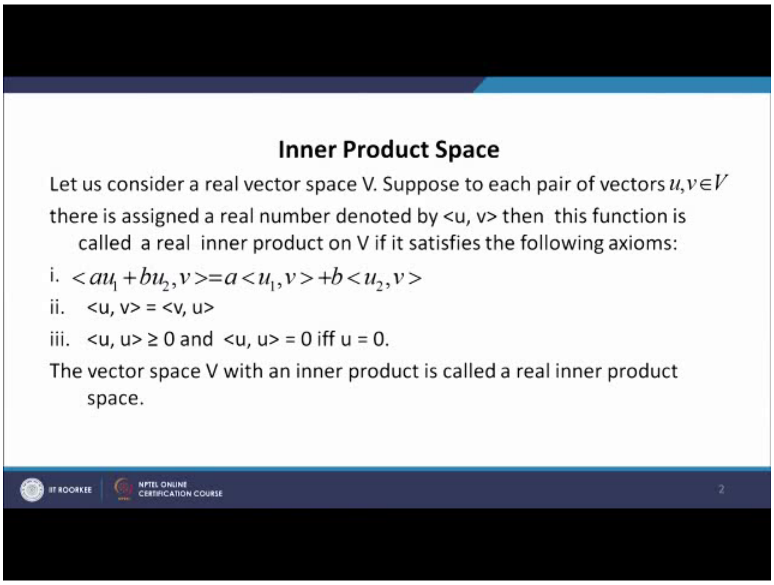


Now, if let us suppose that if we can assign to each pair of vectors u v belonging to B a real number donated by this, then the this function by which we are able to associate to each pair of vectors u v a real number given by this notation is called in a product on b provided it satisfies the following axiom.

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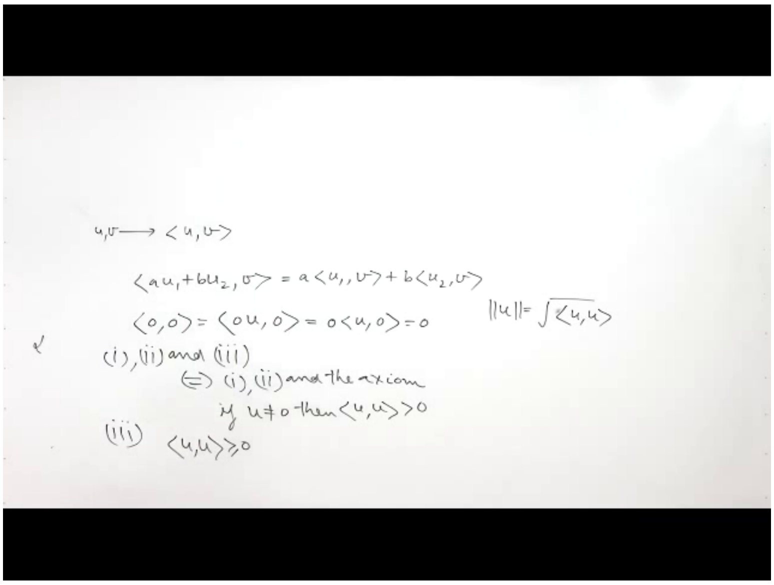


The first axiom is the inner product of au 1 plus bu 2 with v is equal to au 1 v plus bu 2 v and the second one is u v equal to v u the third one is u u is greater than are equal to 0 and then u u equal to 0 if and only if u equal to 0. So, the vector space b equipped with this inner product is called a real inner product space.

Now, from the if we look at the axioms 1 and 2, if we look at the axiom 1 and 2 au 1 plus bu 2 v equal to au 1 v plus bu 2 v and u v equal to v u then from these 2 axioms, we can easily a prove that u c v 1 plus d v 2 is equal to c u v 1 plus d u v 2 because u cv 1 plus dv 2 is equal to cv 1 plus dv 2 u by the axiom 2 and then applying axiom 1 we have c v 1 u plus d v 2 u.

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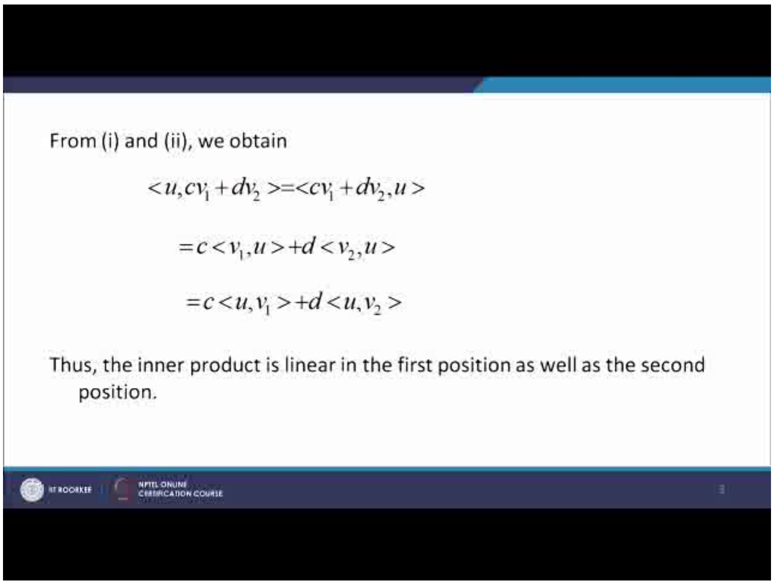


Now, we apply the axiom 2 and we get c u v 1 plus d u v 2 that the inner product is linear in the first position as well as in the second position.

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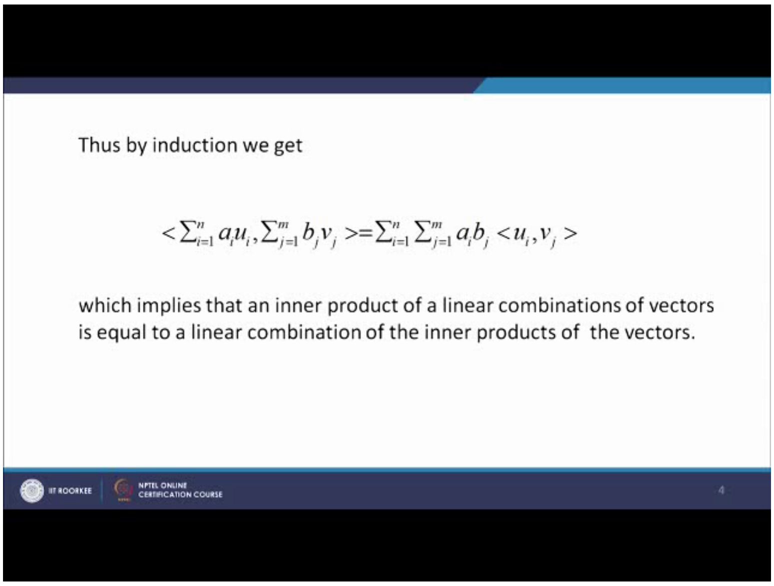


And if we apply the mathematical induction we can easily show that the inner product of sigma i equal to 1 to n, a i u i and sigma j equal to 1 to m b j v j is equal to sigma i equal to 1 to n, sigma j equal to 1 to m, a i b j inner product of u i v j which implies that an inner product of a linear combination of vectors is equal to a linear combination of the inner products of vectors.

Now, from axiom 1 let us see the axiom 1 this is axiom 1, au 1 plus bu 2 v is equal to au 1 v plus bu 2 v. From this axiom we have the following 0 0 see 0 0 equal to the inner product of 0 vector with 0 vector I can write as 0 u 0, because 0 u is equal 0. So, 0 u 0 and then I apply the first axiom. So, I can write the 0 u 0, 0 u 0 and this is equal to now, u 0 is a real number. So, 0 into u 0 equal to 0. So, this means that the first, second and third axioms of the inner product are equivalent to first and second and the axiom.

