Numerical Methods

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Lecture No 8

Newton-Raphson Method

Hello everyone so today welcome to the 3rd lecture of module 2 and in today's lecture we will discuss one of the most popular technique or solving non-linear equation. This technique is called Newton Raphson method, so basically why I am saying this technique is quite popular because this technique is easy to apply moreover the convergence rate of this technique is faster than the earlier discussed techniques like bisection method, Regular Falsi method or Secant method. In particular this method is having quadratic order of convergence which is higher than bisection method which is linear in terms of convergence and Secant method which is having super linear that is there in the convergence at p is around 1.61 that is golden ratio.

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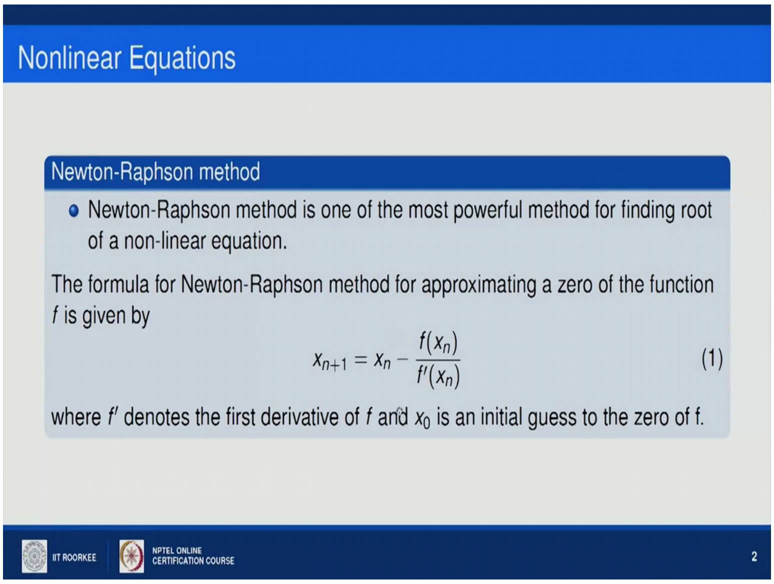

So in this method we use this formula for solving the non-linear equation suppose f of x is equal to 0 is an nonlinear equation and x 0 is the initial solution which is given to us. The next solution can be obtain by this formula x of n plus 1 equals to x n minus f of x n divided by f prime x n. Here f prime x n is the derivative of a prime fx n and x 0 is an initial guess and we will find the subsequent iterates of the sequence x n using this formula.

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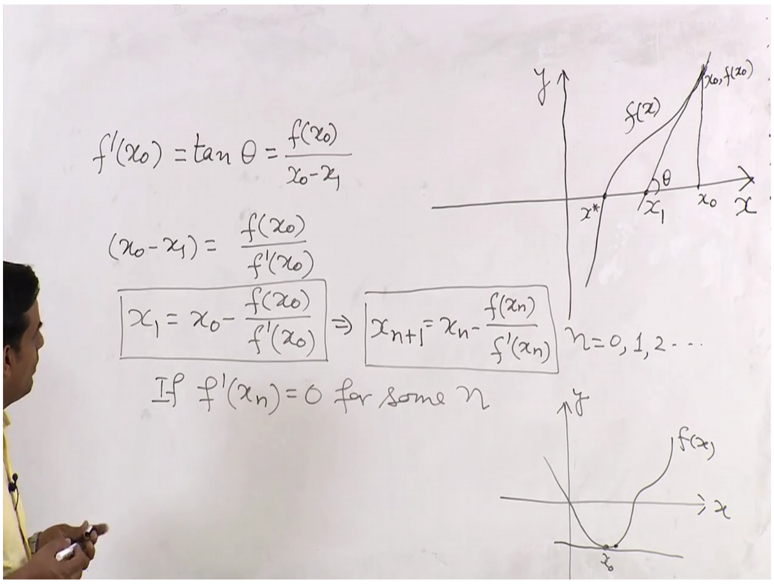

Geometrically this formula can be described as that we are having a function like this, so it is my x axis, it is y-axis and this is the curve fx. As I told you that we are having an initial solution let us say this is my initial solution x naught and the root of this equation fx equal is to 0 is here let us say this point is x star, so I will start with this x naught and using the formula of Newton Raphson method I need to converge here, so what I will do? If I draw a perpendicular here this point is x naught f x naught on the curve y equals to fx.

