Numerical Methods
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Lecture No 5
Iterative Methods II

Hello everyone so today in this lecture we will continue from the previous lecture where I have introduced you 2 particular iterative schemes one is called go Jacobi sorry Jacobi method and another one Gauss Seidel method. It has been seen that Gauss Seidel method converge to 2 solutions is less number of iterations when compared to the Jacobi method.

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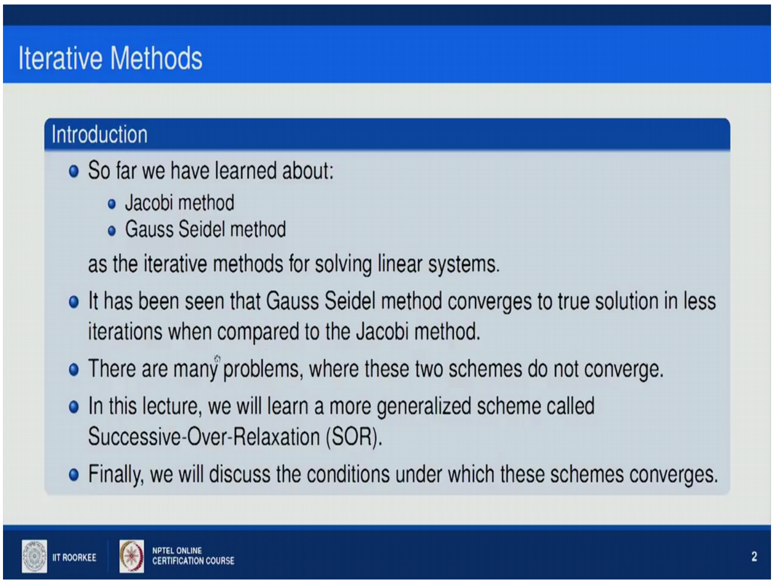

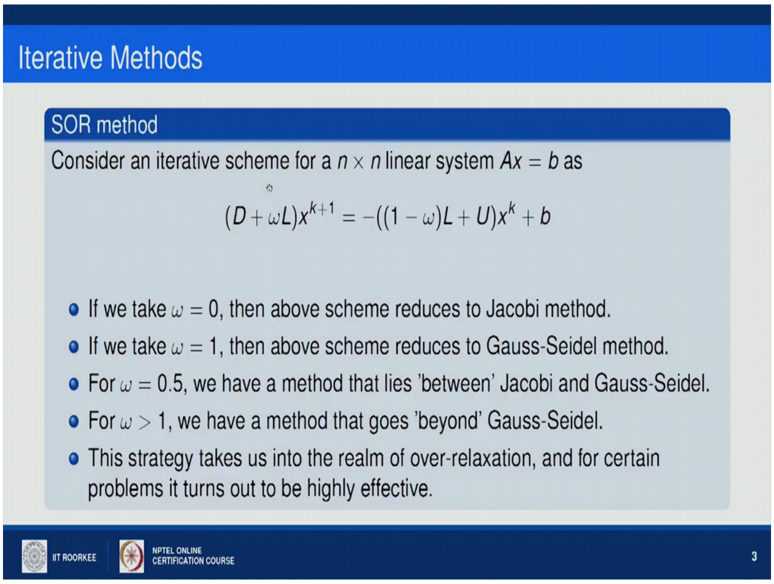

However there are many problems where these 2 schemes not converge at all, in this lecture we will learn a more generalised scheme called successive over relaxation or SOR scheme.

Finally we will discuss the conditions under which these schemes converge, so consider an iterative scheme for a n by n linear system Ax equals to be of this form, so it is something D plus omega L x k plus 1 is equal to minus 1, 1 minus omega L plus U x k plus b. If I put omega equals to 0, what will happen this particular scheme will convert into the Jacobi scheme?

So it will become D times x k plus 1 and so on in the right-hand side it will be minus L plus U x k plus b. If I take omega equals to 1 and this scheme becomes the Gauss Seidel method. For omega equals to 0.5 we have a method that lie between somewhere Jacobi and Gauss Seidel. If omega is greater than 1 we have a method that goes beyond the Gauss Seidel method. This particular thing when omega is greater than 1 takes us into the realm of over relaxation and for certain problem it turns out to be highly effective in terms of convergence.

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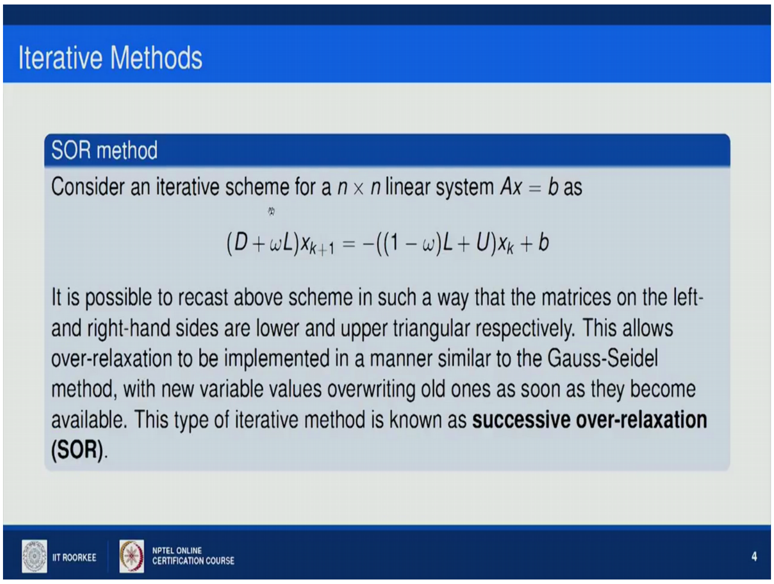

If I take this particular scheme which is Jacobi for omega equals to 0 and Gauss Seidel for omega equals to 1, can be converted this particular scheme such a way that the matrices on the left-hand and right-hand side are lower and upper triangular respectively. If we will be able to do it, what will happen? We can do this over relaxation method in such a way that we can update few of the unknown parameters or unknown variables by their updates from the current iteration itself like we have done in Gauss Seidel method. This type of iterative method is known as successive over relaxation.

