Numerical Methods
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Lecture No 9
Fixed Point Iteration Method

Hello everyone so welcome to the 4th lecture of the module 2 which is the module for nonlinear equations, so in this lecture I am going to introduce you another method of solving non-linear equations and the method is called fixed point iterations method. Why we say it fixed point iteration Method? Because it is based on the concept of fixed point for a given functions. In the past three lecturers we have learned about bisection method then Regular Falsi and Newton Raphson method.

So in all those methods what we were doing, we are finding the iterations, we were establishing an iterative equations and based on that iterative equations we are finding a sequence which is going to converge to the root of the equation. In Newton Raphson Method we were calculating the derivative of a function however in earlier method like bisection or Regular Falsi method, we did not calculate the derivative. Since in Newton Raphson Method we are calculating derivatives and we were doing some extra efforts, we were having the assumption n function should be twice differentiable and hence based on that we were getting a good convergence of Newton Raphson method compared to the earlier methods.

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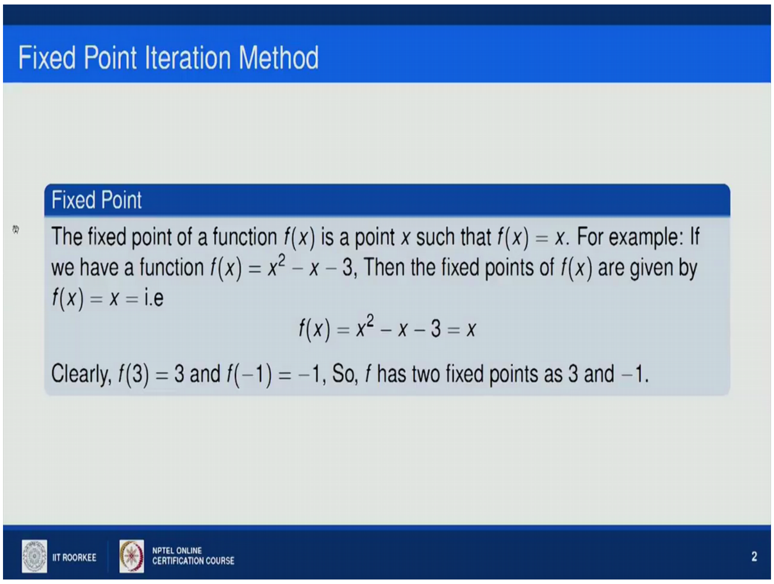

Now let us discuss about this fixed point iterations method, so first of all what is a fixed point? So the fixed point of a function of fx is a point x such that f of x equal to x. For example if we have a function fx equals to x where minus x minus 3, then the fixed point of fx are given by fx equals to x that is x square minus x minus 3 equal to x and when we solve it, we can check that we are having to fixed points this particular equation that is one is 3 and another one is minus 1 because if we calculate f of 3 so it is coming out 3 into 3 9 minus 3 minus 3, so 3 and if you calculate f of minus 1 it is coming out minus 1.

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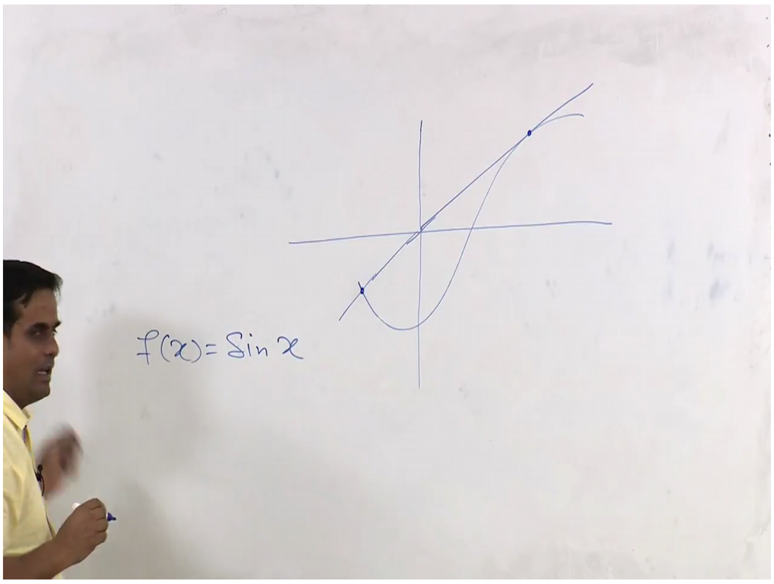

So in general we can say a function the fixed point of a function is given by the intersection of this function with the line y equals to x for example this function is having these 2 fixed points. If I take a another function fx equals to sorry fx equals to sin x, so x equals to 0 is the only fixed point of this function. Similarly we can get another function for which we can have and fixed point, 2 fixed points or more than 2 fixed points. For example if you take the identity function all the points of these functions are the fixed points. Now based on this concept of fixed point, we will develop our fixed point iteration Method.