Now what I do, I will draw tangent at this particular point and the tangent line intersect at intersect the x axis and let us say at x 1, so this will become the next iterate of the sequence x n. If I consider this angle is theta so tangent on the curve fx at this point x naught can be written by f prime x naught and it is equals to tan theta, so if this angle is theta in this right angled triangle I can write this equals to fx naught means this distance upon this distance, so this will become x naught minus x 1.

Now I can write it x naught minus x 1 equals to fx naught upon f prime x naught or x 1 equals to x naught minus fx naught upon f prime x naught. So in this way I get the 1st iterate of the sequence x n using the initial solution x naught and if I generalise this can be written as x n plus 1 equals to x n minus f of x n upon f prime of x n where n equals to 0, 1, 2 et cetera. So as you can see this is the Newton Raphson formula for solving the non-linear equation. Here we are having one drawback of this method that is in the denominator of this we are having f prime x n. If f prime x n becomes 0 for some n then the method fails and we cannot get the next iterations.

Hence this is drawback of the method and geometrically we can see this thing like this. If I am having a curve like this so let us say this is the curve fx this is again x axis this is y axis. If I choose my initial guess at this point, so at this particular point tangent will be parallel to the x axis and hence this tangent will never intersect the x axis, okay and in this way I cannot get the next iteration or if it is very close to x naught let us see here then what will happen? This quantity will be very small and if it is very small this particular term will become very large and hence next iteration will not diverse, so this is one of the drawbacks of Newton Raphson method.

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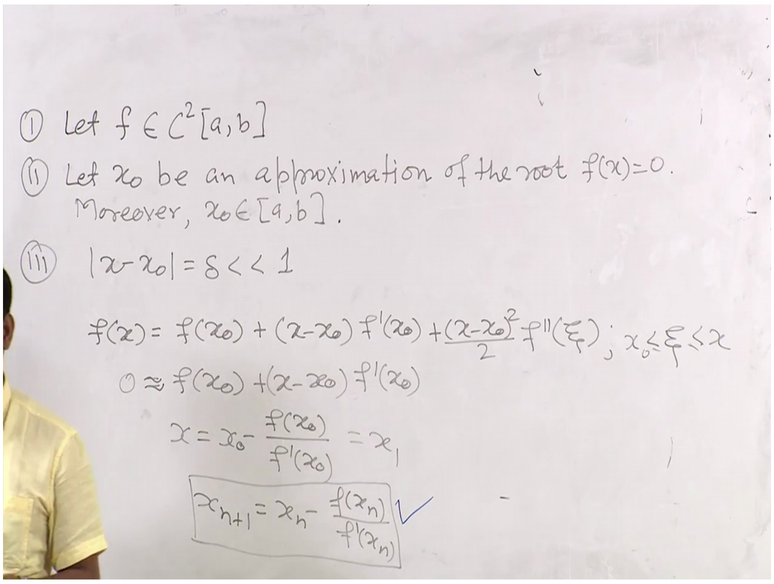

Now we can drive the Newton Raphson formula using the Taylor series expansion also for this we to assume that f is twice differentiable, so let f belongs to C 2 ab on a close interval a 2 b. Second let x naught be an approximation of the root fx equals to 0. Moreover this approximation lies in the close interval ab then what we can do. If we have this x this x naught very close to x then I can write x minus x naught is a small number let us say delta which is very small when compared to 1.

So it means x naught is very close to x then if I write the Taylor series expansion of fx around f 0 then I can write fx equals to f of x naught plus x minus x naught into f prime of x naught plus x minus x naught whole square upon 2 f double prime xi where xi is a number between x and x naught. So here xi now you can see the left-hand side is fx and our given equation fx equal is to 0, so left-hand side is 0 equals to fx naught plus x minus x naught into f prime x naught. Now as I told you x minus x naught is very small number, so the square of this number will be more small and hence I can neglect the 2nd order term here.

So I can say it is a approximate value, so from this I can write x equals to x naught minus fx naught upon f prime x naught and this will be the next iteration of the Newton Raphson Method. So here you can see that we can generalise this formula again x n plus 1 equals to x n minus f of x n upon f prime x n. So this is the again the formula for Newton Raphson method for solving non-linear equation. So we start with an initial approximation x naught then we find the approximation x 1 which is the x intercept of the tangent line to the graph of f at the point x naught f of x naught.