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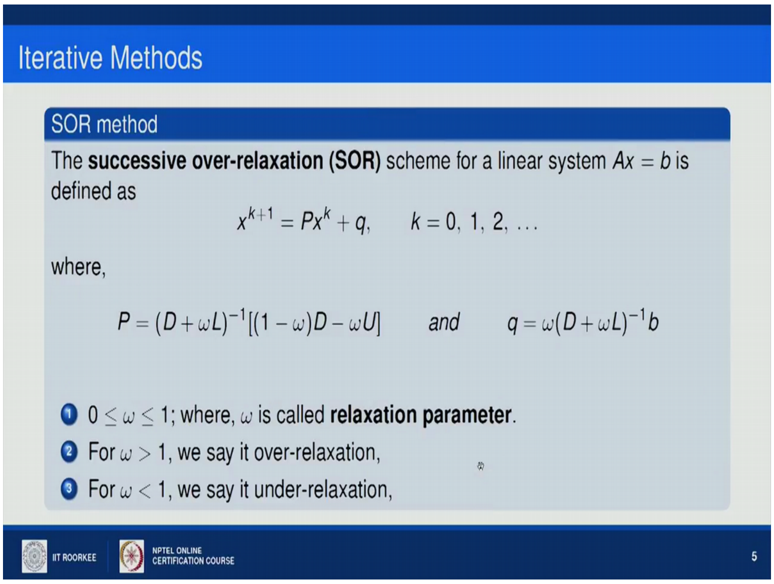

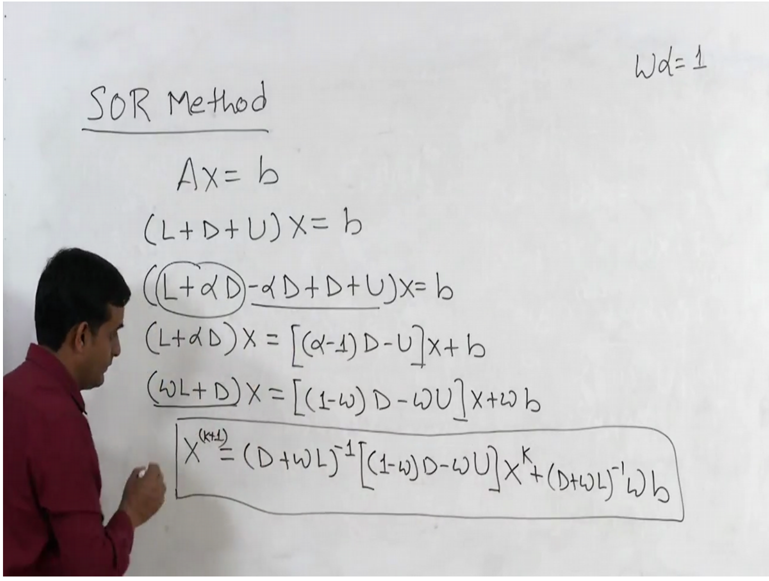

So let me drive this particular scheme that is SOR method, so in short we say it SOR method which is for successive over relaxation. So as you know we will start with a linear system Ax equals to b where A is n by n coefficient matrix x is n by 1 unknown variable vector and b is right-hand side column vector of size n. I will write it as the sum of 3 matrices L, D and U where L is lower triangular matrix, D is a diagonal matrix and U is an upper triangular matrix. I can write this equation in this way also, so what is happening? I can write L plus alpha D into x equals to so what I have done this particular thing will remain the left-hand side and these 3 terms have taken in right-hand side.

Now just consider a relaxation parameter omega in such a way that omega into alpha equals to 1, if I multiply this whole system with that particular omega, the system will become omega L plus Alpha into omega which is one so D x equals to again Alpha into omega will become 1 minus omega D minus omega U x plus omega b. Here omega is a scalar as I told you it is a relaxation parameter. Again if this is non-singular matrix I can find out the inverse of this matrix and I can multiply pre-multiplying by the inverse of this matrix in left-hand right-hand side. So if I will do it what will happen? In the left-hand side you will be having x then what we are having just I am writing this D plus omega L inverse into 1 minus omega D minus omega U x plus D plus omega L inverse into omega b. Now you can see it is in the step form of an iterative scheme and I can write here the value of x in k plus 1 iteration and here I will take the value of x in k iteration, so this is called the successive over relaxation iterative scheme.

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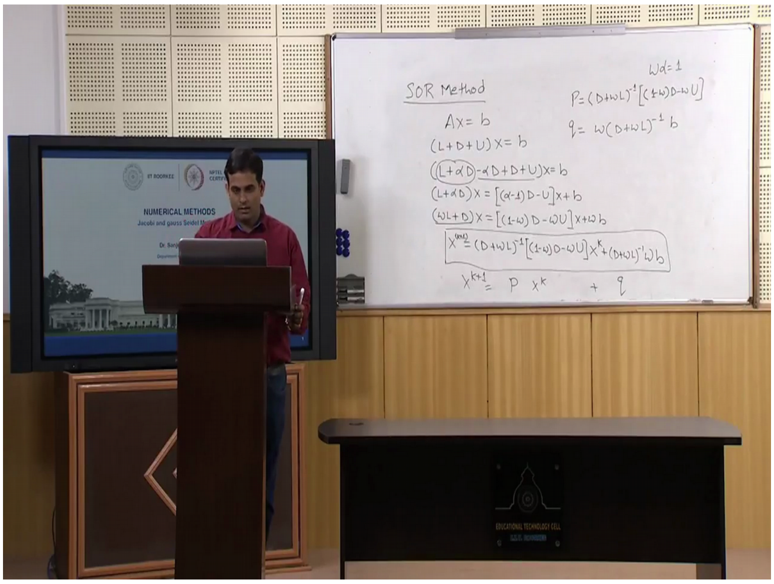

Here the tradition matrix P is given by D plus omega L inverse into 1 minus omega D minus omega U while the column vector q is given by omega into D plus omega L inverse into b. So this is the derivation of this particular scheme as I told you it is called successive over relaxation scheme.

