Numerical Methods
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Lecture 40 Multi-step Method for solving ODEs

Hello everyone so welcome to the last lecture of this module and it is the last lecture of this course also. So in this lecture I will talk about multistep method for solving ordinary Differential Equation numerically. So in the past few lectures we have talked about Euler's method and then Runga Kutta method of order 2 and order 4.

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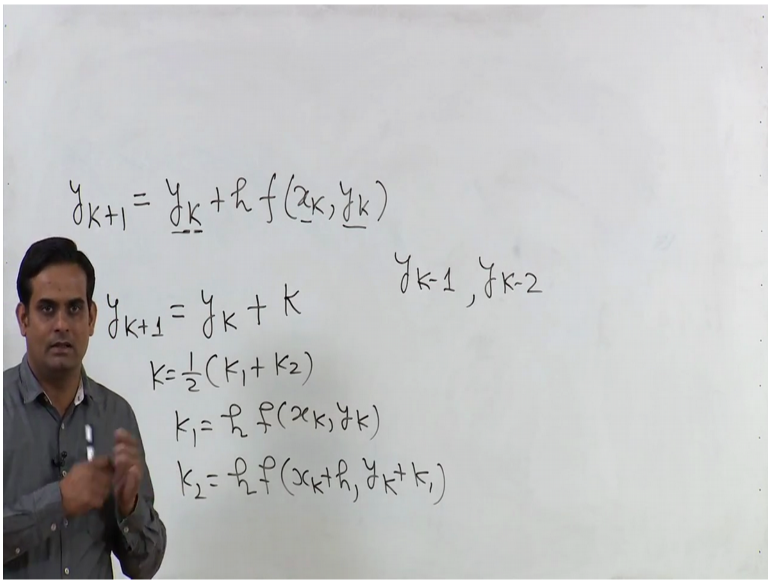

So in all those methods like in Euler's method we Have taken our approximation of Y at X equals to X K plus 1 as YK plus H times F of XK YK. So Here we are assuming that our X equals to XK and at XK YK, the value YK Y is YK and these two values are known to us and from these two values we are moving to the next value of Y at next point that is YK plus 1. In the same way, in the Euler's Runga Kutta method what we have done?

We have calculated the value of Y at X equals to XK plus 1 as Y at X equals to XK plus K. Where K is coming from the average of K1 and K2 and K1 is given as H time F of XK YK and K2 is given as H time F of XK Plus H YK plus K1. Now please look Here that for finding the value of Y at X equals to XK plus 1 that is YK plus 1. We are using the value of Y at X equals to XK and XK and YK.

So for finding the value in the next approximation or in the next iteration what we are doing? We are using only the value of current iteration. Similarly we are doing the same thing in Runga Kutta method. So here we are moving one step that for finding the value in the next iteration we are taking the value of current iteration only. However in multistep method we will use the value not only the value of current iteration.

But we will use values of some previous iterations also. Like the value at X equals to XK minus 1 at X equals to XK minus 2 and so on. And XY for calculating the value of Y at X equals to XK plus 1, we are using value at Kth iteration in K minus 1 iteration, in K minus 2 iteration and that's why it is called multistep method.

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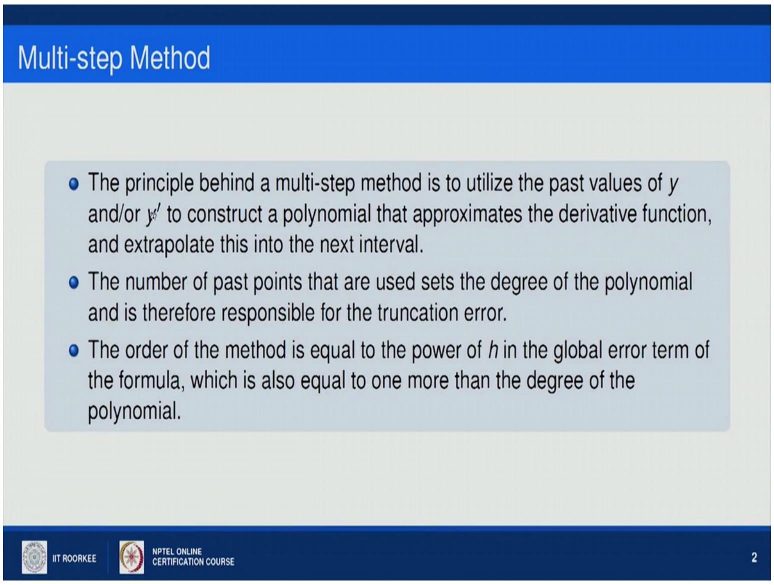

So the principle behind the multistep method is to utilize the past values of Y and or derivative of Y to construct a polynomial that approximates the derivative function and then extrapolate this into the next interval. So basically using four to five points we will construct approximate a polynomial of degree 4 or 5 and for the value of in the next iteration or at the point we will extrapolate the value using this particular polynomial.