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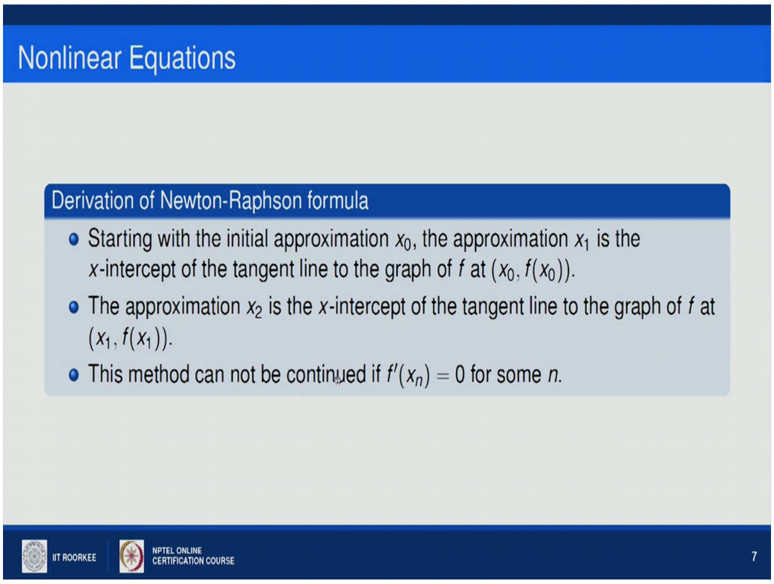

Then similarly we find x 2 which is x intercept of the tangent line to the graph of f at x 1 f of x 1 and finally this method continued continue in this way for finding the approximate or numerical solution x star where the sequence x n is going to converged.

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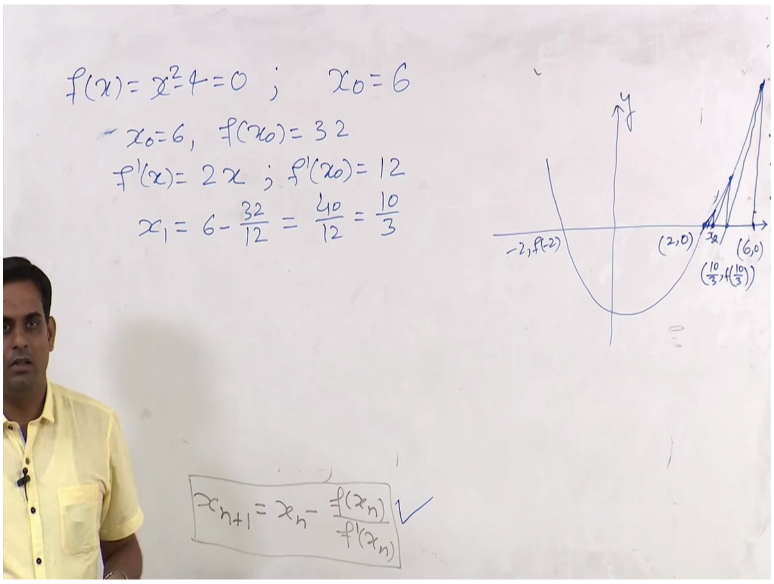

Let us take a very simple example so the example is like that you are having a non-linear equation fx equal is to x square minus 4. We can manually see that the root of this non-linear equation or a 2nd order polynomial is x equals to plus 2 and minus 2, so let us check the root of this with an initial solution x naught equals to 6. Graphically this will be a parabola like this, so this is 2 and f 2, this is minus 2 and f minus 2, so my x axis, y axis and I am going to start. So this will be 2 and f 2, f 2 will become obviously 0 since it is a root, so for this particular curve let us say I start with x equals to 6 so this point is 6 0.

So if I draw a perpendicular line, it intersect this curve at point 6 n 32, so here x naught is 6, fx naught is 32, if I get f prime x it is coming out to be twice of x and hence f prime of x naught is 12. So using the Newton Raphson Formula I got x 1 equals to 6 minus 32 upon 12 and this is…which is nearly hash 10 upon 3. So a bit bigger than 3 so if it is 3, so we are having a…this tangent which intersect here at 10 by 3 and f of 10 by 3. Again I will draw a perpendicular line again I will find the tangent on this point that is x 1 and f of x 1. This tangent line intersect x axis at x 2 and in this way again I will draw a perpendicular line again I will get the tangent line and this way I will converge towards my exact solution in this particular example using the Newton Raphson techniques.