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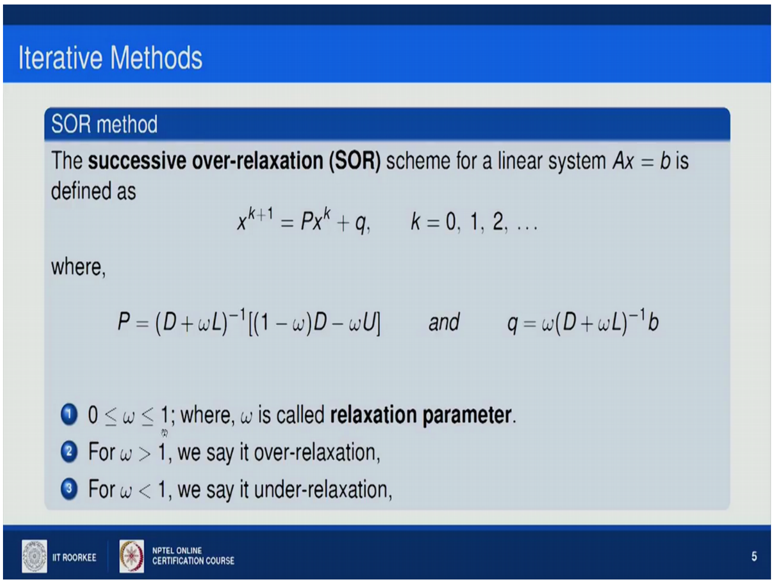

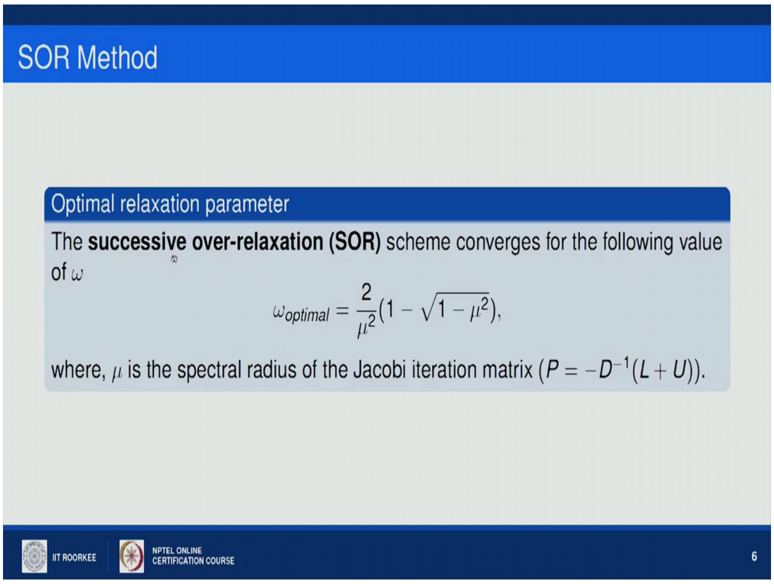

If omega is between 0 to 1 omega is called the relaxation parameter, if omega is greater than 1 we say it over relaxation and if omega is less than 1 we say it under relaxation. The successive over relaxation scheme converges, following value of omega, so for the convergence of this particular scheme we need to find out an optimal value of omega and that will be twice upon Mu square 1 minus square root of 1 minus Mu square. Here Mu is the spectral radius of the Jacobi iteration matrix and if you can remember the iteration matrix in Jacobi iterative scheme is minus D inverse into L plus U.

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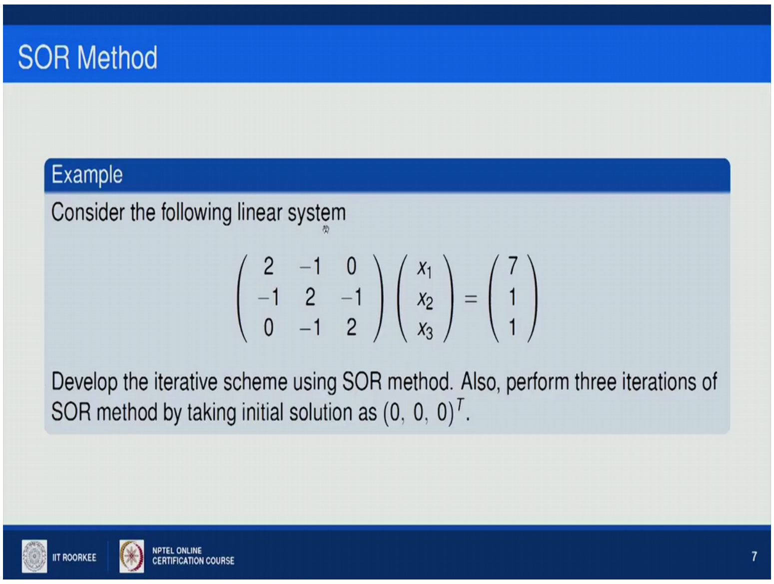