So the number of past point that are used sets the degree of the polynomial which we are going to approximate or going to fit and is therefore responsible for the truncation error. The order of the method is equal to the power of H in the global error term of the formula which is also equal to one more than the degree of the polynomial.

So if we are having 4 degree polynomial we will be having the error of order H rest to power 5. So generally using an implicit linear multistep method, there is an additional difficulty because we cannot solve simply for the newest approximate Y value that is YN plus K.

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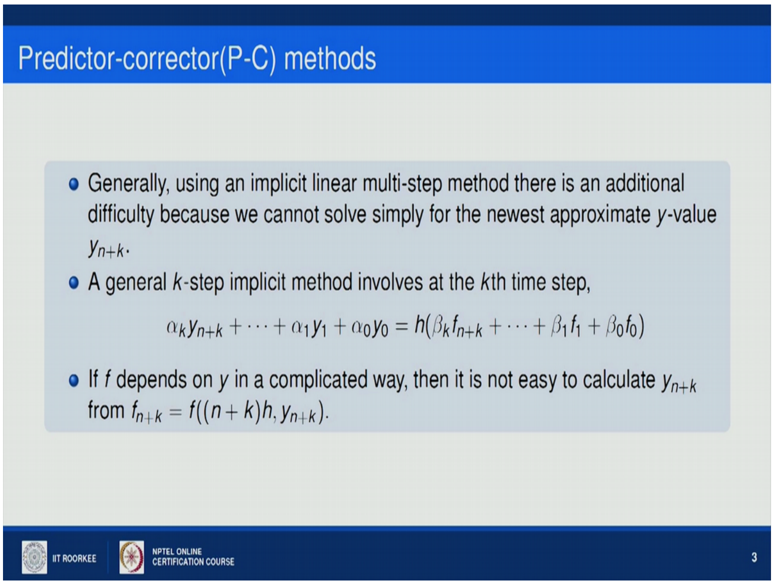

So because we are having an implicit formula and here we are having the value of Y at X equals to XN plus K in left hand side as well as in right hand side and hence how to get out the value explicitly that is the problem here. So a general K step implicit method involves at the Kth time step by this equation.

So alpha KYN plus K plus plus plus like that alpha 1 Y 1 plus alpha 0 Y 0 and this is equals to H times beta K F of N plus K plus plus plus beta 1 F1 plus beta 0 F0. So now here you can note down that here in this equation I am having the value of YN plus K in the left hand side and FN plus K in the right hand side.

Basically this FN plus K is also involving the term YN plus K because FN plus K is F of XN plus K that is NK plus times H into comma YN plus K. So in this way we are having YN plus K in the both sides and that's why we are saying it an implicit scheme.

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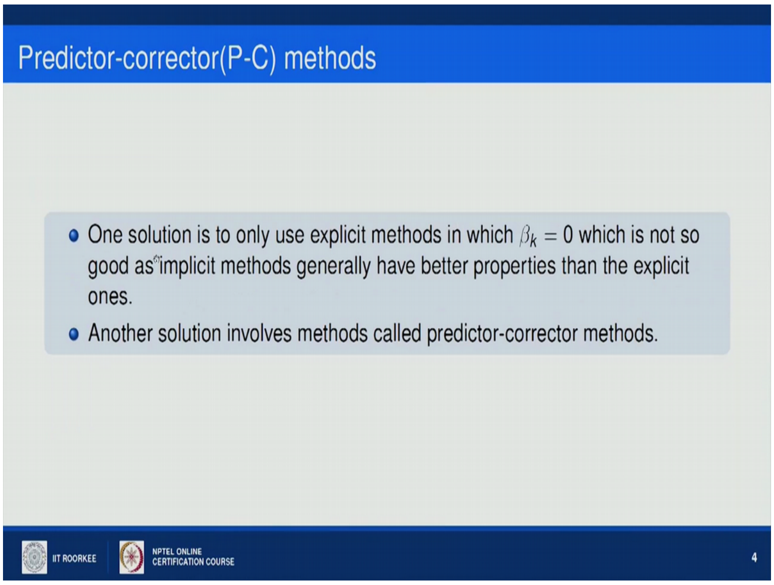

So to get rid of this particular thing one solution is to only use explicit method.