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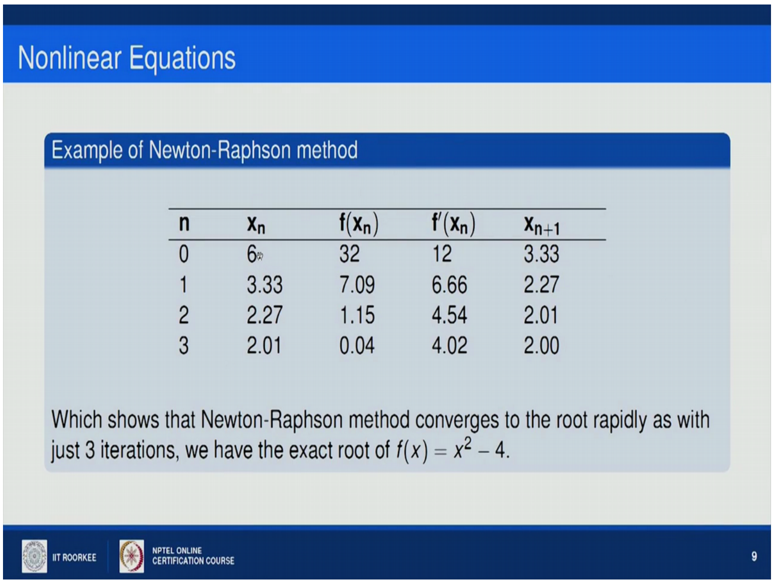

Numerically if I solve this problem the iteration will be like this the 1st x naught is 6, x 1 will be 3.33, x 2 will be 2.27 and then x 3 will be 2.01. So which shows that Newton Raphson method converge to root rapidly as with just 3 iterations we are going to have the value very close to the exact root of fx equals to x square minus 4.

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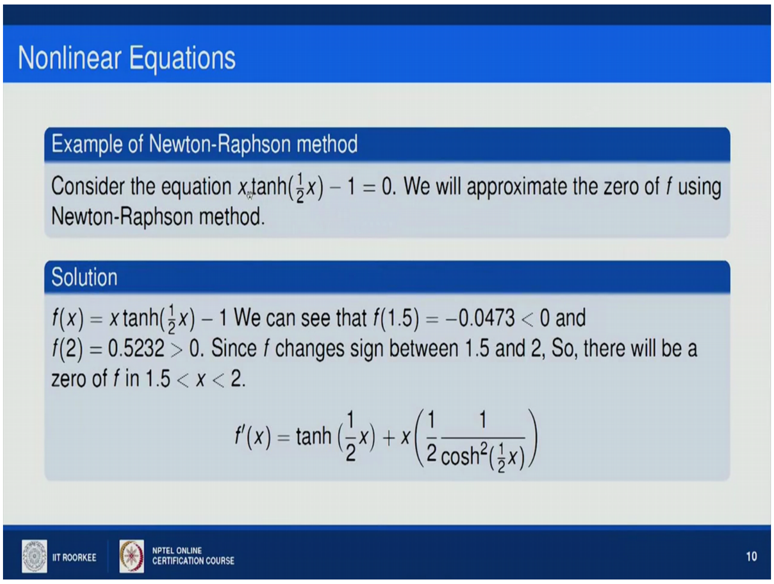

If we take a bit difficult example let us say my function is a transcendental function which involve tan hyperbolic terms, so f x equals to x into tan hyperbolic x by 2 minus 1 equals to 0 is the equation, so we will approximate the 0 of f using Newton Raphson method. So we can see that this function at a point 1.5 is giving you the negative value and at f equals to 2 it is going to be a positive value. So since f changes sign between 1.5 and 2, so there will be a 0 or root of fx equals to 0 in this interval calculate f prime x that will be tan hyperbolic half x plus x times 1 x 2 1 upon cos hyperbolic half x whole square.