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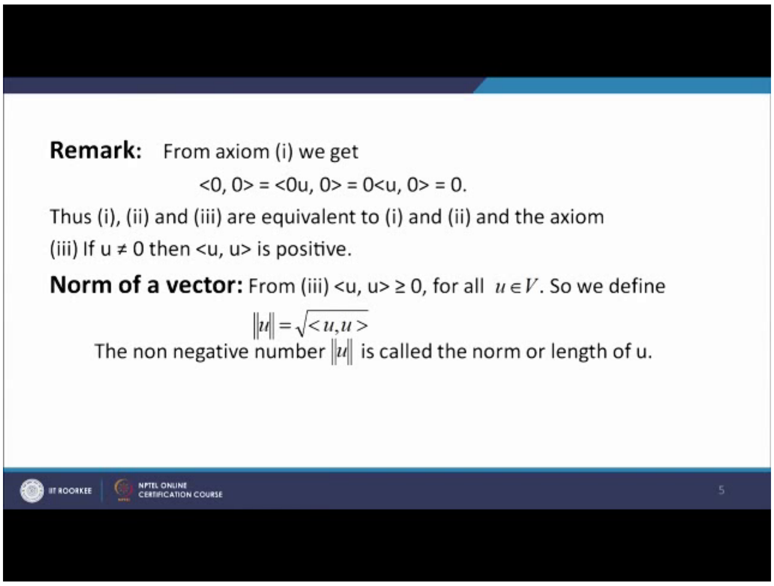


So, 1 2 and 3 the 3 axioms of the inner product are equivalent to 1 2 and the axiom, if u is not equal to 0 vector then the inner product of u with u is strictly greater than 0.

So, a vector space with the first axiom, second axiom and this third one if u not equal to 0 then u u inner product of u u greater than 0 this will call as I will v a real inner product spaces. Now, let us define the norm of a vector from the third axiom the third axiom of the inner product space tells us that the inner product of u with u is a non-negative number. So, we can take the square root of this and define norm of u norm of u is equal to norm of u is the length of the vector u. So, this is multi square root of u with u. Now, this notation is called at the norm or the length of the vector u.

Now, we have the examples of inner product space, let us say the vector space R n over the real field R which is known as the fluid in any space. So, R n R let us consider.

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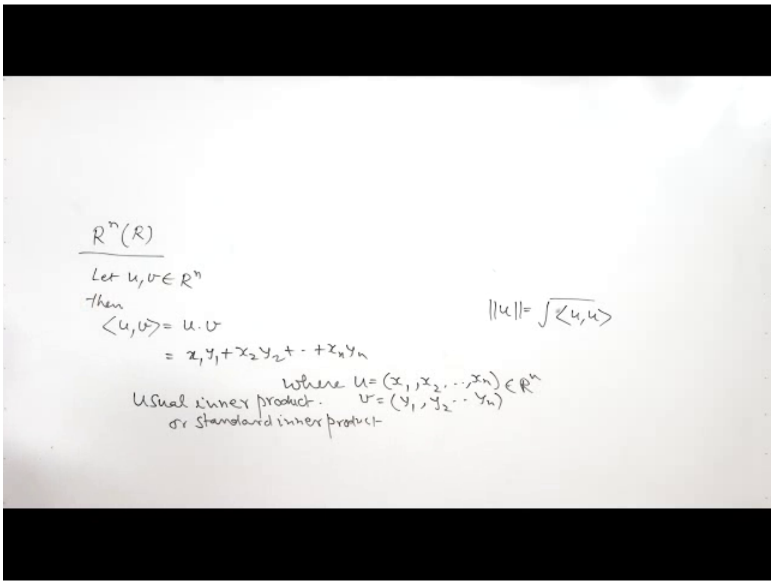


Let us define the inner product in R n as the dot product or scalar product in R n. So, let u v belong to R n then the inner product of u with v in R n v defined as the dot product of u with v are the scalar product of u with v. So, if I write u as x 1, x 2, x n and v as y 1, y 2, y n then I right x 1, y 1 plus x 2, y 2 and so on x n, y n where u is the vector which are elements of R n and we can easily check that all the axioms of the inner product are satisfied by when we define the scalar product as the inner product in R n. So, with this inner product R n becomes a inner product space.

Now, there are many other ways in which we can define an inner product in R n, but throughout our future discussion we shall be considering this inner product in R n and this inner product in R n is called we usual R standard inner product, and because of this and inner the inner products this space R n with this scalar product has the inner product this also called as the usual inner product space.

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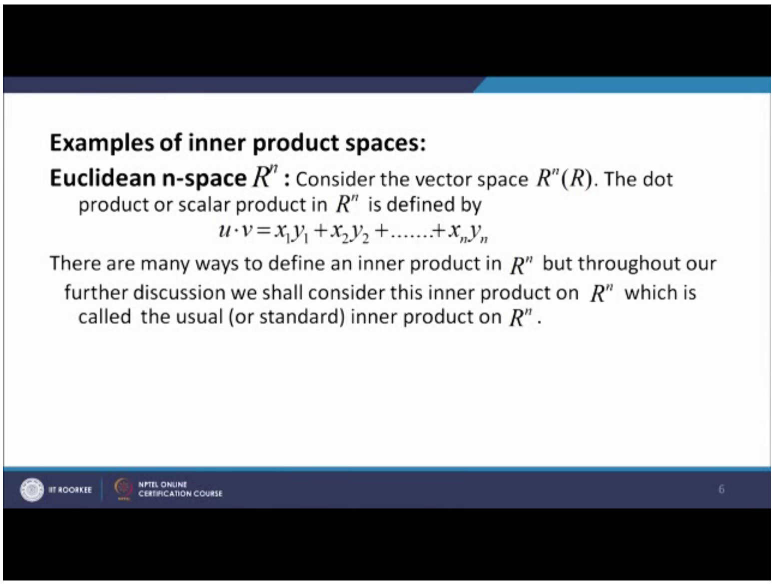


Now, they other space where we have we define inner product let us take as C a b function is space C a b.

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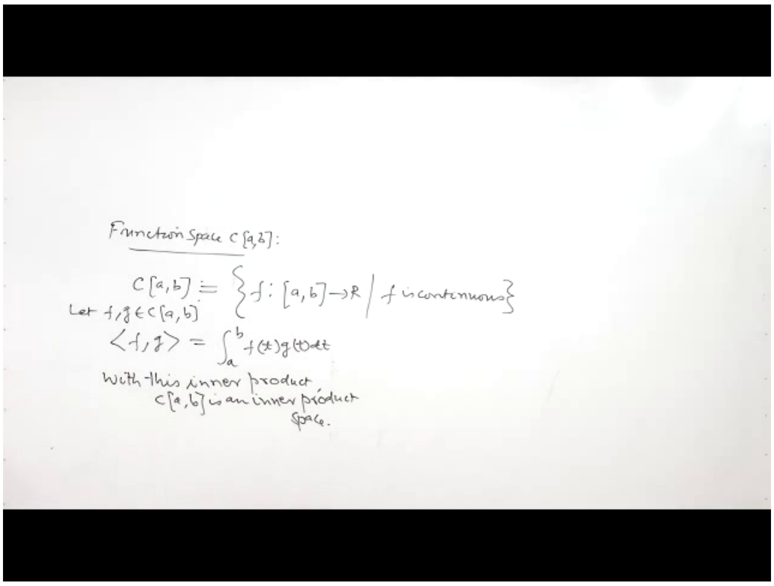


So, C a b is the space of all continuous functions on the bounded and closed interval a b we can write C a b as all functions defined from a b into R such that f is continuous function. And then let us define plus take any 2 functions f and g in C a b and define the integral of f t g t as the 2 each f and g belonging to C a b let us associate a number f g in this manner. We can again check that all the axioms of the inner product space are satisfied. So, this define say inner product in C a b and so with this inner product C a b is a inner product space.

Now, let us move to a orthogonality, let say we have the concept of orthogonality if we take 2 vectors x y belonging to R n. They are called orthogonal if there dot product is 0, so if x and y are there in R n then they will be orthogonal if x dot y equal to 0 and then we have the next definition a vector x belonging to R n will be called orthogonal to a set by which a subset of R n provided x that y is equal to 0 for any y belonging to R n.