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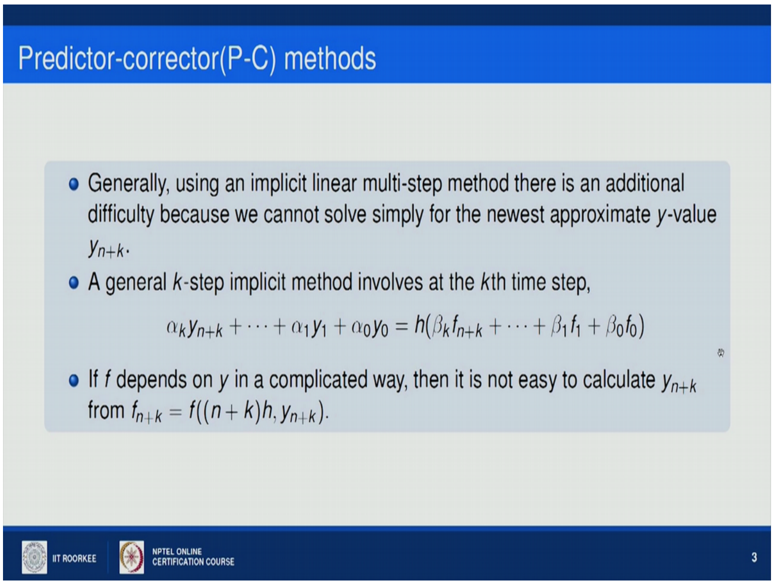

In which right hand side is 0 that is all betas are 0.

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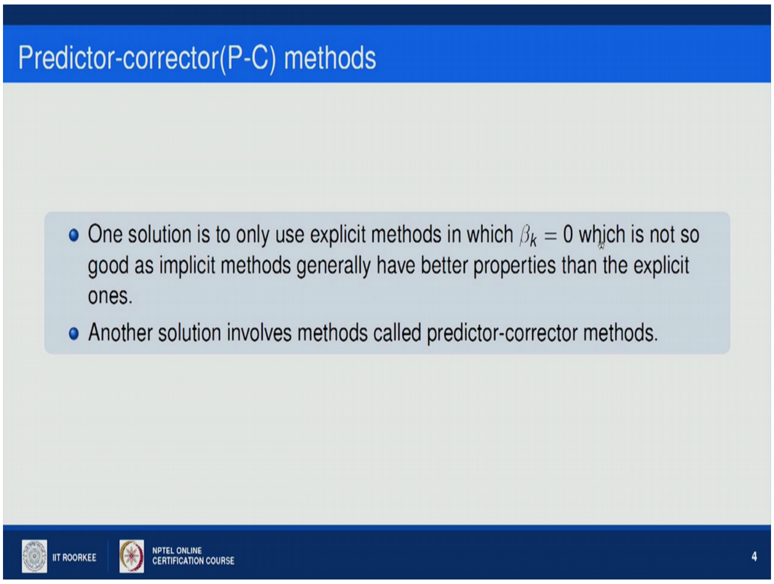

However this is not so good as an implicit method generally but it can have a simplification in terms of calculation or as I told you it is simple in calculation but not so good in terms of approximation. So what is the solution? The solution is to use predictor corrector method so explicity scheme with predictor corrector formulation.

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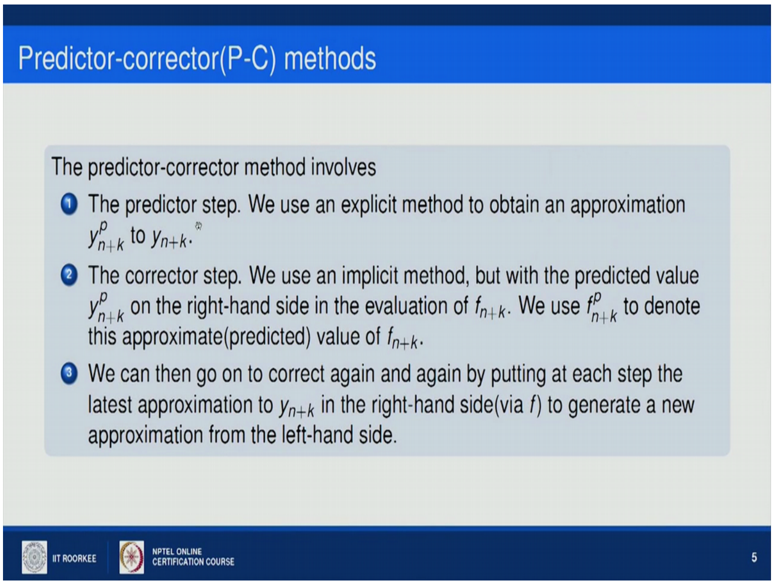

So the predictor corrector method involves the predi predictor step in which we use an explicit method to obtain an approximation YN plus K rest to power P to YN plus K. So or you can say it star also.