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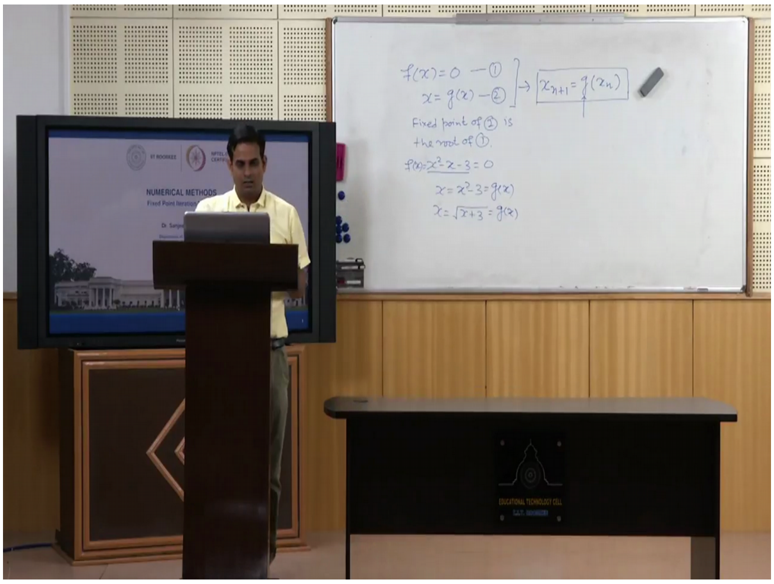

So let us say we are having a non-linear equation fx equals to 0. Now we need to write this function f equal to 0 function f in such a way that I can write it as x equals to gx in such a way that the fixed point of this particular equation becomes used root of this equation means if this is the equation one, this is the equation 2, so the fixed point of the equation 2 becomes the root of equation one, so fixed point of 2 is the root of 1. Now if I give you a non-linear function f of x I can write this function in this form in several ways, for example consider a function fx x equals to x square minus x minus 3 equals to 0.

So here f of x is x2 at minus x minus 3, so I can write this function as x equals to x square minus 3, so here and that is equals to gx. So g of x is x square minus 3 another way of writing this, I can write this x equals to square root of x plus 3 and that is plus minus, so here gx can be square root of x plus 3 or gx maybe minus of square root of x plus 3, so this is my gx, so these are the 2 ways of writing this function fx in terms of x equals to gx, we can have several other ways of writing function x equals to gx. Now after writing equation one into equation 2 then what we will do? From the 2 I will generate and iterative scheme that x of n plus 1 equals to g of x n.

If x or x star is a fixed point of this particular g then what will happen when this sequence or this iteration scheme will converge towards the x star finally this will become x star equals to 0 x star and hence x star is a fixed point of this which gives the root of fx equals to 0. So this particular formula is called the iterative scheme or formula for the fixed point iteration Method. The only thing you need to take care in this that choice of this function g how to write this g from the given equation f of x equals to 0.

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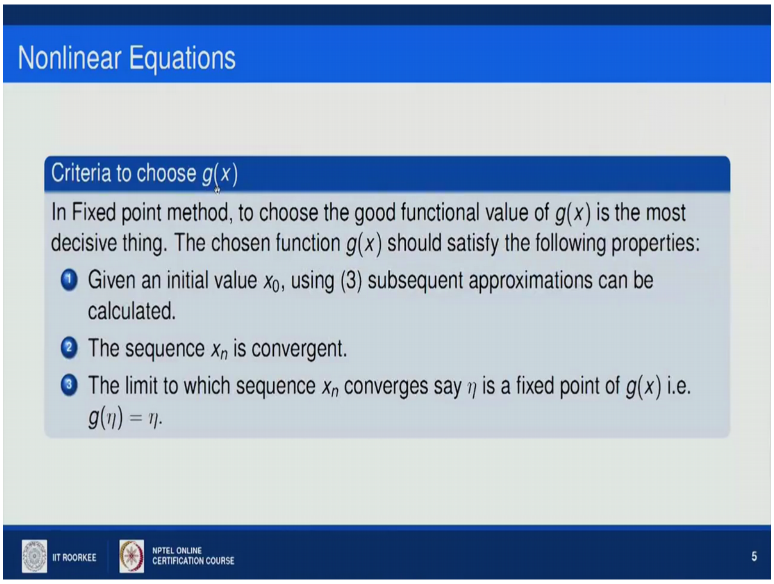