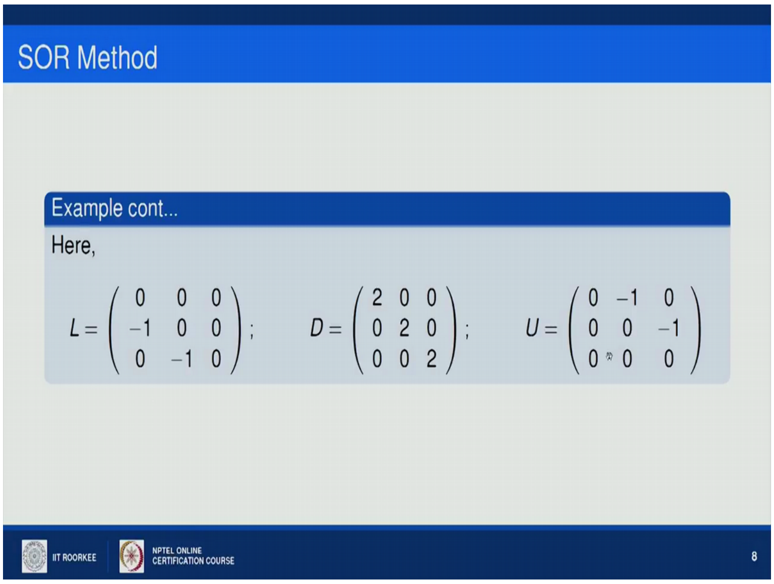

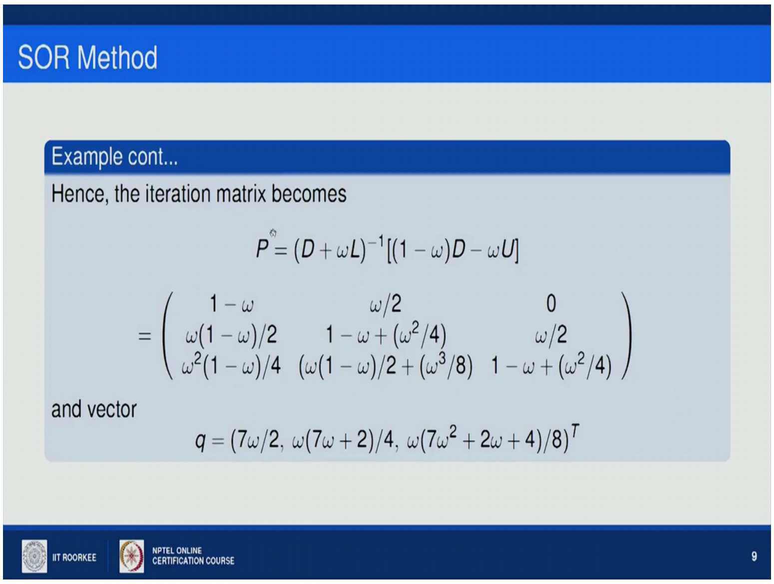

If we solve this particular example using the successive over relaxation scheme it is a 3 by 3 system and here we are going to perform 3 iterations of SOR method by taking the initial solution as 0, 0, 0, so here from this particular coefficient matrix I can write letter L, D and U in this way then the iteration matrix P will become this 3 by 3 matrix and vector q is a 3 by 1 column that is 7 omega by 2 this will be the 2nd element and this will be the element in the 3rd row. However you can see here the iteration matrix P as well as q is having omega in all terms, so what we need to do we need to find out the optimal value of the omega from the by using the spectral radius of the Jacobi iteration matrix.

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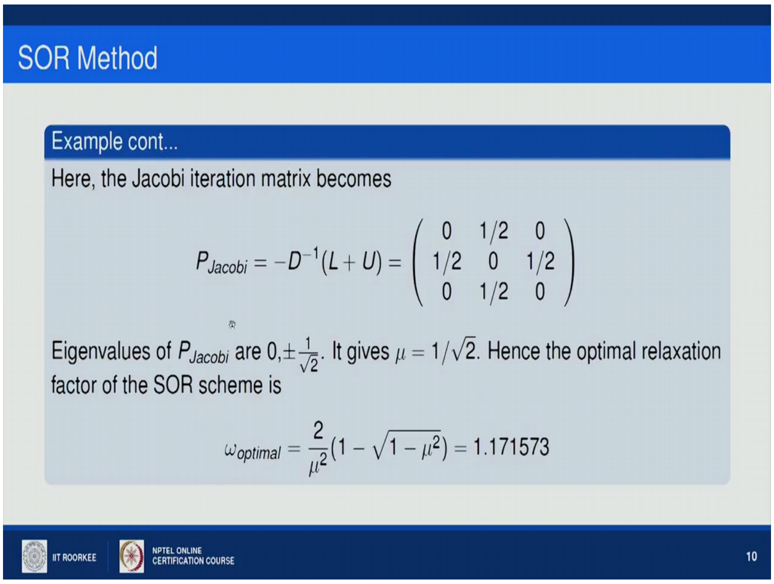

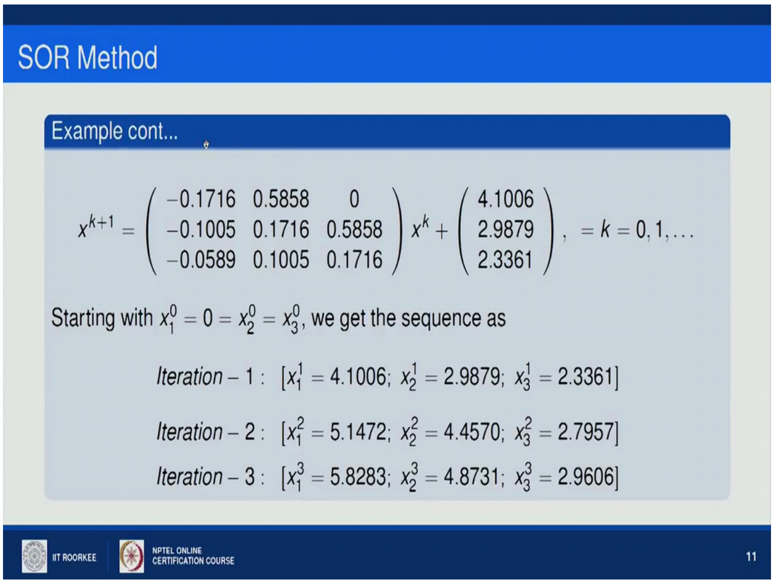

And if I use it Jacobi iteration scheme matrix for this particular scheme comes out to be this 3 by 3 matrix if I calculate the eigenvalues of this matrix, eigenvalues are 01 by root 2 n minus

1 by root 2. It means the spectral radius of Jacobi iteration matrix for this particular example is one upon root 2 and hence the optimal relaxation parameter for the SOR scheme is 1.171573. If I use this value for these 2 matrices the final iteration matrix is this one and the column vector q is this one and scheme can be written as x k plus 1 p into x k plus q. Starting from k equals to 0 to 1 et cetera.

So if I use the 0, 0, 0 as the initial guess in the iteration 1 I got x 1, x 2 and x 3 this one, in iteration 2 this one and in iteration 3 the values are coming out this one. For the convergence we can do the further calculations. So this is all about successive over relaxation scheme, so this scheme depends on a relaxation parameter omega and based on that omega optimal value of omega it converge quite faster towards the solution exact solution when compared to the Jacobi and Gauss Seidel scheme. Now we will discuss about the convergence for iterative methods. What sorts of conditions are responsible for the convergence of any iterative scheme? So here how to drive the convergence we will show it here in an abstract way in a general setting and then we will take some specific example for it.