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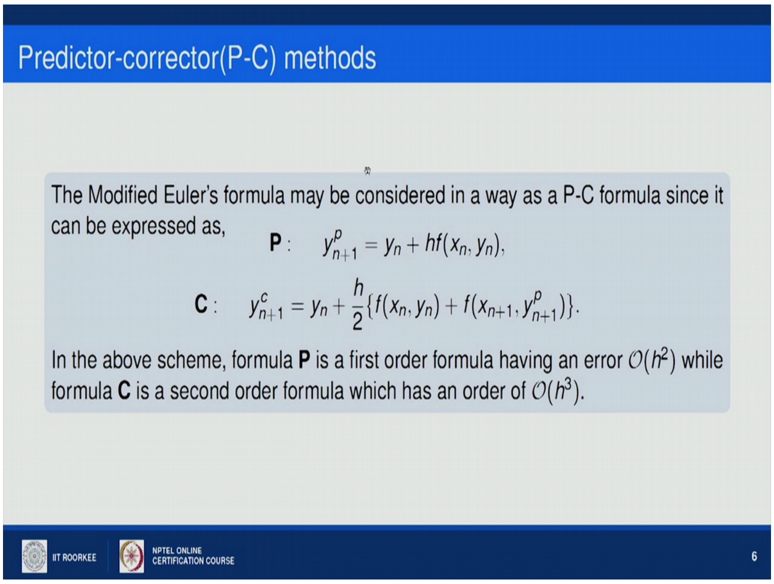

If you see this Euler's formula as I told you it may be consideredin a way as a P-C formula that is predictor corrector formula. Since we are going to express it in this way and here only we are using P equals to YN plus 1 P equals to YN plus H times F XN TN. While in corrector form YN plus 1 C can be obtained at YN plus H by 2 F of XN YN plus F XN plus 1 YN plus 1 P.

And which is the accurate Euler's for modified method in which we are taking the average of the slope in the whole interval. In the above scheme that is in this scheme the formula P is a first order formula having an order error of order H while formula C that is the corrector is a second order formula which has an order of H cube.

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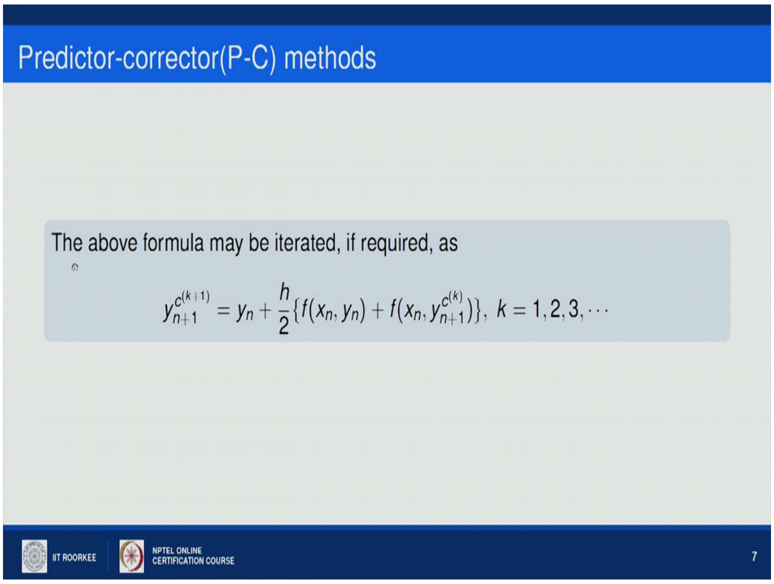

So the above formula may be iterated if required as YN plus 1 CK plus 1 equals to YN plus H by 2 F of XN YN plus F of XN YN plus 1 CK. Where K equals to 1, 2, 3. So you can iterate and iterate again and again. Now we will come to multistep predictor corrector formula and this is the Milne's method.

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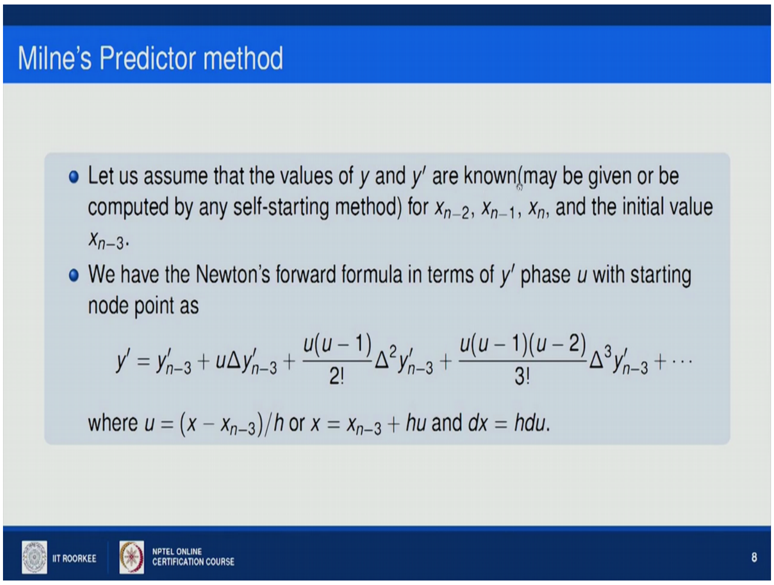

So in Milne's method let us assume that the value of Y and Y dash are known may be given to us or it may be computed by any self starting method like Euler's method or any other method. For the points like XN minus 2, XN minus 1, XN and the initial value XN minus 3.