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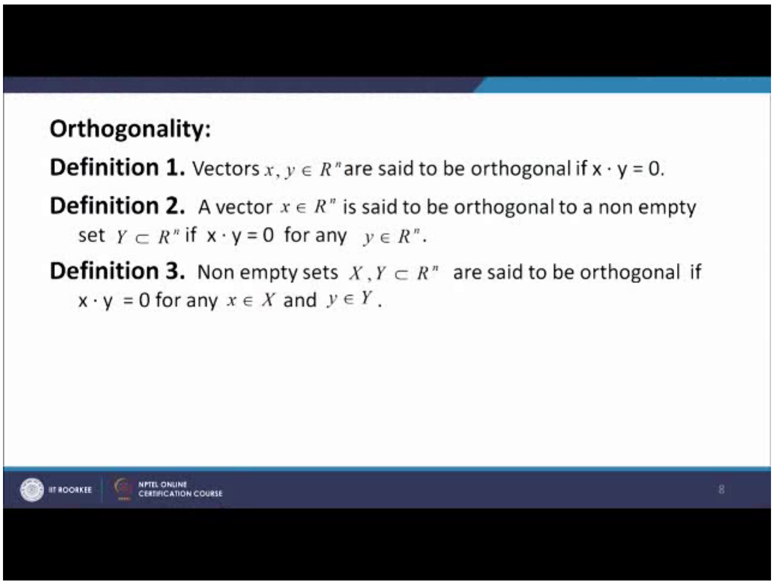


Now, if we take 2 sets x and y which are subsets of R n they will be orthogonal provided x dot y equal to 0 for any x belonging to x and y belonging to y. Now, let us take some examples.

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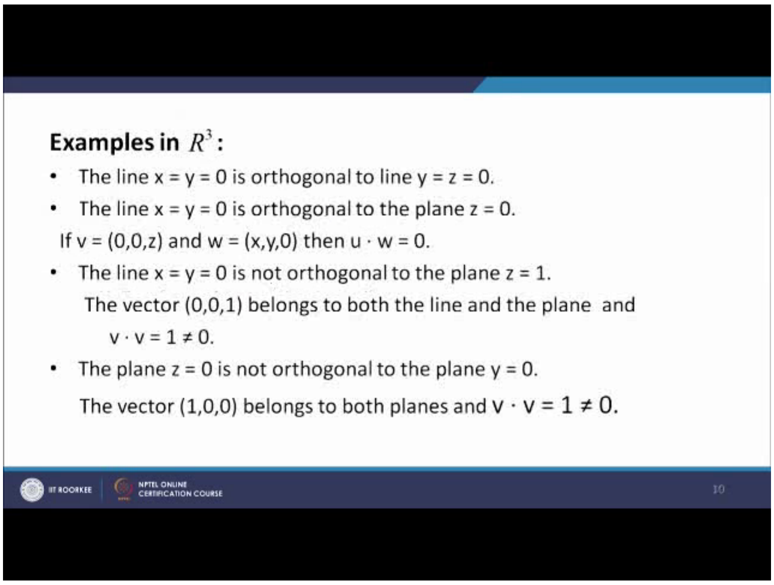


Let us take the space R q and see the examples where you will see that if you take the line x equal to y equal to 0, x equal to y equal to 0 means z axis.

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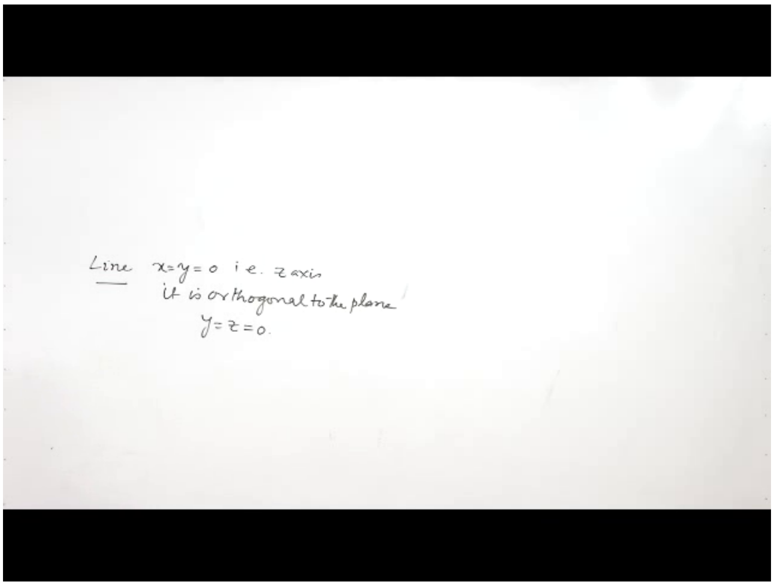


This line is nothing, but z axis. So, it is orthogonal to the plane y equal to z equal to 0, it is it is orthogonal y equal to z equal to 0; y equal to z equal to 0 means y equal to z equal to 0 it sorry not plane it is line sorry, it is line orthogonal to the line.

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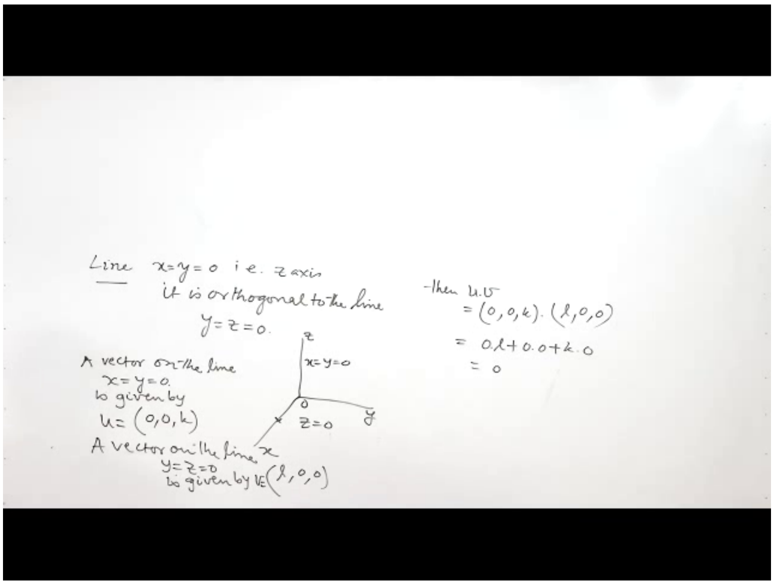


Now, we can see this is very easily this is you are x axis, this is y axis and this is z axis. If you take any point on the x axis a point on the x axis can be written as are a vector on the x th e line y x equal to y equal to 0 is given by on the line x equal to y equal to 0 is given by 0 0 let us say k. And a vector on the line y equal to z equal to 0 is given by l 0 0, where l is a real number. Now, let us take the dot product of the 2 vectors say this is u and this is v. Then u dot v is equal to 0 0 k dot l 0 0 which we have defined as the scalar product. So, this is 0 into l plus 0 into 0 plus k into 0. So, we get 0.

So, the the line x equal to y is equal to 0 is orthogonal to line y equal to z equal to 0, and then the line x equal y equal to 0 is orthogonal to the plane z equal to 0. So, you can see this is z axis x equal to y equal to 0 line and this is your the plane z equal to 0. So, line x equal to y equal to 0 is perpendicular to the plane z equal to 0 the plane z equal to 0 we can easily show this. So, if you take any vector here a vector will be of the form at say l m n 0 a vector; in z equal to 0 is of the form l m 0 ok.

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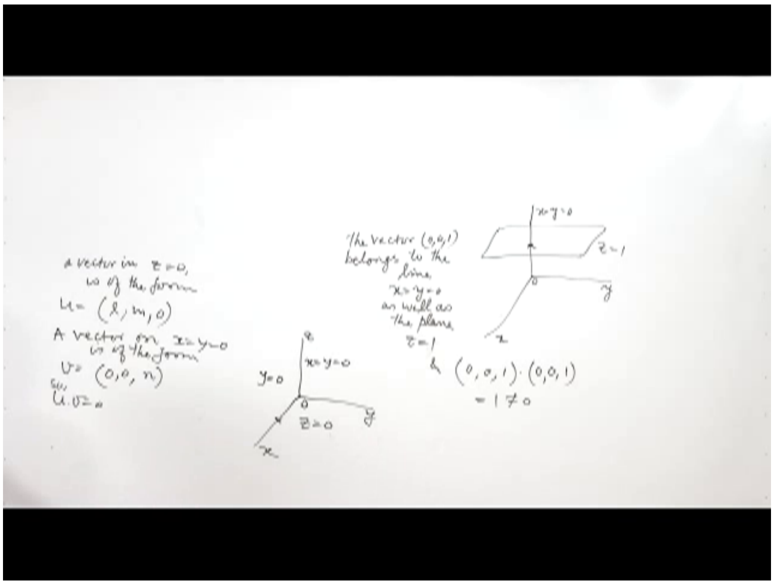


So, let me call this as u and a vector on x equal to y equal to 0, is of the form 0 0 n. So, u dot v is equal to 0 and therefore, there orthogonal to each other. But if you can see that the line x equal to y equal to 0 that is z axis is not orthogonal to the plane z equal to 1 plus is how it is. These your line this z axis x equal to y equal to 0 and say this is your plane z equal to 1. Then the line x equal to y equal to 0 is not orthogonal to the plane z equal to 1. Why because the vector 0 0 1, the vector the vector 0 0 1 belongs to the line x equal to y equal to 0 as well as the plane z equal to 0 z equal to 1. And the dot product of 0 0 1 with itself and 0 0 1 dot 0 0 1 is equal to 1 which is not 0. So, the line x equal to y equal to 0 is not orthogonal to the plane z equal to 1.