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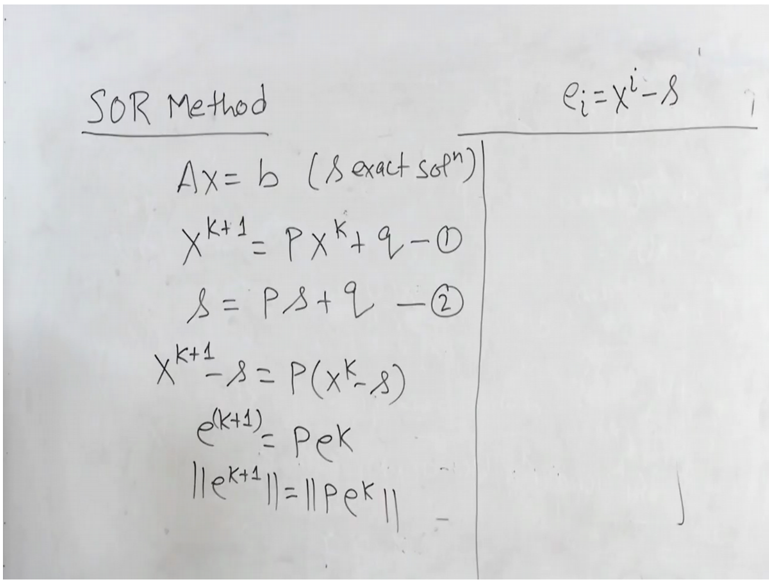

So let us say we are having a system a x equals to b and this system we are solving using an iterative scheme Px k plus q. So convergence of this scheme means if we are having s as the exact solution of this system our scheme can be written as like this for the convergence means after a finite number of iterations I got this s and further if I use this s in subsequent iteration there is no change in the solution because it is the exact solution, so let us say equation 1 and equation 2.

If I subtract the equation 2 from the equation one I can write it x k plus 1 minus s as the left-hand side and in the right-hand side it will be P x k minus s, q will be cancel out. Now if I denote that e i be the error in ith iteration so error will be the difference between the value of x in that particular iteration and the exact solution. So it will be something like x i minus s. So this system I can write in terms of error, the error in k plus 1 iteration equals to P times error in k iteration. Now if I use the norm on this equation I can write norm of e k plus 1 equals to norm of P into e k.

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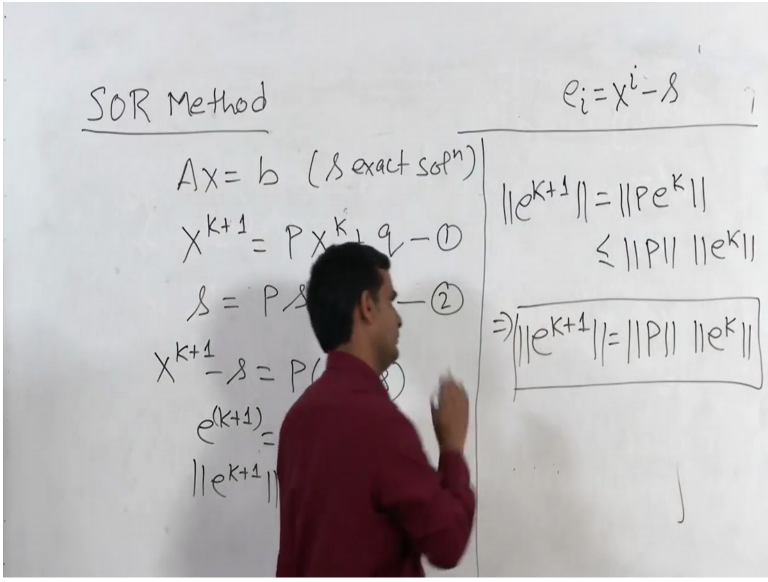

As we know that norm of Ab will always be less than equals to norm of A into norm of p, so I can write it left-hand side will be same as the previous line in the right-hand side I will be having Pe k and this will be less than equals to P into e k or I can write e k plus 1. Now this particular equation tells us about the convergence of any iterative scheme and as you can see this equation is totally dependent on the iteration matrix P.

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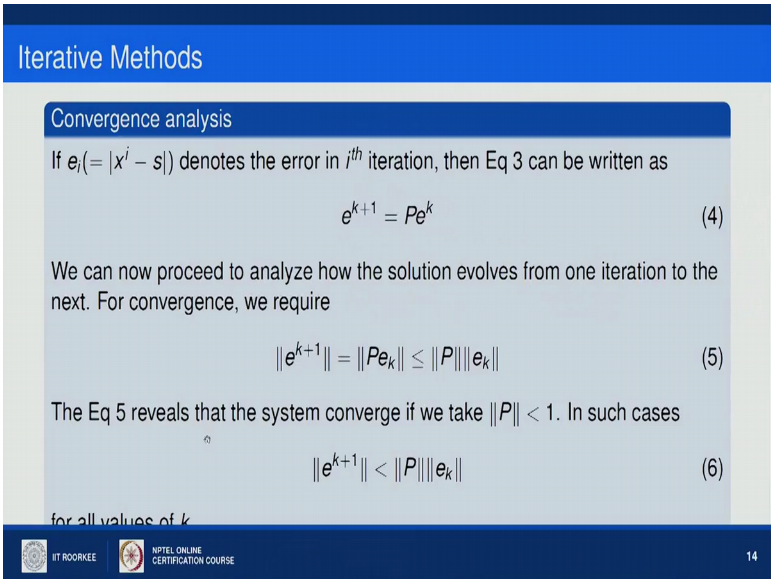

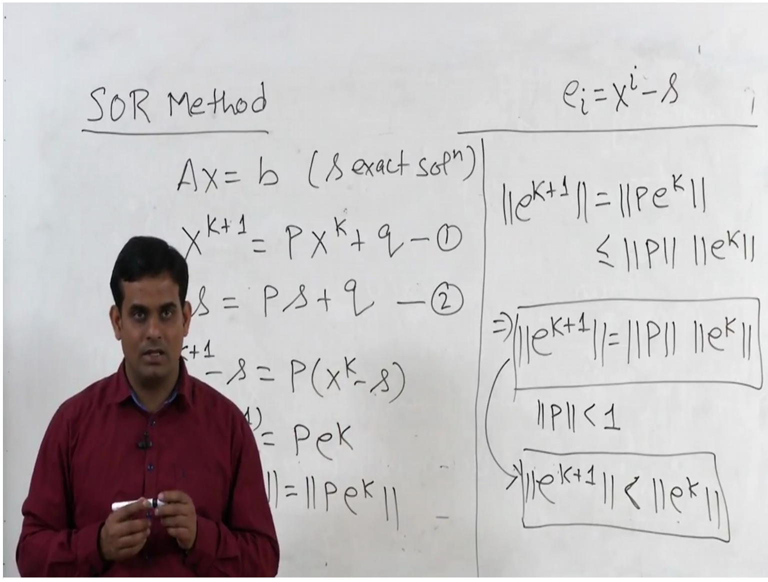

So this particular equation reveals that the system converge if we take the norm of P less than equals to 1, in such cases I can write so if I take here norm of P less than equals to 1 this equation can be written as it means error is reducing in each iteration, so if we are having a very large error in the initial solution in 1st iteration it will reduce in 2nd iteration, it will further reduce and so on.