So we have done Newton's forward formula in terms of Y prime phase U with starting node point as Y prime equals to Y prime N minus 3 plus U times delta that is the forward difference operator Y prime N minus 3 plus U into U minus 1 upon factorial 2 del square Y prime N minus 3 plus U into U minus 1 into U minus 3 2 upon factorial 3.

And then del cube of Y prime N minus 3 and so on. Where this U is X minus XN minus 3 upon H where H is step size. So we can write X as XN minus 3 plus H times U and from here we can write DX equals to H times DU.

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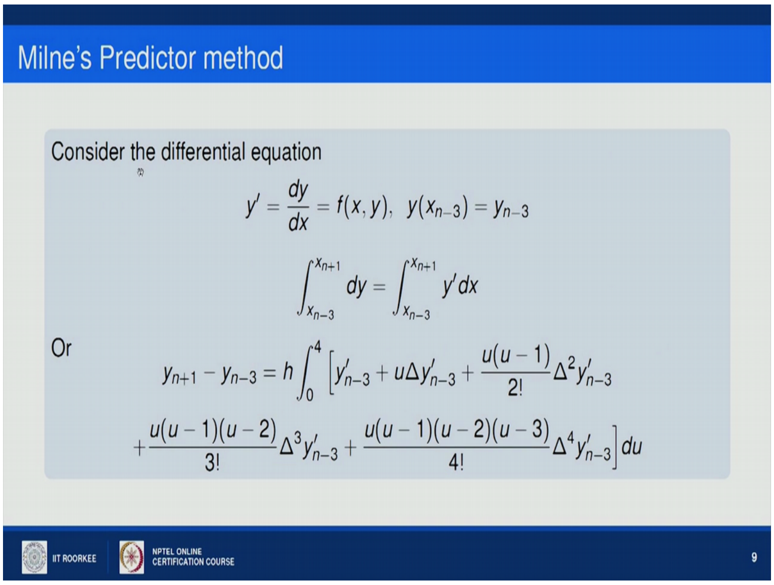

Now consider the initial value problem Y prime equals to F of XY and the initial value is given at XN minus 3 that is Y at XN minus 3 is YN minus 3. Now if we differentiate it sorry integrate it both sides from XN minus 3 to XN plus 1 then we can have this iteration that integration over XN minus 3 to XN plus 1 DY equals to XN minus 3 to XN plus 1 Y prime DX or when this integration will be Y when I will substitute limits.

So it will become YN plus 1 minus YN minus 3 from the lower limit equals to H times 0 to 4 and from the Newton's forward formula it can be written as Y dash Y prime N minus 3 plus U into delta of Y prime N minus 3 and so on up to fourth order and finally DU of this.

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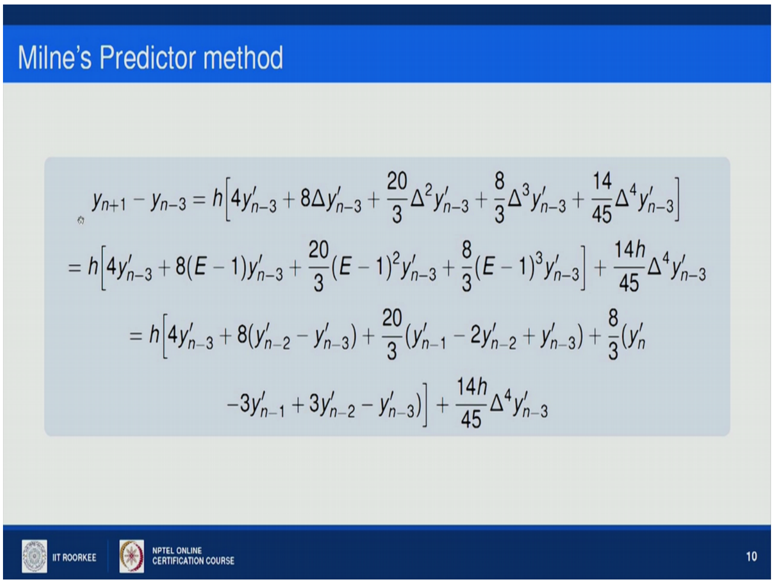

So YN plus 1 minus YN minus 3 equals to we are writing as is this formula.