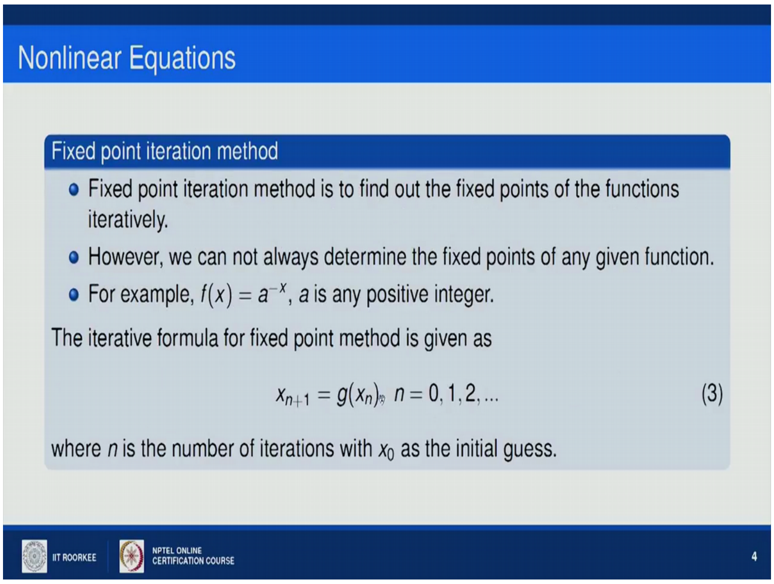

So criteria to choose this function g should be given any initial x naught then the iterative scheme of this equation or iterations of this equation give you can be calculated very easy. The 2nd criteria for choosing g is the sequence x n is convergent using that particular g and 3rd one is the limit to which sequence x n converge let us say eta it should be a fixed point of gx that is g of eta equal to eta. Now if you talk about 1st point that even an initial value of x naught using this method we should be able to find out subsequent approximations of x n.

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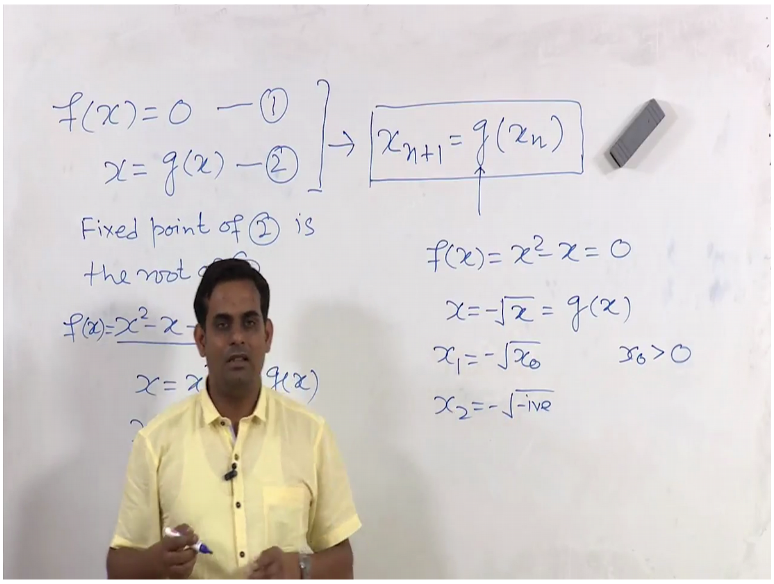

If you take a very simple example let us say fx equals to x square minus x equal to 0 suppose I need to solve this equation. Now what will happen? If I choose x as minus of square root of x equals to gx that can I calculate I take this x the right-hand side and then I can take the square root of the both side, so it will be plus square root as well as minus square root. Suppose I take this minus square root, so then my x 1 will become minus square root of x naught.

Now this gx is define only for positive x or non-negative x, so if x naught is greater than 0, what will happen? I will be able to find out x 1 which will be a negative number. Now when I will calculate x 2 at will be minus square root of a negative number which is my coming from the 1st iterations, I will not be able to find out x2 hence I am not able to find out any subsequent approximation of x n and hence we should choose gx in such a way that we can find on the approximation in subsequent iterations of x n.

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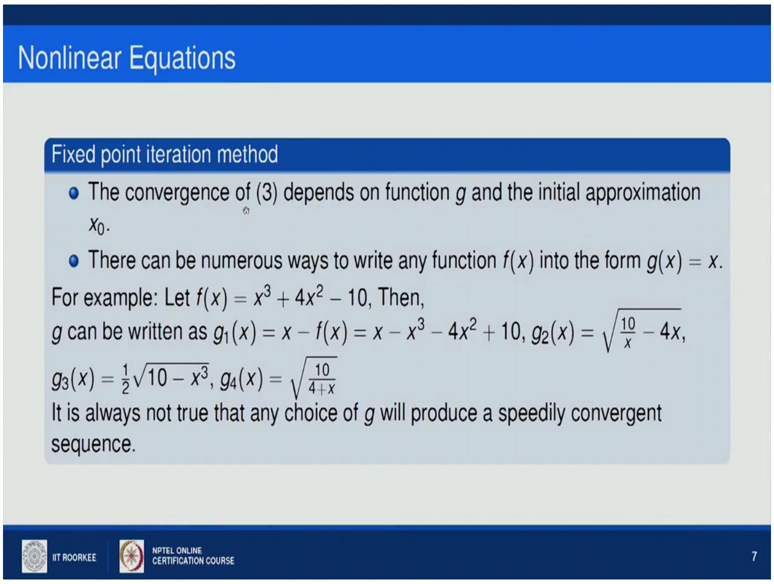