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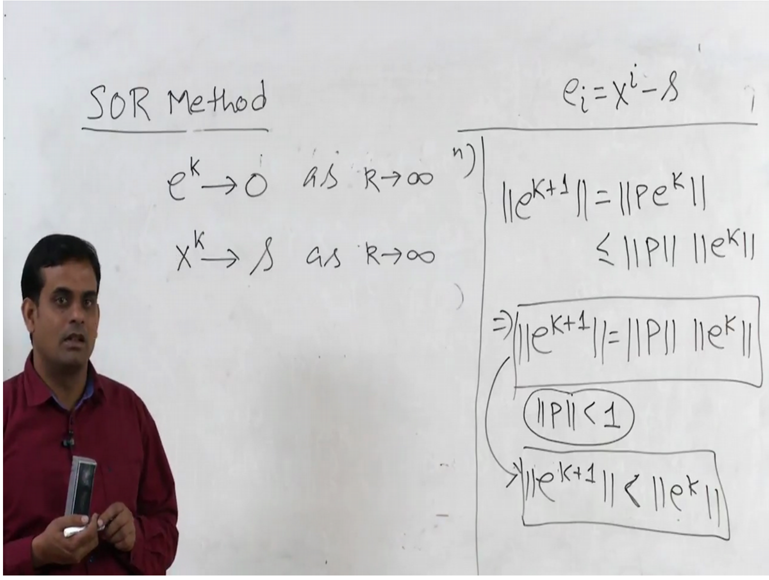

So basically when you having large number of iterations then what will happen, this particular equation tells us that error tends to 0 as this k tends to infinity and this is the convergence of any iterative scheme. Basically what it is telling? It is telling to me that the solution if error is going to 0 when k is infinity the solution is going to converge to the exact solution as k tends to infinity. It means this particular condition that is norm of the iteration matrix is less than 1 gives a guarantee for the convergence of an iterative scheme.

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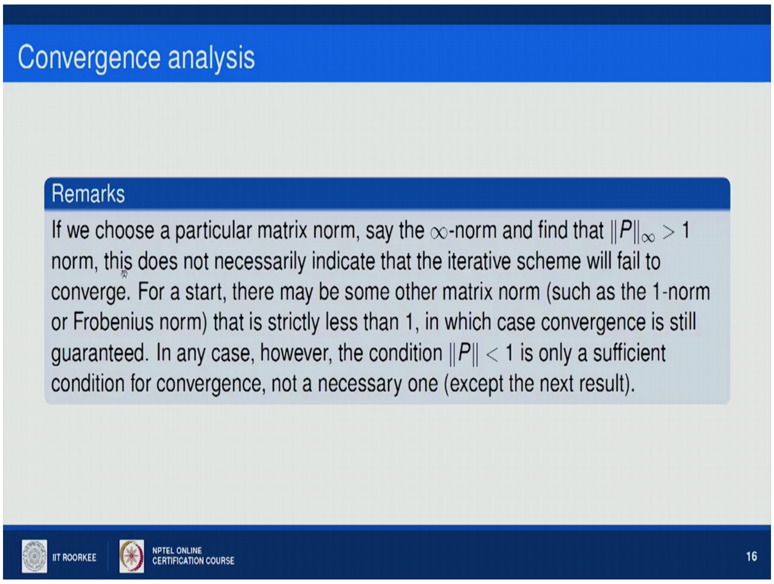

It means however I want to put a remark here that if we choose a particular matrix norm say the infinity norm and find that in this norm P the norm of P is greater than 1, this does not necessarily indicate that the iterative scheme will fail to converge because it is a sufficient condition and here we are not talking about any particular norm. For a start there may be some other matrix norm such as column sum norms or Euclidean norm that is strictly less than 1 in which case convergence is still guaranteed. So in any case condition norm of P less than one is only a sufficient condition for convergence, not a necessary one.

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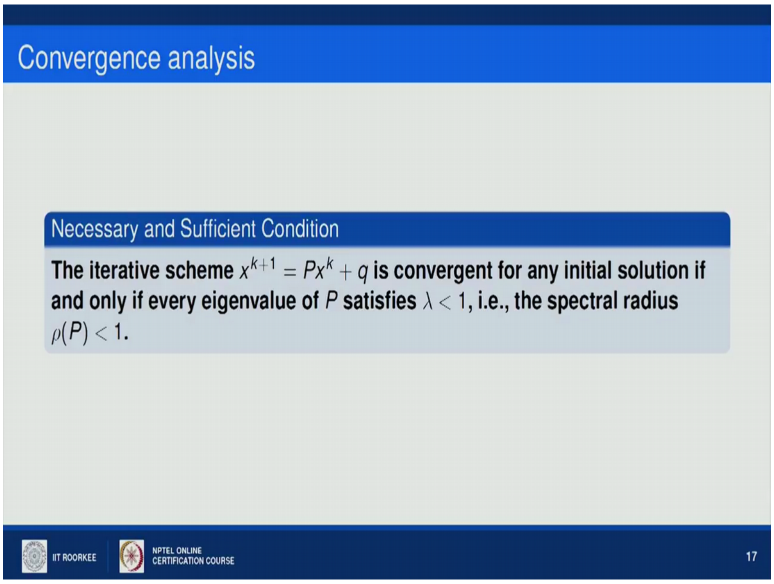

If we need to tell necessary as well as sufficient condition for the convergence of an iterative scheme then the iterative scheme x k plus 1 is equal to P times x k plus q is convergent for any initial solution if and only if every eigenvalue of P satisfy lambda less than 1. Every eigenvalue is less than one means the spectral radius of that particular matrix is less than 1. Here we have established a condition on the iteration matrix and as you know original problem was given in the form Ax equals to b and from here you need to drive the iteration matrix in which you need to do some computation. Can we have some condition on A itself which reveals about the convergence of the scheme that this particular system will converge in case of n iteration scheme.