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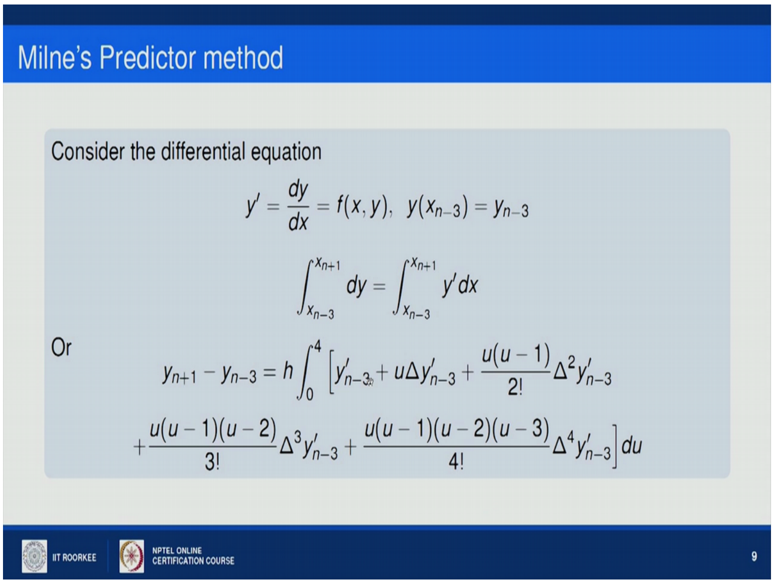

And when I will di different integrate it the U terms will come here because that integration of 1 will become U with respect to U. So when I will substitute limit so upper Limit is 4.

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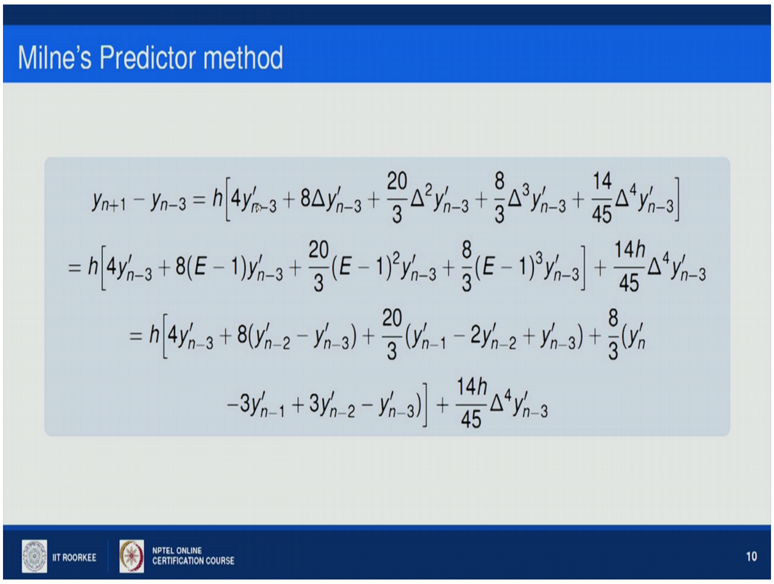

So it will become 4 times Y prime N minus 3 which is Y first term.

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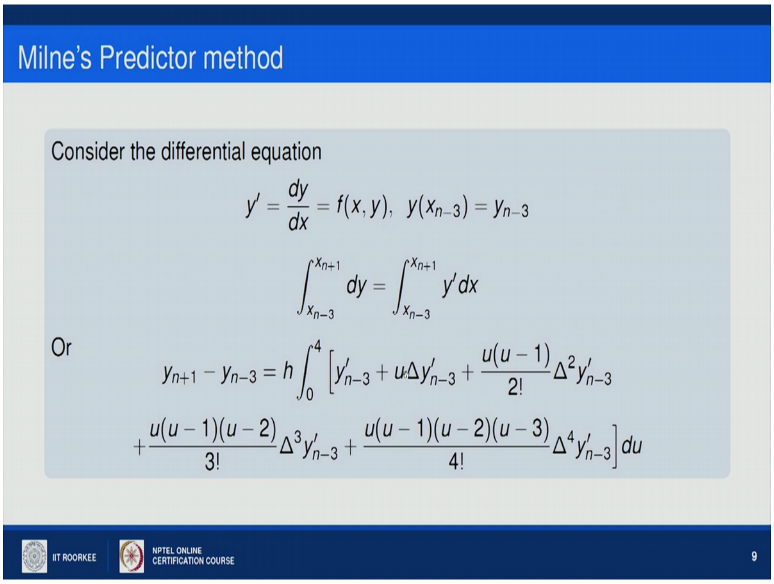

Then in the second term it will become U square upon 2. So when I will substitute the limit U square will become 4 into 4 16. So 16 by 2 will become 8. So 8 times delta of Y prime N minus 3 and similarly we got another 3 terms that third term will become 20 upon 3 del square Y prime N minus 3 plus 8 upon 3 del cube Y prime N minus 3 plus 14 upon 45 del 4 Y prime N minus 3.