Moreover the convergence of the fixed point iterations method depends on function g that is how you are choosing your function g and the initial simulation x naught.

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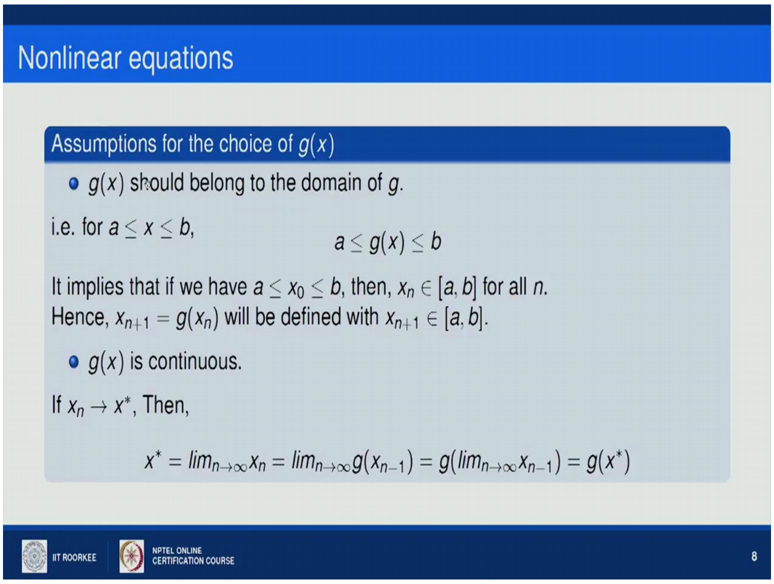

Now I am going to explain few of the assumptions for choosing g and the 1st assumption is gx should belong to the domain of x that is if x belongs to a closed interval a to b, then gx should belongs to close interval a and b. Why I am saying this or why I am taking this assumption? That if we have a x naught between a to b, then for all n, x n should lie in the close interval a to b because x n plus 1 equals to g of x n and hence x n plus 1 should be defined in the close interval a to b and it can happen only when your domain of g is between a to b.

The another is g should be a continuous function that is if the n^{th} approximation of x n tending to x star then x star can be written as limit n tending to infinity x n that is equals to limit n tending to infinity g of x n minus 1. Since x n equals to g of x n minus 1, now since x in minus 1 is a fixed point of g I can write it, I can take g out and limit n tending to infinity x of n minus 1 and it is equal to g of x star.

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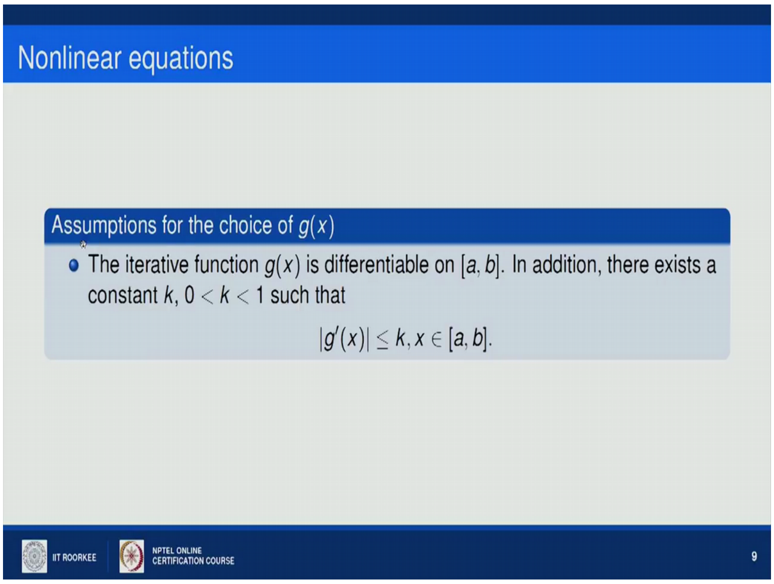

Now this is very important assumptions and this is a condition of the choice of g which subsequently tells us that whether the method is going to converge for this particular choice of g or not, so the iterative function g is differentiable on a, b. In addition there exist a constant k between 0 to 1 open interval 0 to 1 such that the absolute value of g prime x should be less than equals to k for all x belongs to a to b interval a to b. Hence I can write this condition like this I can state like this, the absolute value of g prime x for all x belongs to a to b should be less than 1. If you are having such a g then your fixed point iteration scheme converge for any initial approximation chosen from the interval a to be.