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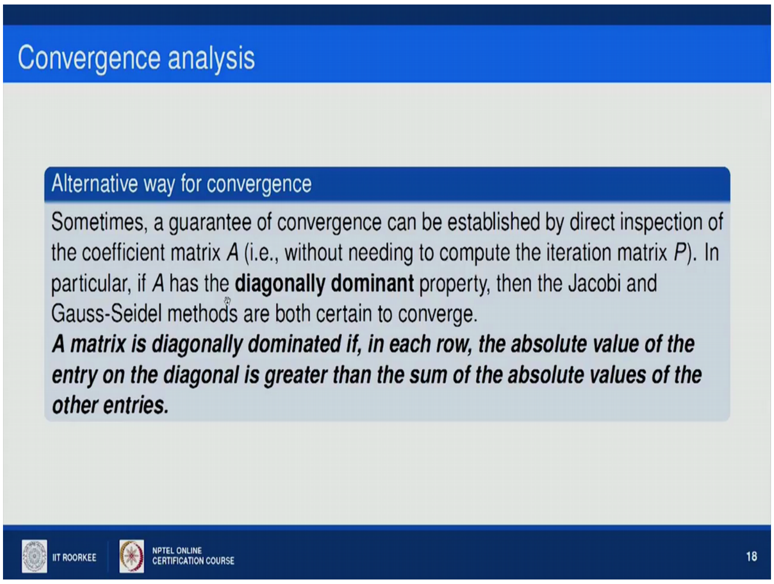

So yes we are having such condition on A, if A as the diagonally dominant property, then the Jacobi as well as Gauss Seidel method are both certain to converge. Now what we mean by diagonally dominate, a matrix is a diagonally dominate if, in each row, the absolute value of the entry on the diagonal is greater than the sum of the absolute value of the other entries.

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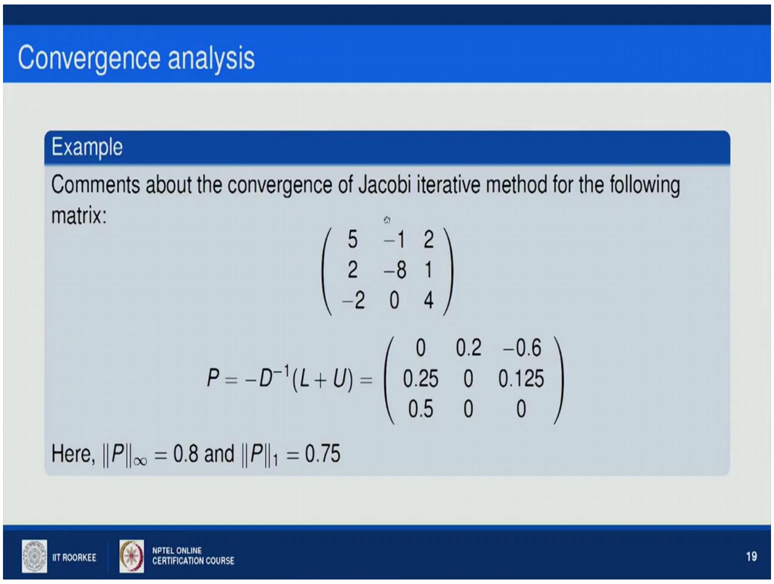

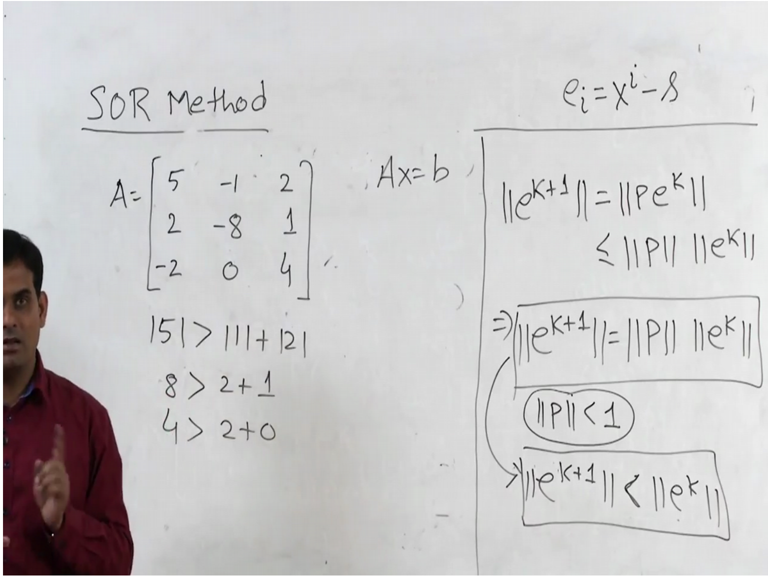

If we take the example of this 3 by 3 coefficient matrix let us say let me explain here so if a is 5, minus 1, 2 the 2nd row is 2, minus 8, 1, minus 2, 0, 4. Then what you can say about the solution of this system Ax equals to b for any right-hand side vector b using Jacobi or Gauss Seidel method whether Jacobi or Gauss Seidel method will converge to the true solution for any right-hand side vector be and for any initial solution. Yes, I can see yes because you can see the matrix A is strictly diagonally dominant, here you can see this 5 is greater than 1 plus 2 that is coming from the 1st row.

In the 2nd row the absolute value of minus 8 is 8, 8 is greater than 2 plus 1 and in the 3rd row the diagonal element 4 is greater than 2 plus 0 where 2 is the absolute value of minus 2, since the matrix is a diagonally dominant hence Jacobi as well as Gauss Seidel method will converge for initial solution when you are using this method, when you are solving this particular problem. Moreover if you see the iteration matrix and Jacobi iteration matrix for this particular problem it is coming out like this and here the norm is the row norm is row absolute row norm is 0.8 and column sum norm is 0.75.