Then we go to the plane z equal to 0 is not orthogonal to the plane y equal to 0. So, these your plane z equal to 0 and then we consider the plane y equal to 0 this plane. The plane z equal to 0 is not orthogonal to the plane y equal to 0. Because the vector 0 0 1, 1 0 0 the vector 1 0 0 lies in both the planes and the dot product of 1 0 0 with itself is 1 which is not 0. So, the 2 planes are not orthogonal to each other.

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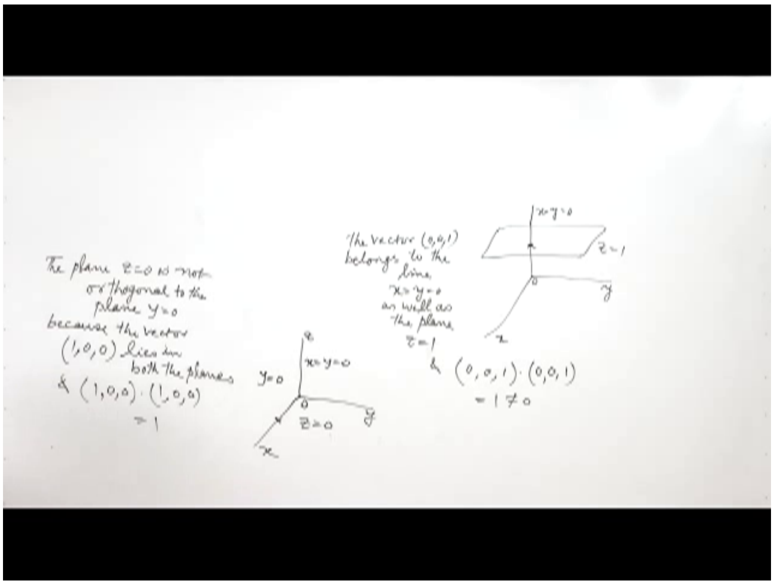


Now, the vector 1 0 yeah, so this is what I have explained in the next line we go over to. Now, proposition 1 if x and y are 2 orthogonal sets, that means you take any element in x an element in y they are perpendicular to each other that is the dot product is 0 then either they are disjoint R x intersection y is equal to singleton set 0. Let see how we get this.

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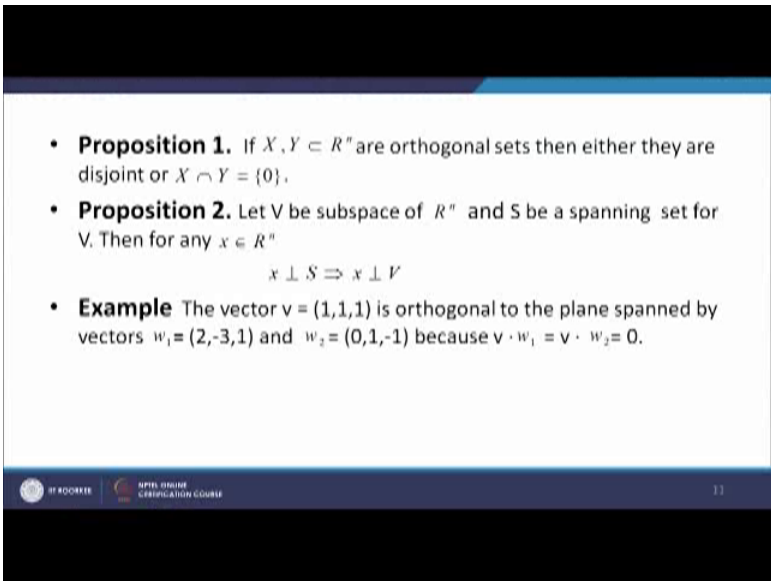


So, let X and Y be two subsets of R n which are orthogonal.

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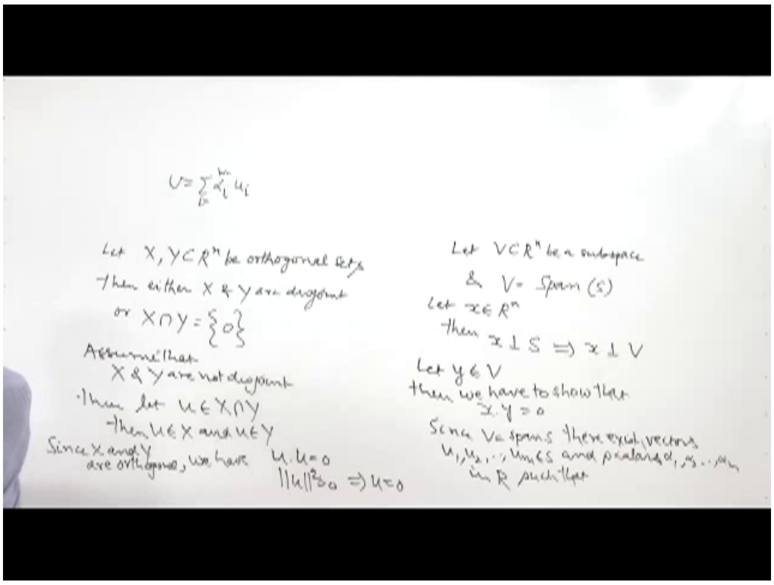


Let this be orthogonal sets then either X and Y are disjoint or X intersection Y is the singleton set 0. So, let us to prove this assume that X and Y are not disjoint to joint. So, then let us say let there let there be element let U belong to X intersection Y. So, then U belongs to X and U belongs to Y. Now, since X and Y are orthogonal sets since X and Y are orthogonal we have the dot product of any element of X with any element of Y equal to 0. Now, u belongs to X as well as u belongs to Y.

So, inner product of u with u is equal to 0 or u dot u equal to 0, u dot u equal to 0 means norm of u square equal to 0 and norm of u square is equal to 0 implies that u equal to 0 in the axioms of the inner product space we have said that u equal to 0 if u dot u equal to 0 if and only if u equal to 0. So, if X and Y are not joint then there intersection is the singleton set 0.

Now, let us go to proposition number 2 let V be a subspace of R n and S b a spanning set for V then for any x belonging to R n x is perpendicular to S implies x is perpendicular V. So, if V is subspace of R n let us say and V is equal to span of S, span of S. Now, it says that if you take any x belonging to R n, for any x belonging to R n then x is perpendicular to S this sin is x is perpendicular to S implies that x is perpendicular to V. So, x is perpendicular to V means if take any elements in V let us tell y then x has its inner product with y equal to 0. So, to prove this let y belongs to V then we have to show that x is orthogonal to y that is x dot y equal to 0. Now, since S is a spanning set of V, so since v equal to span of S their exists vectors u 1 u 2 say u m in S and scalars alpha 1 alpha 2 and so on alpha m in R the field R such that V is equal to sigma alpha i u i, i is equal to 1 to m.