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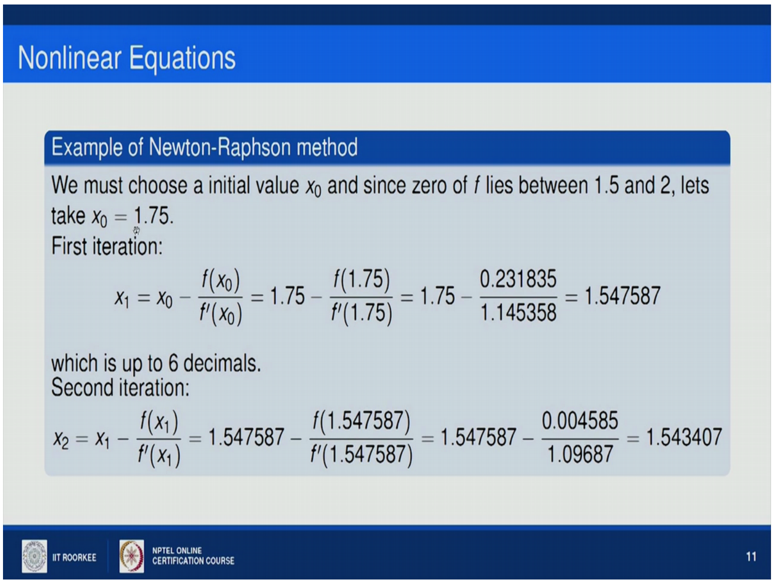

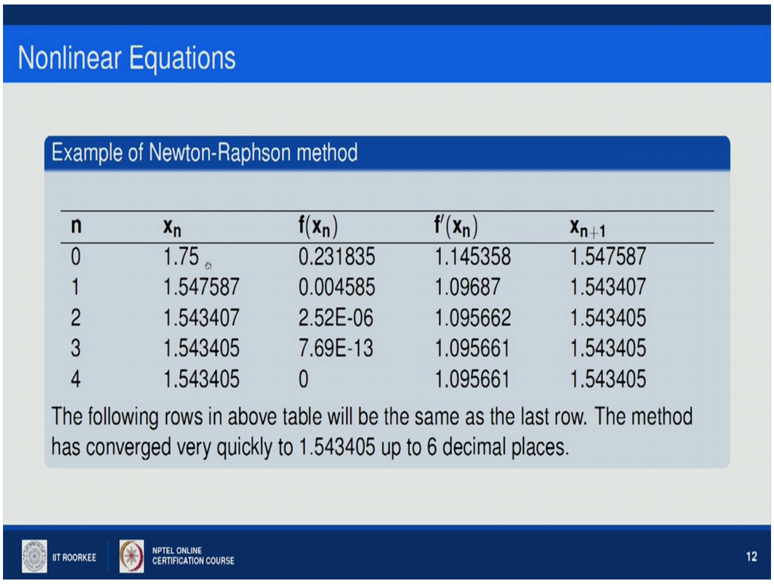

So if we start with initial solution 1.75, the 1st iteration gives the value of x 1, so x 1 will become x naught minus f x naught upon f prime x naught which is 1.75 minus this number and it is coming out to be 1.547587. In the 2nd iteration x 2 becomes 1.543407 in the same way we can x 3 as 1.543405 and finally x 4 as 1.543405. So here you can see these 2 consecutive iterations are same and hence we are having accuracy in the solution up to 6 decimal places. So we are having several good things about Newton Raphson method.

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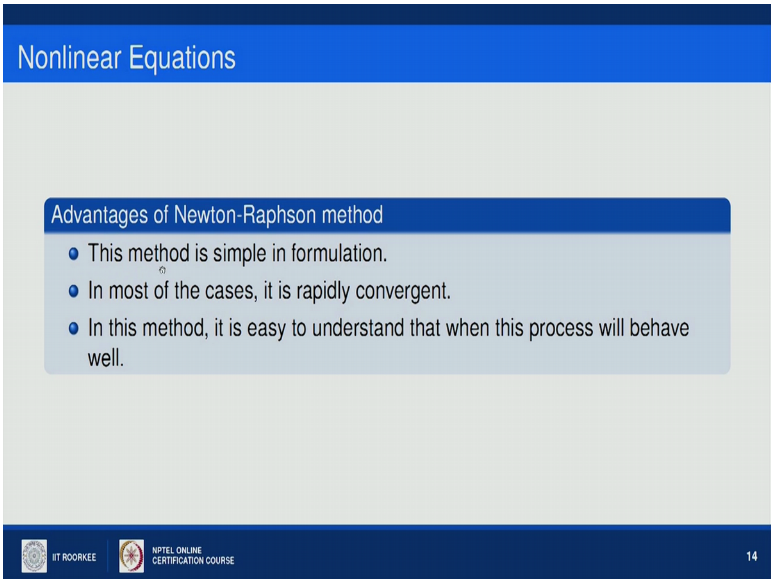