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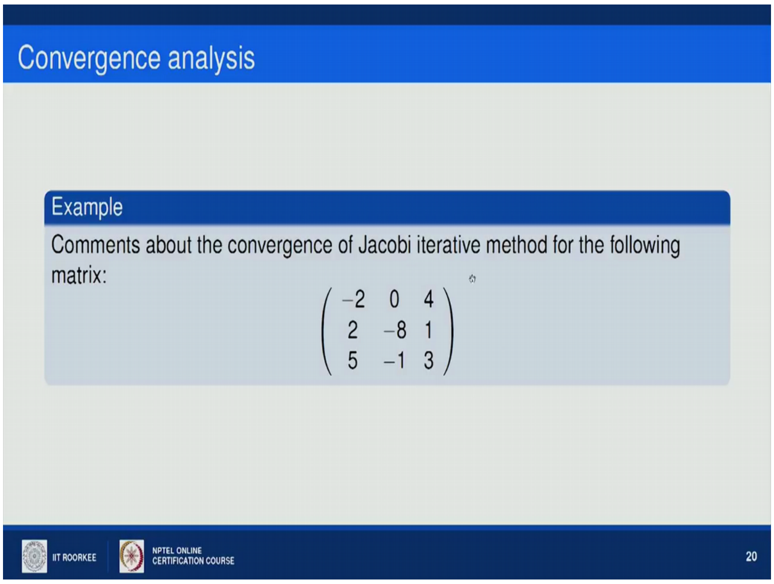

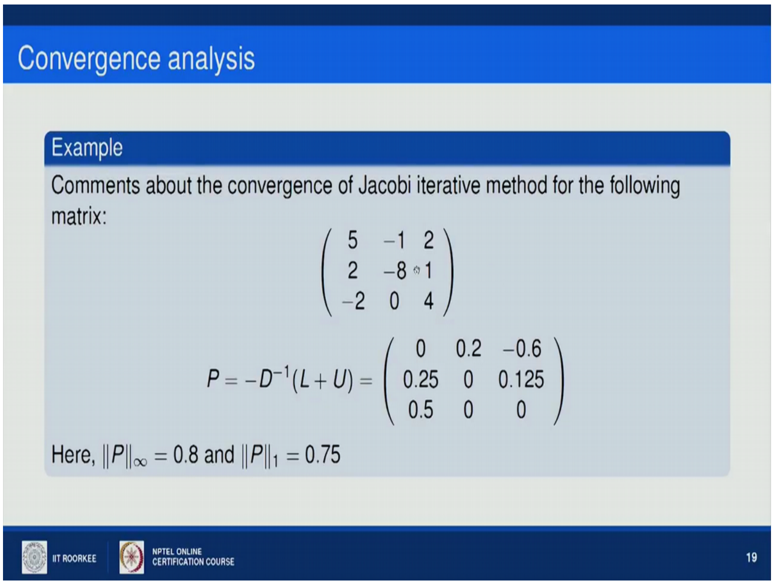

If you consider this example this is the coefficient matrix and if someone ask you that what can you comment about the convergence of Jacobi iterative method in this case then in this form the matrix is not diagonally dominant hands since you can see 2 is less than 4. However if I interchange 1st and 3rd equation of the system, the system convert into the previous system and the coefficient matrix will become the diagonally dominant and hence the Jacobi scheme will converge for this system.

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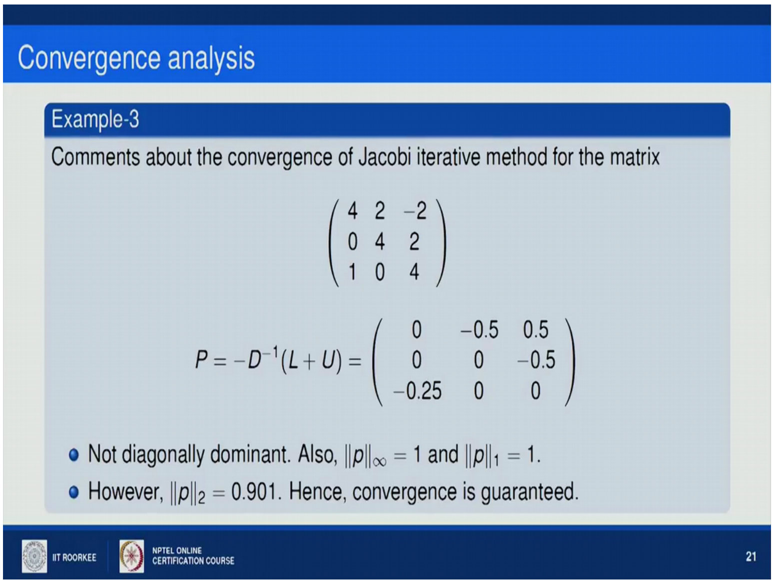

Now consider one more example this is the coefficient matrix and here in this coefficient matrix we can see that the matrix is not strictly diagonally dominant because from the 1st row 4 equals to 2 plus 2 hence we cannot comment anything about the convergence of Jacobi iterative method for this particular problem from the property of the diagonally dominant, so what we need to do? We need to find out the iteration matrix, so iteration matrix in case of Jacobi method will be like this and here you can see that the row sum norm is 1 as well as column sum norm is 1.

Hence the condition that row sum norm is less than 1 or column sum norm is less than one fails here as well as we cannot comment anything from the diagonal dominant property but if you calculate the Euclidean norm of Frobenius norm it is coming out to be 0.901 and as I told you if one norm is coming greater than one of the iteration matrix, it does not mean that method will not converge, you need to find out a norm which is less than 1 and here we are able to find out Frobenius norm which is having value 0.901 and hence convergence is guaranteed for this particular problem. So in this lecture we talked about successive over relaxation method and then in the later part of the lecture we discussed few conditions about the convergence of iterative scheme.

There we found that if any norm of the iteration matrix is less than 1 then the scheme will converge. Moreover we have seen that if the coefficient matrix of the system Ax equals to b is a diagonally dominant then Jacobi as well as Gauss Seidel method will converge for any initial solution and right-hand side vector b. So this is all about iterative schemes and the end of this particular unit that is linear system of equations. Here we learned 2 categories of methods one is direct method another one has a iterative method. In the next class we will discuss methods for solving non-linear systems or 1st we will start with non-linear equation, a single non-linear equation and then we will also learn that how to solve non-linear system of equations system of non-linear equations. Thank you.