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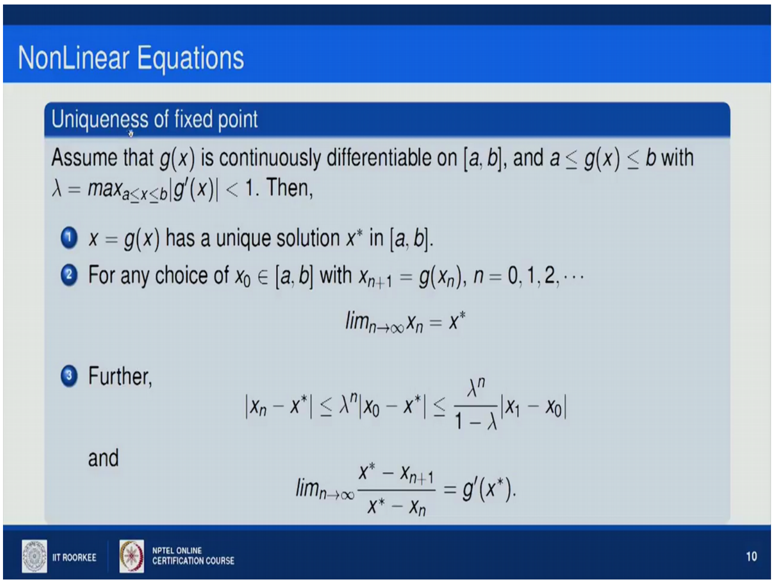

Moreover we are having a condition on the uniqueness of fixed point that is assumed that g is a continuously differentiable on a close interval a to b and the domain of g is in close interval a to b with lambda that is the maximum value of g prime x for all x belonging to a to b and this value is less than 1 then x equals to gx has a unique solution x star in a b that is the fixed point will be unique in this particular interval a to b. The another one is for any choice of x 0 in this interval a to b with x n plus 1 equals to g of x n, n equals to 0, 1, 2 et cetera. The sequence x and will converge to the this unit fixed point.

Further the absolute value of x in minus x star will be less than equals to lambda raise to power n x 0 minus x star because one lambda will be added in each iteration like x n minus x star will be less than equals to x n minus 1 lambda times x n minus 1 minus x star less than equals to lambda square x n minus 2 minus x star and so on and that will be less than equals to lambda raise to power n upon 1 minus lambda into absolute value of x 1 minus x 0 and limit n tending to infinity the error in n plus 1 iteration and error in n^{th} iteration will be equals to g prime x star.

It means when n is quite large, the limit value of the error in n plus 1 iteration that is g n plus 1 upon e n should be less than 1 because we are taking this x star will be in this interval and we are taking lambda as the maximum value of g prime x for all x in this interval, so x star will be somewhere in this particular interval and hence this value will be always less than 1, so limiting value of this e n plus 1 will be less than e n strictly less than. After this let us take you examples which we will solve using this fixed point iteration method.

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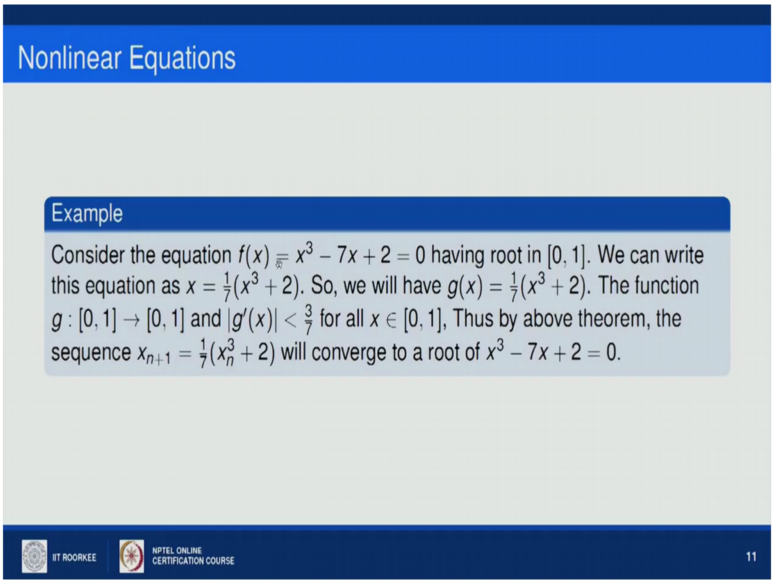

So consider the equation fx equals to x cube minus 7 x plus 2 equals to 0. This equation is having root in 0 to 1 because when we test it at x equals to 0 it is coming out 2 which is a positive number when I test it at x equals to 1, so f of 1 coming out to be 1 minus 7 plus 2 that is minus 4 which is a negative number, so here f 0 into f 1 is a negative number and according to intermediate value theorem there will be a root between 0 to 1. We can write this equation as x equals to 1 upon 7 x cube plus 2. So we will have gx equals to 1 upon 7 x cube plus 2 and you can see the function g belongs to 0 to 1 will be always 0 to 1.