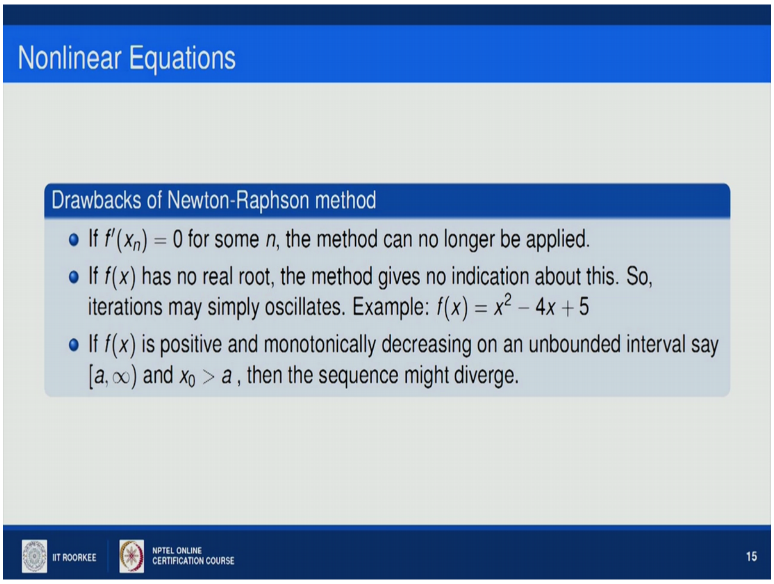

One of them is this method is quite simple in formulation as I have done in this lecture. In most of the cases it is rapidly convergent moreover in this method it is easy to understand that when this process will behave well. Moreover we are having certain numbers of drawbacks of this method one of them have told you earlier that if at any x n, f prime of x n becomes 0 then we cannot apply this method further. Moreover if f of x has no real root, then method gives no indication about this and so iterations may simply oscillates.

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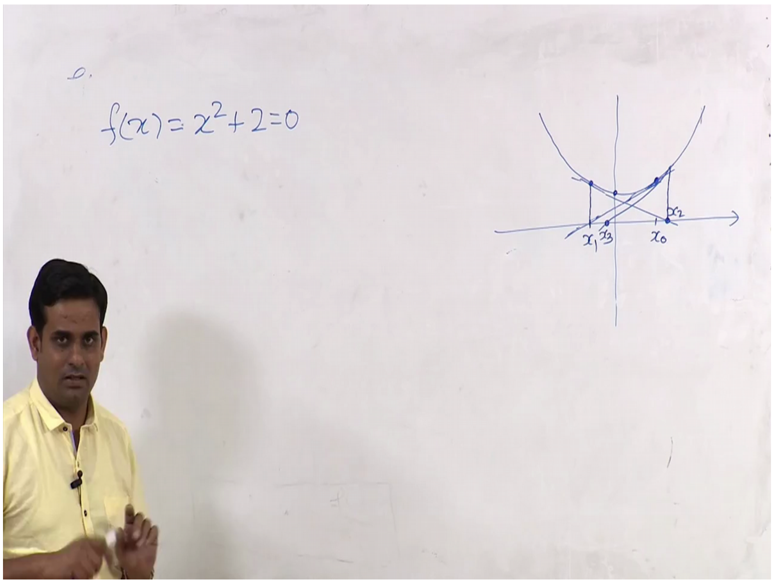

For example take a very simple example to illustrate this particular drawback of Newton Raphson Method, so let me take a simple function f of x equals to x square plus 2 and it is equals to 0, so you can see that we do not have any real root of this equation and this equation can be drawn as a parabola and this parabola never intersect the x axis. Now if I take an initial solution here which is some positive value of x naught, then I will find out the tangent at this point, so tangent line becomes like this and if this is x naught I get x 1 which is a negative number.

So I start with a positive number I got a negative number in the next iteration. Now at this particular point I need to find out tangent at this point of the curve I got this as x 2 which will become again a positive number. Similarly at this point if I find out a tangent I get another negative number as the next iterate is of x n and in this way I will service oscillates between a positive value negative value, positive value negative value or sometimes 2 consecutive positive values and negative values like that and hence this for this particular problem Newton Raphson method will never converge, it will always oscillates and the problem is that the equation is not having any real root and hence if reapply this particular method using our formula, we do not have any feeling of this thing.