Now we will replace this forward difference operator by shift operator. So we know that from the interpolation that del delta equals to E minus 1. So I can write this delta 8 8 delta into 8 times E minus 1. Similarly for del square it will become E minus 1 square. Del cube will become E minus1 cube and this term I will take out in terms of delta only.

So when I will use it and so 4 times Y prime N minus 3 will remain as such. From here I will get 8 times E of Y prime N minus 3. So V E of Y prime N minus 3 will become Y prime N minus 2.

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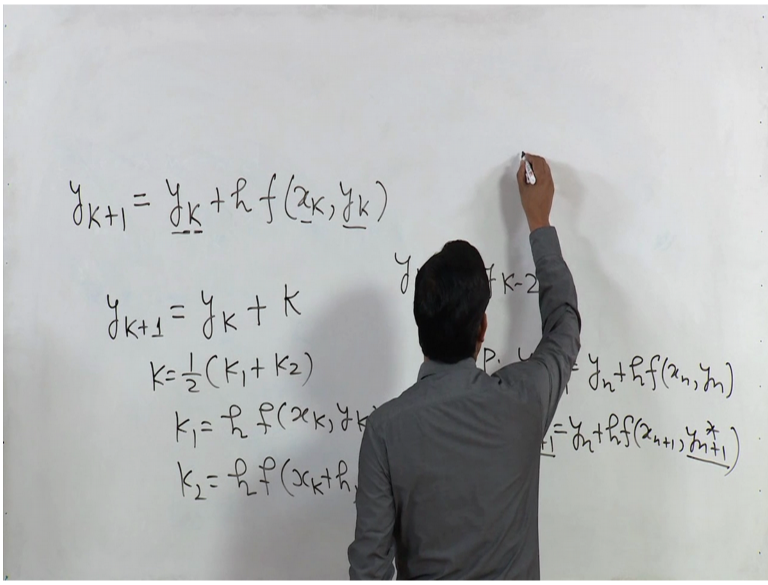