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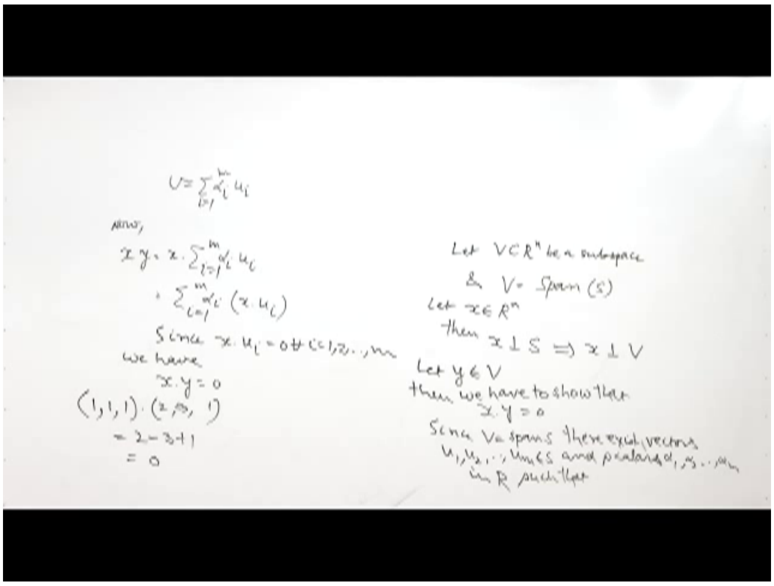


Now, x dot y is equal to x dot sigma i equal to 1 to m alpha i u i. So, by the since the dot product are the inner product is linear in first as well as second position I can write it as sigma i equal to 1 to m alpha i x dot u i. x is orthogonal perpendicular to S. So, x that u i is equal to 0 for every i. So, we have, we have x dot V y is equal to 0. So, for any y belonging to V x dot y equal to 0 in therefore, x is perpendicular to V.

Now, let us take an example let us consider the vector 1 1 1 it is orthogonal to the plane is spend by w 1 and w 2. So, if we can show that these orthogonal to w 1 and these orthogonal to w 2 then we will be orthogonal to the plane is spend by the vector w 1 w 2 because any vector in the plane will be a linear combination of w 1 and w 2. So, you can see that dot product of 1 1 1 with 2 minus 3 1 dot product of 1 1 1 with 2 3 minus 1 is equal to 1 into 2 2 1 into 3 3 and then minus 1.

So, this is not coming out, this is a 2 minus 3 1 is 2 minus 3 1. So, this is 2 minus 3 plus 1. So, these equal to 0. So, the 1 1 1 is orthogonal to 2 minus 3 1 and similarly 1 1 1 is orthogonal to 0 1 minus 1. So, it is orthogonal to any linear combination of w 1 and w 2 and therefore, 1 1 1 vector is orthogonal to the plane spend by 2 3 minus 1 and 0 1 minus 1.

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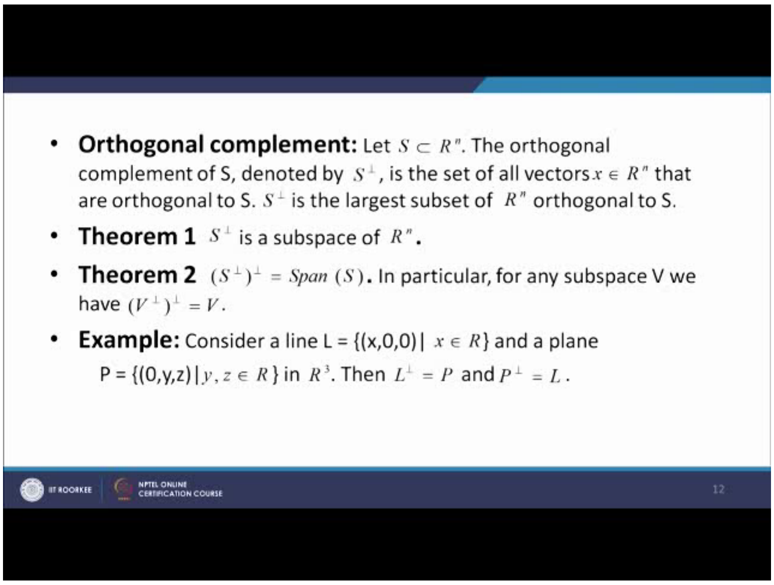


Now, orthogonal complement, let say we have a subset S of R n. The orthogonal compliment of S is donated by S perpendicular and it is the set of all vectors x belonging to R n that R orthogonal to S. So, S perpendicular is defined in this manner.

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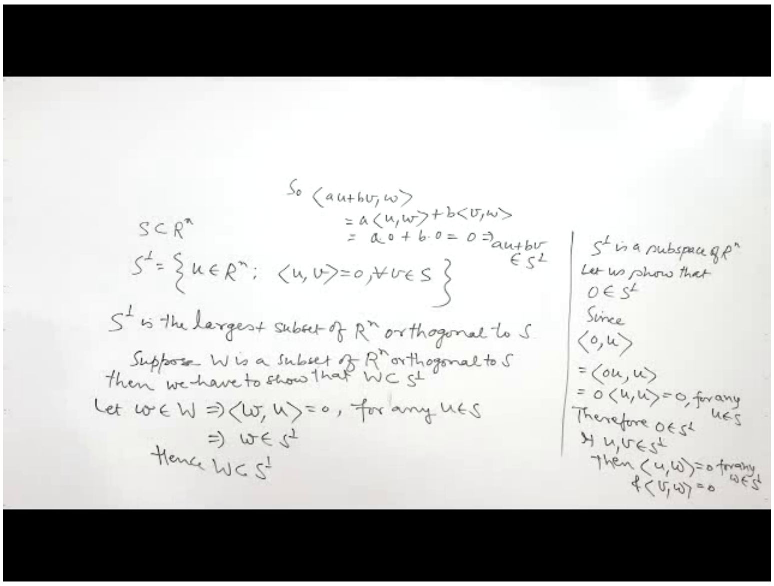


So, S is a subset of R n, then S perpendicular is the set of all u belonging to R n such that the inner product of u with v equal to 0 for every v belonging to R n for every v belonging to S. So, it is the set of all those vectors u belonging to R n whose a inner product with any v belonging to S is equal to 0 and as we have already said the inner product when we write here it will mean that the dot product are scalar product. So, this the dot product are scalar product of u with v equal to 0 for every v belonging to S.

Now, it is easy to see that S perpendicular is the largest subset of R n orthogonal to S. S perpendicular is the largest subset of R n it is the largest to subset of R n orthogonal to S. So, is this means that if you take any subset of R n which is orthogonal to S then that is sets will be contend in S R S perpendicular. So, you can say that suppose W is a subset of R n which is orthogonal to S, then we have to show that then we have to show that W is (Refer Time: 29:08) W C is a subset of S perpendicular. So, S perpendicular the larger subset of R n which is orthogonal to S means if we take any subset of R n which is orthogonal to S then that is set has to be a subset of S perpendicular.

So, to prove this let us say suppose I take any element belonging to W. Let w belongs to W then we have to show that w is this w also belongs to S perpendicular. Now, let w belong to W, W is a subset of R n which is orthogonal to S this will mean that w has its inner product with any u belonging to u S equal to 0 for any u belonging to S, w u is equal to 0 for any u belonging to S.

Now, S perpendicular defines the those vectors of R n whose inner product with any vector v belonging to S is equal to 0 and w is a vector who is inner product with any vector u belonging to S is equal to 0. So, this may mean that w belongs to S perpendicular. So, it will mean that W is hence, W is a subset of S perpendicular. It is the largest subset of R n which is orthogonal to S if we take any subset of R n which is the orthogonal to S then we have to show that that is contained in S perpendicular. So, if we do this yes. So, W will be a subset of S perpendicular.

Now, let us; so let us, so that S perpendicular is a subspace of R n, S perpendicular. So, first of all we show that 0 vector of R n and belongs to S perpendicular let us show that 0 vector the 0 vector of R n belongs to S perpendicular. Since the inner product of 0 with u let us say u is any vector belonging to S then the inner product of 0 with u I can write as 0 u u which is equal to 0 u u.

So, this is equal to 0 for any u belonging to S and therefore, 0 belong to S perpendicular. So, 0 vector is there in S perpendicular and if you take if u and v belongs to S perpendicular then u w is equal to 0 for any w belonging to S perpendicular and v w is also 0 u w is equal to 0 and v w equal to 0 for any w is belonging to S perpendicular and, so, a u plus b v w by linearly in the first position we can write it as a u w plus b v w. So, a into 0 plus b into 0. So, we get 0. So, a u plus b v belong to S perpendicular.

Similarly we can show that if u belong to S perpendicular and see is any scalar in the field R then c into u also belongs to S perpendicular. So, next let us show that S perpendicular is close with respective scalar multiplication. So, let us say that u belong to S perpendicular and c belongs to the field R then by definition of S perpendicular u w equal to 0 for any w belonging to S.