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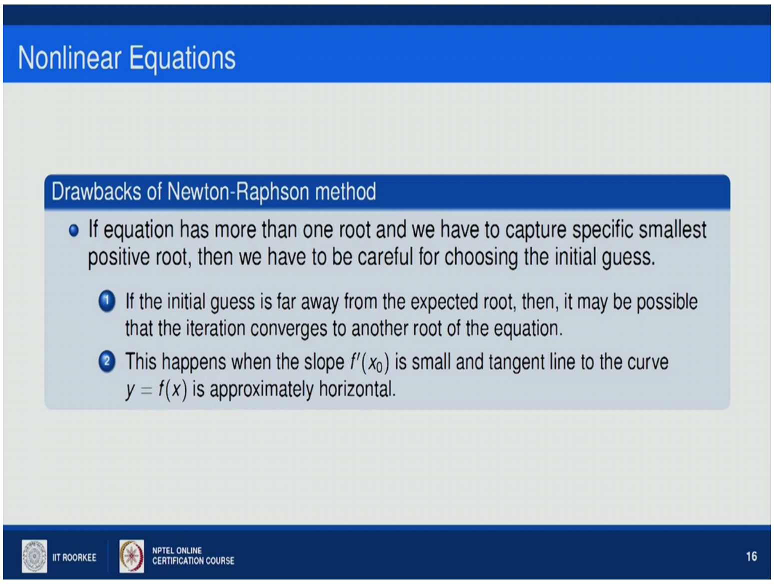

Now another drawback of this particular method is, generally we speak that if an equation is having several roots and you need to find out a specific root, so take any initial approximation close to that particular root, so that method will converge towards that root.

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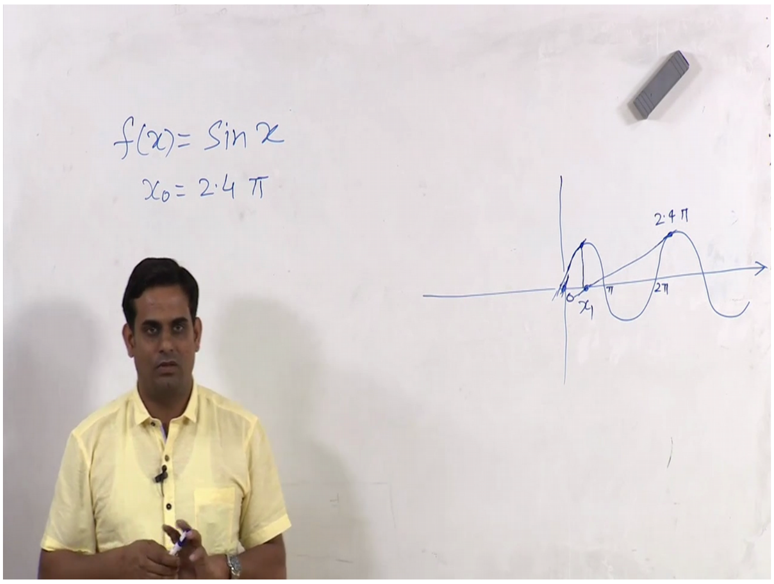

For example if I take a simple transcendental equation fx equals to sin of x then the curve of this function will be like this in the for the positive values of x. Now this is x equals to 0 this is pi this is 2 pi. Now if I take an initial approximation here which is let us say something around 2.4 pi so if my x naught is 2.4 times pi then what will happen this particular initial guess should converge to 2 pi. However if we draw a tangent here according to the Newton Raphson method, what I will get? I will get this as my next iterate and then if I draw a line here, this converge towards 0. So I have taken and initial guess near to the root 2 pi, however it will converge to the 0 using the Newton Raphson formula, so sometimes it does not converge to the nearest root.