So domain equals to range it means that whatever domain you are having for x same domain I am having for g of x that is our 1st assumption in this method. Moreover if you check g dash x that will be always less than 3 upon 7 for all values of x between 0 to 1 thus by the convergence condition of the fixed point iteration method, the sequence x of n plus 1 equals to 1 upon 7 x n cube plus 2 will converge to root of x cube minus 7 x plus 2 equals to 0. That is the fixed point of this equation will give you the root of this equation.

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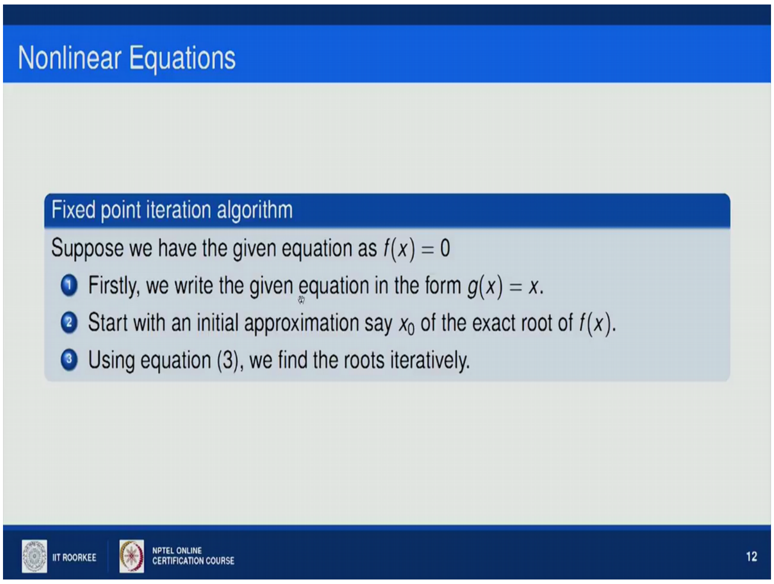

The algorithm works like that suppose we are having equation fx equals to 0, first we write the given equation in the form gx equals to x then we start with an initial approximation say x naught and then we find the successive iterate approximation of x n in different iterations.

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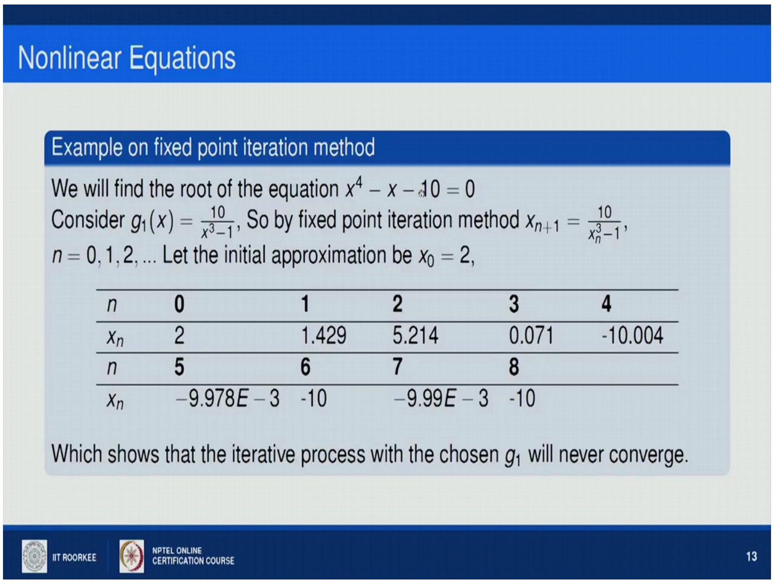

Now let us take another example of fixed point iteration method, suppose we are having an equation that is the 4th order polynomial equals to 0 and polynomial is x 4 minus x minus 10, so consider g of x as 10 upon x 3 minus 1, so by the fixed point method the iterative formula will become x of n plus 1 equals to 10 upon x n cube minus 1. For an equals to 0, 1, 2 is start with an initial approximation x naught equals to 2 and then x 1 will become 1.429, x 2 will become 5.214, x 3 will become 0.071, x 4 will become minus 10.004, x 5 will become minus

9.97 into 10 raise to power minus 3 then x 6 will become minus 10, x 7 will become minus 9.99 into 10 raise to power minus 3, x 8 will become minus 10 and then it will always oscillates between minus 9.99 raise into 10 raise to power minus 3 and minus 10 and which shows that the iterative process will never converge.