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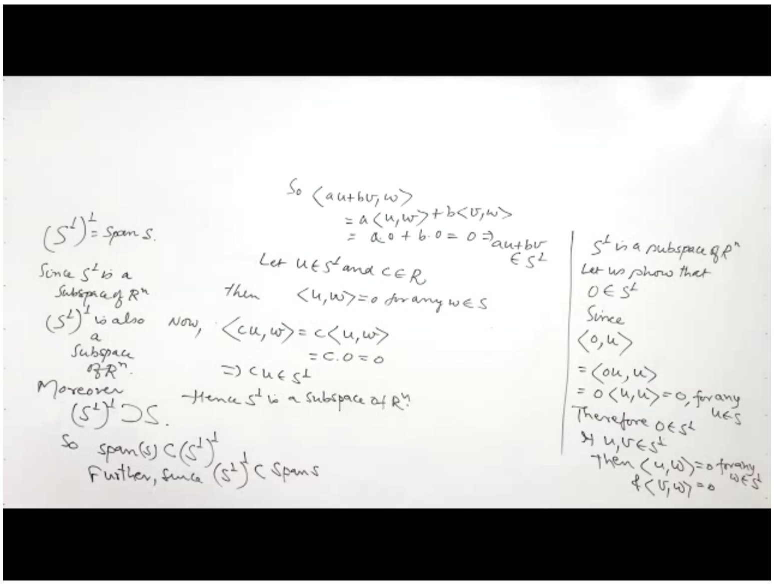


Now, c u w by linearity in the first position this is equal to c times u w which is equal to c into 0. So, 0 and. So, c u belongs to S perpendicular hence S perpendicular is a subspace of R n it is subspace of R n. Now, S perpendicular perpendicular is equal to span of S, S perpendicular perpendicular we can show that S perpendicular perpendicular is equal to span of S. S perpendicular we have shown that it is a subspace of R n. So, S perpendicular perpendicular is also a subspace of R n ok.

So, since S perpendicular is a subspace of R n S perpendicular perpendicular is also a subspace of R n moreover S perpendicular perpendicular contains S moreover it contains S. So, span of S span of S is the smallest subspace of R n which contains S. So, span of S will also contained in S perpendicular perpendicular and we can show that S perpendicular perpendicular is contained in span of S. So, further sense, they are equal they S perpendicular perpendicular is equal to span of S.

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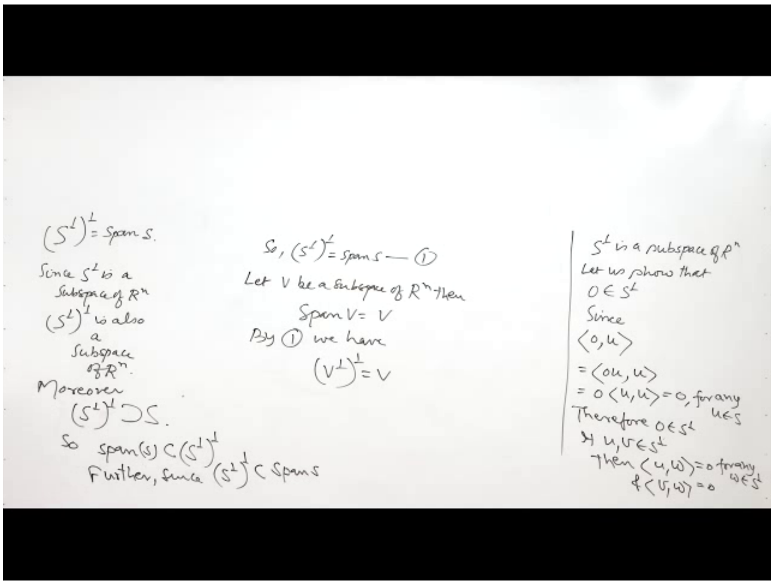


Now, in particular if we take any subspace. So, let us say let V be a subspace of R n then span of V is equal to V. So, is span of V is equal to V. So, let us apply this result. So, by the equation 1 by 1 we have V perpendicular perpendicular equal to v because span of V is equal to V.

So, we have this now, considered a line that is x axis we are taking it its elements as x 0 0 and where x belongs to a and plane p where the plane p is the y z plane because the coordinates of the point they R 0 y z. So, it is y z plane in argue and then we can see that perpendicular is equal to V, x axis, this y z plane.

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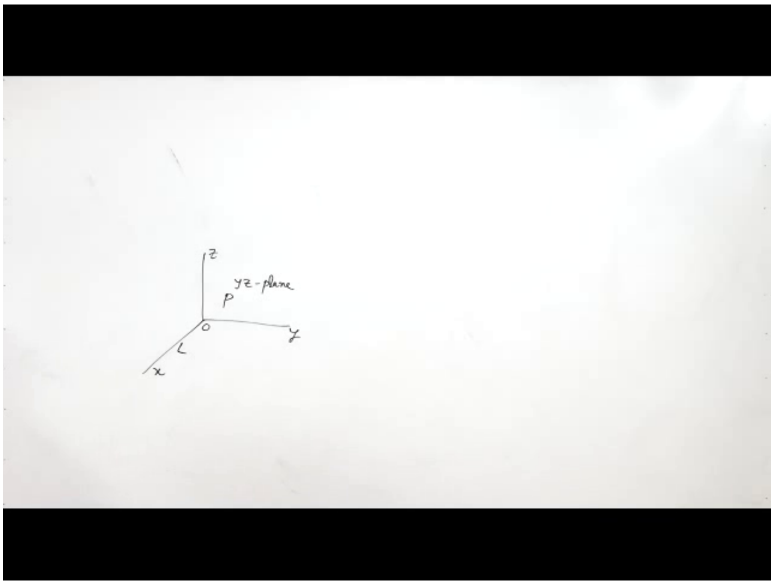


So, this is y z plane and this is x axis. So, here our L is x axis this is L and this y z plane we have denoted by P, and we know that this line L is orthogonal to the y z plane which is given by P. So, L perpendicular is P and P perpendicular reason they are orthogonal to each other. Now, we go to let us say take v and w to v 2 subspaces of R n, so that P is perpendicular to W.

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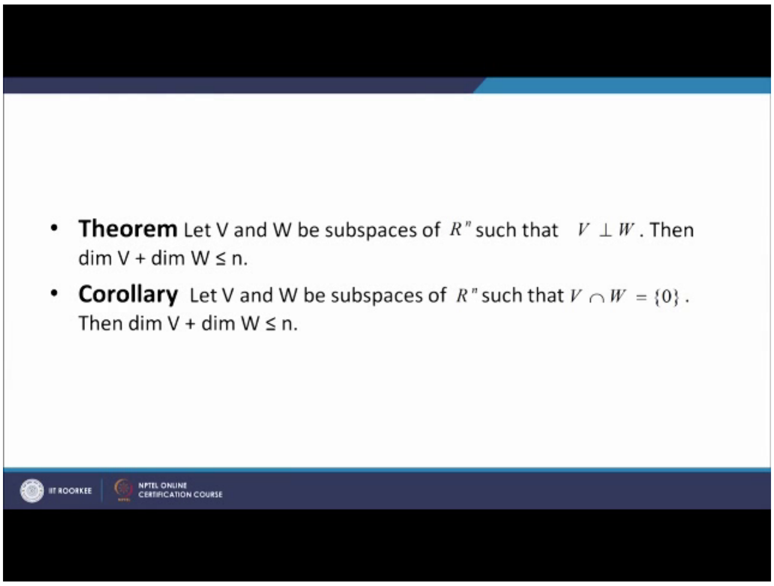


So, V is perpendicular to W means either that disjoint are there intersection is equal to 5 is equal to 0 subspace.