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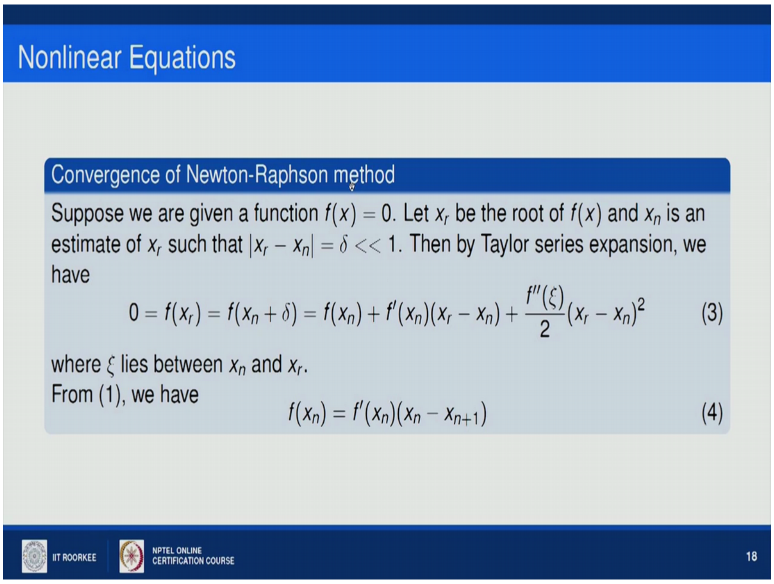

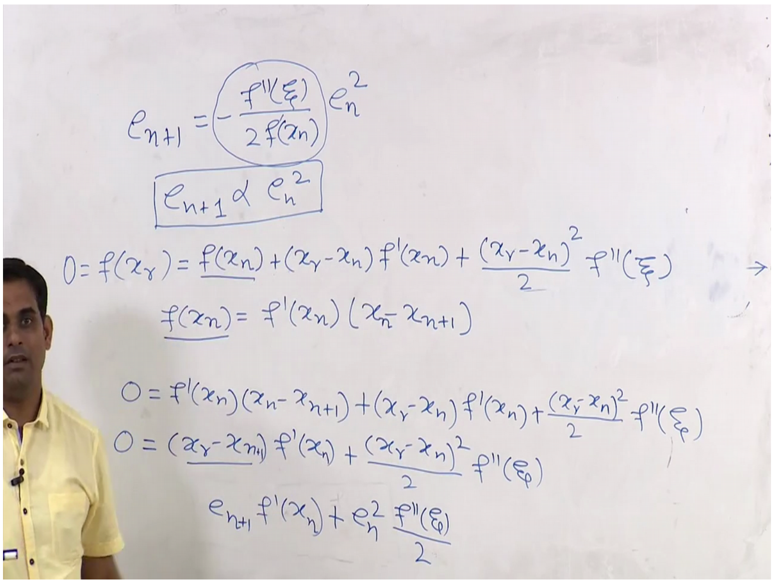

Now if we talk about the convergence of Newton Raphson method and as I commented in the beginning of this lecture at this method is having 2nd order of convergence, so let us prove it. So suppose we are given function f of x which is 0. Let x r be the root of fx and x n is an estimate of x r such that the difference between x r and x n is very small that is less than 1 then by the Taylor series expansion, we can have the expansion of f around about the point x r and this will become f of x n plus x r minus x n f prime of x n plus x r minus x n whole square upon 2 f double prime of xi, so here xi is a number between x r and x n.

Now the Newton Raphson Formula tells us that fx n equals to f prime x n into x minus or x naught minus. So if initial solution is x naught so let us take the solution in nth iteration as x n, so x n minus x n plus 1. Now substitute this value of fx n in the above equation, so here I got 0 equals to f dash x n. This value of if f fx n from here into x n minus x n plus 1 plus x r minus x n f prime of x n plus the second order term of the Taylor series expansion. Now you can see we can cancel f prime x n into x n because it is a positive from this term and negative from this term.

So what I will get, I will get x r minus x n into f prime x plus f double prime xi. Now you can see if I denote this particular term that as error sorry it will be n plus 1, so error in n plus 1 iteration and then so f dash x will become plus then this term will become error in nth iteration, so e n square upon f double prime xi upon 2, so from here I can write that e n plus 1 equals to minus f double prime xi upon twice of f prime x n into e n square. So you can note it will be constant, so I can write e n plus 1 is proportional to e n square. So it means that error will be square of the error in the of the current iteration in the next iteration, so if it is less than 1 it converge to the exact solution quadratically and hence I can say that Newton Raphson is having the 2^{nd} order of convergence.

So in this lecture we have learn about Newton Raphson method, 1^{st} of all you have drive this using a geometric illustration of this method, then we have drive it using Taylor series expansion of a function fx, then what we have done we have solved 2 examples using this method one of them was quite simple, one is having the transcendental equation. Finally we have looked on some drawback of this particular method we have explained them graphically and we have proved that this method will be having quadratic convergence. Thank you very much.