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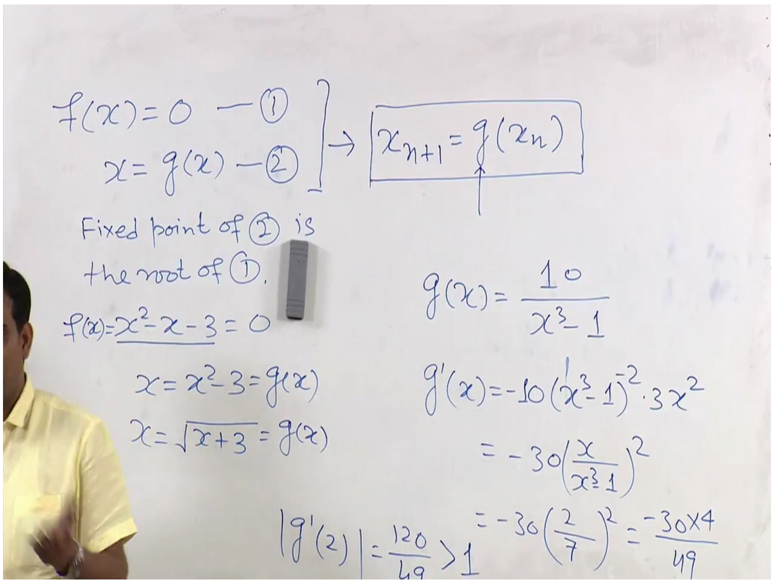

So what is the problem here? The problem is if you see over gx, so here we are choosing gx as 10 upon x 3 minus 1. If I find out g prime x it will become minus 10 into x 3 x cube minus 1 raise to power minus 2 into 3 x square, so basically I can write it minus 30 x upon x 3 minus 1 whole square. Now if you check that when x is 2, so when x is 2 it will become minus 30, 2 upon 7 square which is minus 30 into 4 upon 49. So hence g prime absolute value of g prime 2 will be somewhere 120 upon 49 and which is greater than 1. So this choice of g does not fulfill the requirement of the convergence and hence we are getting such a behaviour of the iterative scheme that is not going to converge.

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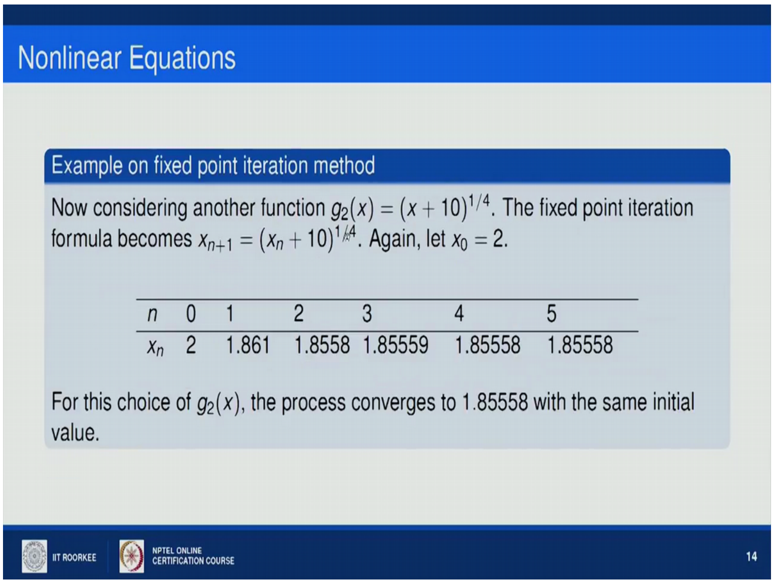

If we take another function g for the same equation x 4 minus x minus 10 equals to 0, so another way of writing gx is x plus 10 raise to power 1 upon 4 then the fixed point iteration formula becomes x n plus 1 equal to x n plus then raise to power 1 upon 4. Again if I start with x naught equals to 2, so my x naught is 2, I got x 1 is 1.861, x 2 as 1.8558, x 3 as 1.85559, x 4 as 1.85558 and then x 5 as 1.85558 which is same up to 5 decimal places in this to iteration and hence my fixed point iteration method converge to the root with an accuracy of 5 decimal places in just 5 iterations and this is happening because if you calculate here for this particular choice of gx.