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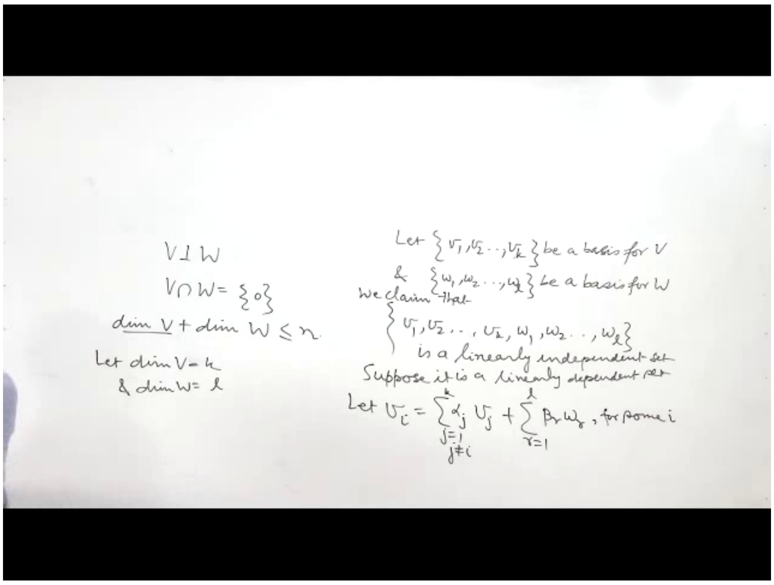


So, then dimension of V plus then dimension of W is less than R equal to n; So, V is perpendicular to n then dimension of V plus the dimension of W is less than R equal to n. Now, this follows because let us say dimension of V is equal to k and dimension of W is equal to L. Let us take a basis for dimension for the v subspace let v 1 v 2 and so on v k, v a basis for the subspace v and w 1 w 2 w l v a subspace v a basis for the subspace w. So, by the definition of basis vectors v 1 v 2 v k are linearly independent and the vectors w 1 w 2 w l are linearly independent. So, then we claim that the set of vectors v 1 v 2 and so on v k, w 1 w 2 and so on w l the setoff vectors consisting of v 1 v 2 v k and w 1 w 2 w l it is a linearly independent set.

Now, to prove this you can assume that is not linearly independent it is linearly dependent. Suppose it is a linearly dependent set then one vector in the set can be written as a linear combination of the other vectors let that vector v some v i here you can also take some w j there. So, the proof will be the same. So, let us say let v i equal to a vector here is a linear combination of the remaining 1. So, sigma alpha i, v o sorry v i can be written as a linear combination of v j, j equal to 1 to k and j not equal to i plus sigma i can write a say R equal to 1 to l, I can write beta r w r. So, this v i for sum i this I will be taking value from 1 to k sum i k.

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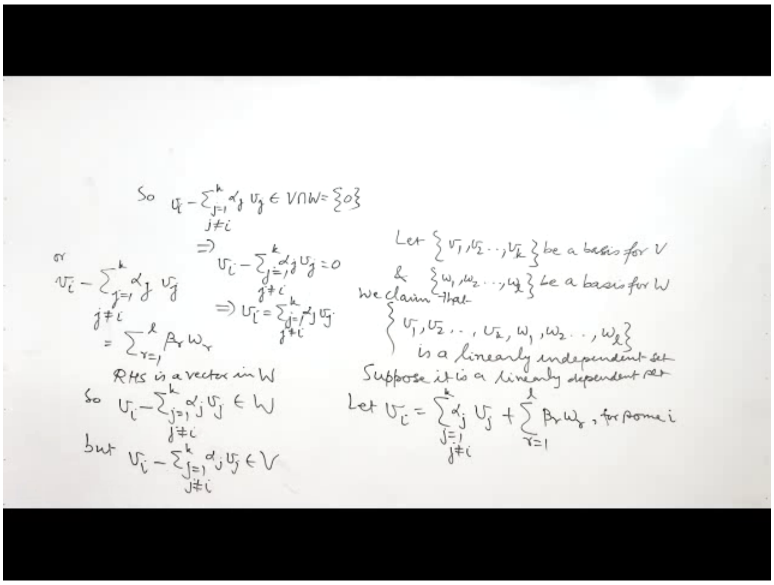


Now, what we can do is we can write like this, or I can write it as v i minus sigma j equal to 1 to k alpha j oh sorry alpha j, v j, j not equal to i equal to sigma r equal to 1 to l beta r w r ok.

Now, this is a linear combination of w 1 w 2 to w l. So, a linear combination of w 1 w 2 w l will be an element of w. So, the right hand side here right hand side is a vector in w and therefore, this is also vector w. So, v i minus sigma j equal to 1 to k j not equal to i will alpha j v j this belongs to w, but this vector is a linear combination of v 1 v 2 v k where the coefficient of v i is 1 and the coefficient of v j is R minus alpha j s. But v i minus sigma j equal to 1 to k j not equal to i belongs to oh sorry alpha j v j belongs to V.

So, this vector belongs to w as well as this vector belongs to be. So, it will belong to their intersection. So, v i minus sigma j equal to 1 2 k j not equal to i alpha j v j belongs to V intersection W. Now, V and W are subspaces they are intersection cannot be empty, the intersection will be I sorry the intersection will be 0 subspace. So, this implies that v i minus sigma j equal to 1 2 k j not equal to i alpha j v j equal to 0 vector which implies that v i is a linear combination of the other vectors in v's ok.

So, sum v i is a linear combination of v 1 v 2 v i minus 1 and then v i plus 1 v i plus 2 and so on v k. So, this means that v 1 v 2 v k is not a linearly independent set which is a contradiction. So, this implies that is a linearly dependent set, so there is a contradiction and hence our assumption was wrong. So, this set is linearly independent. Now, the dimensions of say V dimension of sorry dimension of R n is n and this is a linearly independent set in R n dimension of R n is n means any linearly independent set in R n cannot contain more than n vectors. So, this set will always v a 1 the that there are l plus k vector. So, l plus k will always be less than or equal to n ok. So, l plus k is always less than or equal to n will mean that dimension of v plus dimension of w is less than or equal to n.

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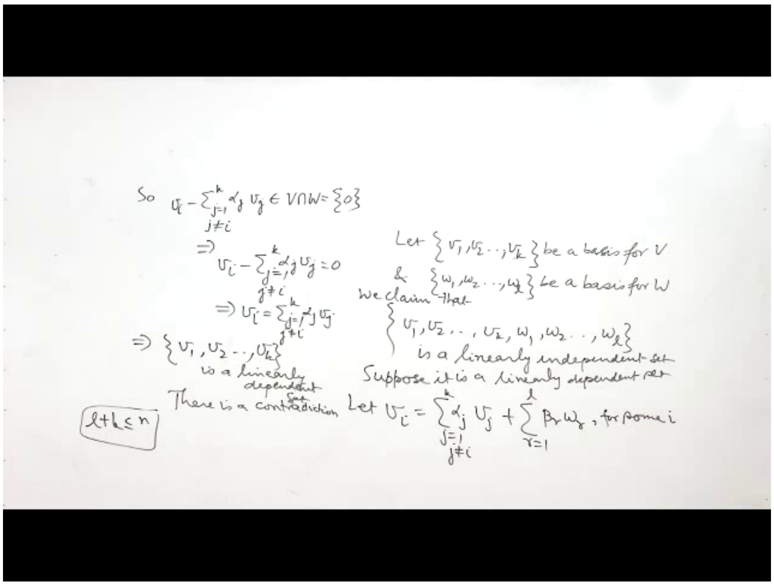


Now, here here assuming that let be W is subspace R n. So, that V intersection W is equal to yeah. So, these a corollary here V is perpendicular to W means the intersection of V and W is 0 subspace. So, that is what we have a written here let V and W subspace R n. So, that V intersection W C 0 subspace the dimension of v plus dimension of W is less than R equal to n. Now, let us discuss fundamental subspaces suppose we have n by n matrix a.

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Ok, I will finish within 5 minutes.

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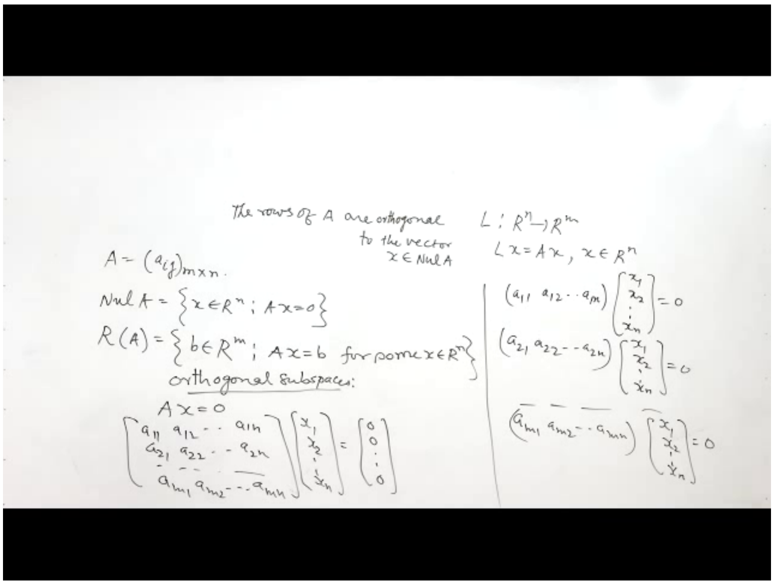


So, let us say we are given n y and m y n matrix A. Then the null space of a is the setup all those vector x belonging to R n, null space of a we have as x belonging to R n such that Ax equal to 0. And the range of a R A range of a R A is the set of all v belonging to how it will written R m such that Ax is equal to v for some x belonging to R A. R A is the range of the linear mapping.

We can say let us corresponding to the matrix A we have the linear transformation L equal to R n to R m where we have Lx equal to Ax for x belongs to R n. So, R A is the range of the linear mapping L from R n to R m where we defined L x equal to x and null space of a is the kernel of L it is called the kernel of L.

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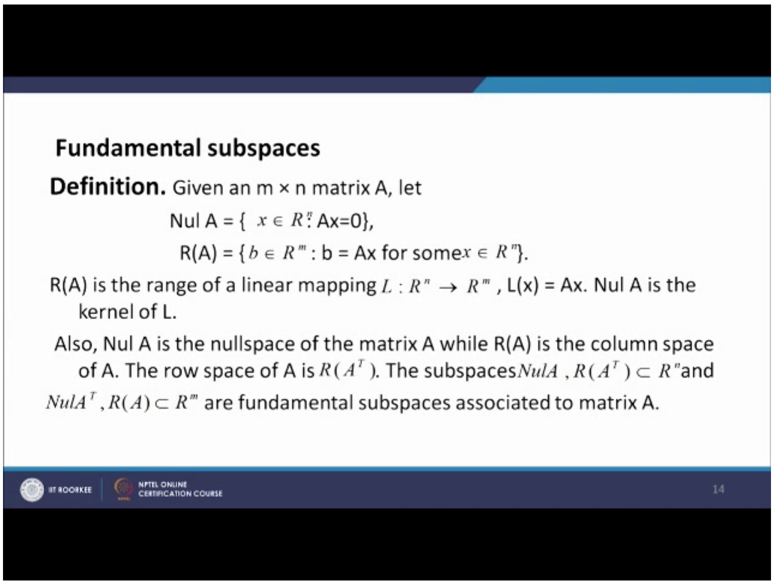


Now, null space of A is the null space of the matrix A and R A is the column space of A. So, when it is the column space of A, the row space of A will be R A perpendicular. Now, let us see which are the orthogonal subspaces here. See Ax equal to 0 means in the definition null space we have all those vectors x for which Ax equal to 0, Ax equal to 0 means A is this vector, A is this matrix a 11, a 12, a 1n; a 21, a 22 and so on a 2n and then we have a m1, a m2 and so on a mn and let us say vector x is x 1, x 2, x n. These equal to 0 vector, so we will have 0 0 0.

Now, from this matrix matrix multiplication what follows when you multiply the first row, the first row the dot product of first row defines the row vector a 11, a 12, a 1n whose dot product with x 1, x 2 is equal to 0. So, what we have? We have the system of equations a 11 I can write them in the form of the dot product a 1n and then x 1, x 2, x n equal to 0 this 1, then a 21, a 22, a 2n and then its dot product with x 1, x 2, x n equal to 0 and so on. So, what we can gather from here that the rows of A are orthogonal to the solution vector orthogonal the to every vector x belonging to R n, the rows of A are orthogonal to the vector x belonging to null of A.

And this is the row of A orthogonal to the vector x belonging to null of A, the linear combination of the rows of A will also be orthogonal to any linear combination of the rows of A will also be orthogonal to x belonging to null of A. Any linear any linear combination of so, we can say range space of A and null space of A are orthogonal to each other.

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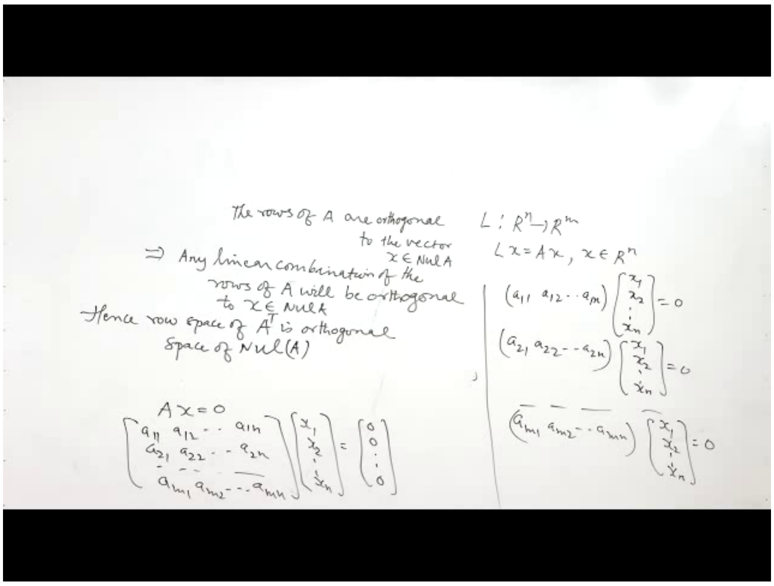


So, row sorry row space of A, null space of A, row is space of A means column is row space of A perpendicular, row space because row rows space of A perpendicular will be orthogonal because here, yes because null space of a contained in R n. So, we have to take the range space of A perpendicular, A transpose, A transpose, range of A transpose, A transpose will have a 11 and 11 and a 21, a 22, a 2n and we will have n we have. So, range space of A perpendicular will orthogonal to null space of A.

So, they are orthogonal subspaces and then we can say range here also we have written ranger space of A is R perpendicular the subspaces null space of A and range space of A perpendicular range; in ranger space of A transpose not A perpendicular range of A transpose is subset of R n, these range space of A transpose range space of A transpose are rows space of A transpose.

So, this orthogonal to null space of A and R A, R A subset of R m null space of a transpose in R A subset of R m they are orthogonal to each other. So, these are fundamental subspaces associated to the matrix A. Null space of A in this theorem we say that null space of A is range is space of A transpose and null space of A transpose is range space of A transpose. That is null space of a matrix is orthogonal to complement of its, orthogonal complement of its row space; null space of a matrix is orthogonal complement of its row space.

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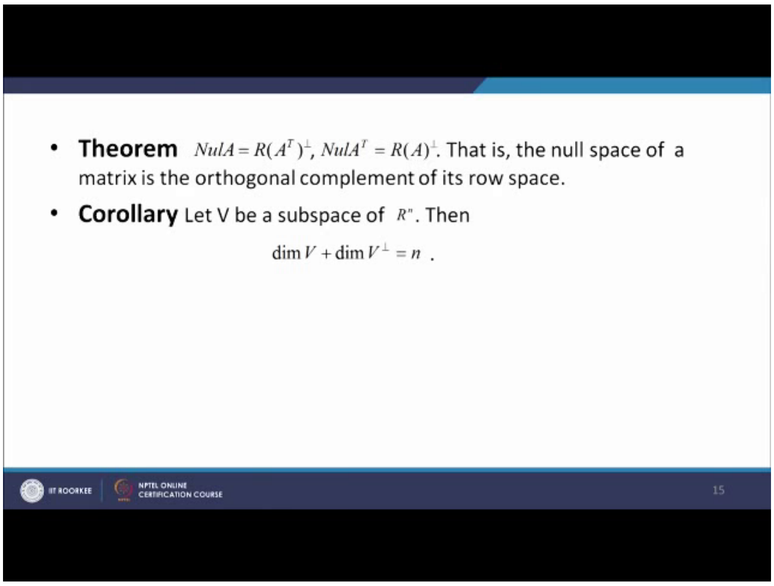


So, V, if V is subspace of R n when dimension of V and dimension of V perpendicular is n, so let us see how we is get this. V perpendicular is orthogonal complement of V and therefore, dimension of V plus dimensions of V perpendicular, if V j subspace of R n must equal to n. With that I would like to end my lecture.

Thank you very much.