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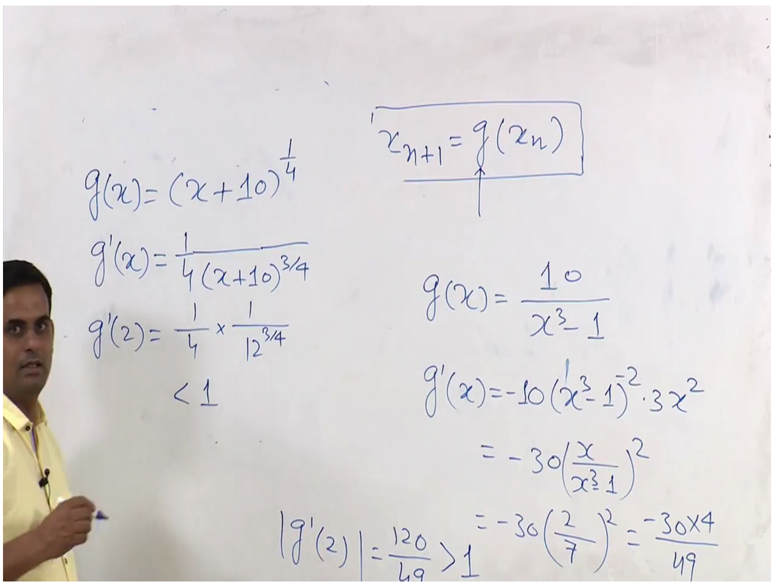

So here am taking gx is x plus 10 raise to power 1 upon 4, so here g prime x become 1 upon 4 into x plus 10 raise to power 3 by 4. If I calculate g prime 2 it will become 1 upon 4 into 1 upon 12 raise to power 3 by 4 and this number will be less than 1 and this is the condition for the convergence on the choice of g is fulfilled here and that is why method is converging quite faster.

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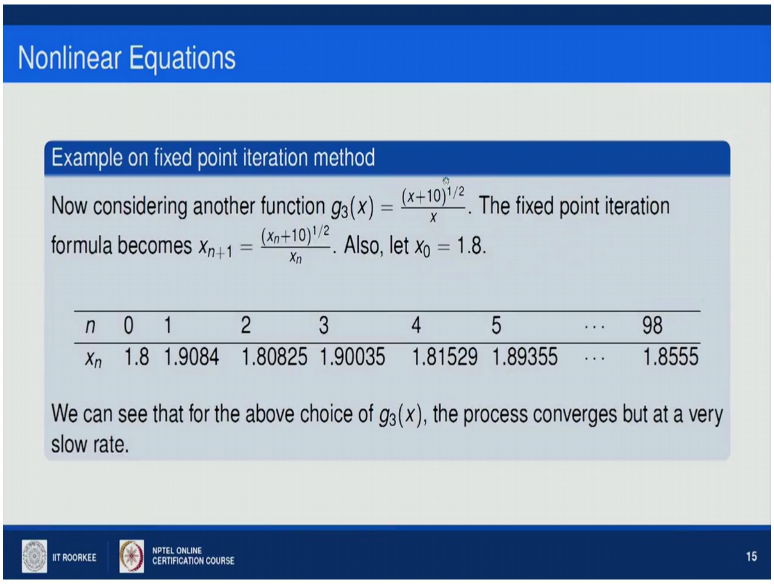

If we take another choice the 3rd choice of g that can be written as x plus 10 raise to power half upon x then the fixed point iteration form becomes x of n plus 1 equals to x n plus 10 raise to power half upon x n and here let us start with x naught equals to 1.8 which is quite close to the exact solution earlier we were taking 2 but now let us take 1.8, so here x 0 is 1.8, x 1 will become 1.9084, x 2 will become 1.80825, continuing in the same manner we see that the solution converge to 1.8555 in 98 iterations, so in the earlier method you are taking the initial solution with far away from the root still we are getting the convergence in just 5 iterations, here we are getting initial solution close to the exact root but we are getting a solution with a large number of iterations.

However method is converging hence the choice of g because and it is happening because you find out g 3 prime x at x equals to 1.8, it is just close to one means it is near around 0.9 something and hence we are taking more number of equations. Hence in this method resolve g is very important for the convergence, you need to check the condition absolute value of g prime x less than equals to 1 for all x belonging to interval a to b for each g and which is giving the value were less than 1 or a smaller 1 we will use that one, smaller one as well as less than 1 or further iterations based on that we will use our fixed point iteration methods.

Moreover this method is simple in formulation if we leave out the choice of G, so thank you very much for this lecture. In the next lecture we will learn about the solution of system of non-linear equations like in module one, we have solved the system of linear equation here we will take the system of non-linear equations. If you take the system of non-linear equation we will have non-linear equation involving more than one variable and therefore we will modify our Newton Raphson method as well as fixed point iteration method for solving such type of systems of non-linear equations. Thank you once again.