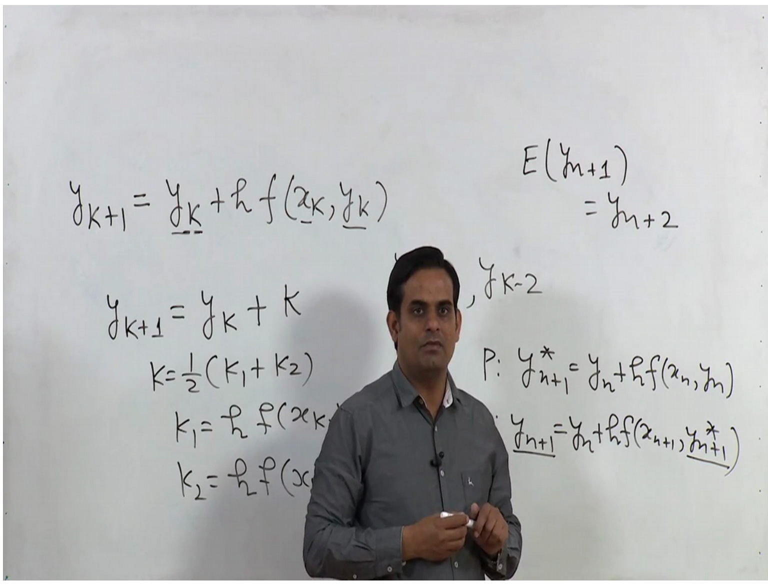

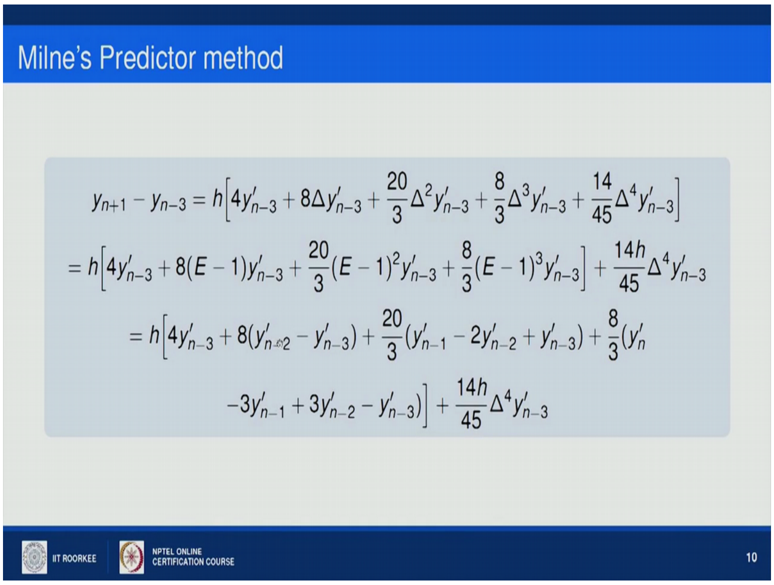

Because we know from the shift operator that E of Y of N plus 1 will become YN plus 2. So hence it will be come 8 times Y prime N minus 2 minus 8 times Y prime N minus 3 plus 20 by 3 Y prime N minus 1 minus 2 times Y prime N minus 2 plus Y prime N minus 3. So this is coming from this E minus 1 square.

And similarly for this term I will get 8 by 3 into Y prime N minus 3 times Y prime N minus 1 plus 3 Y prime N minus 2 minus Y prime N minus 3 and then this term which we have taken out already. That is 14 H upon 45 del 4 Y prime N minus 3. Now after simplification here we are having 4 times Y prime N minus 3. Here we are having minus 8 times Y prime N minus 3. Similarly similarly here I am having 20 by 3 times Y prime N minus 3 and here minus 8 by 3 times Y prime N minus 3.

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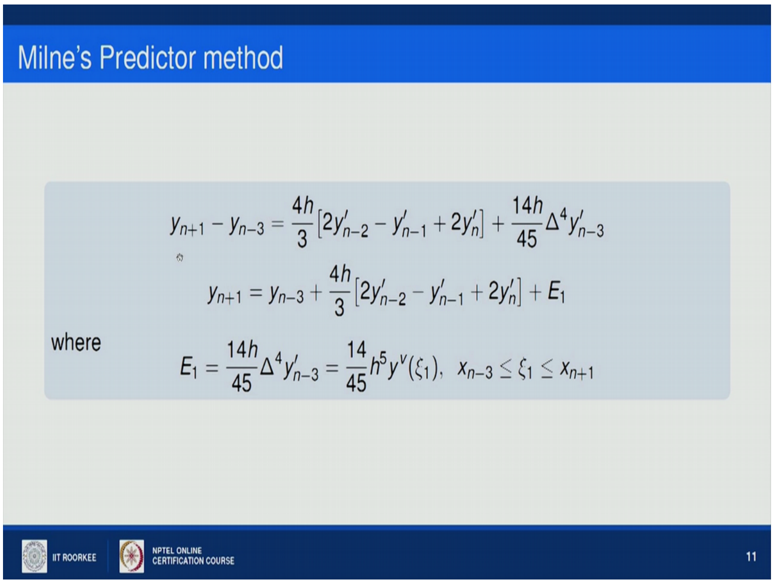

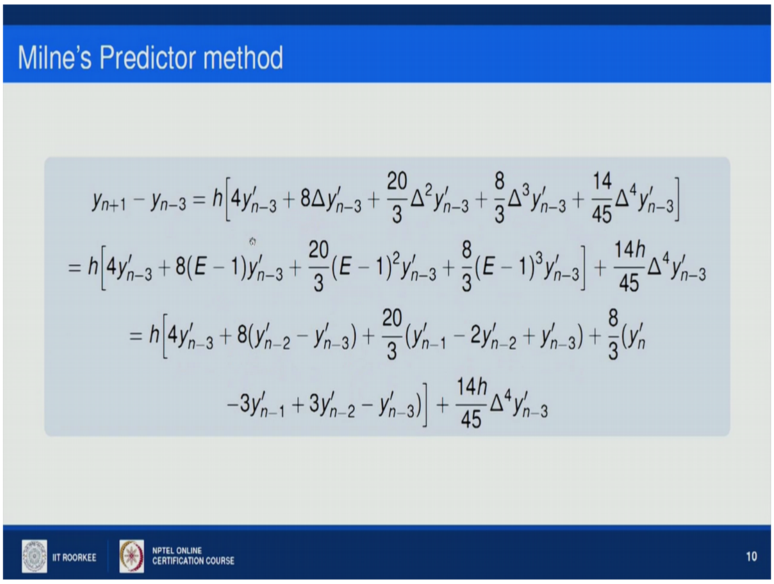

So after simplification I will get that YN plus 1 minus YN minus 3 that is the left hand side equals to 4 times H upon 3 into twice of Y prime N minus 2 minus Y prime N minus 1 plus 2 times Y prime N plus 14 H upon 45 del 4 Y prime N minus 3. Or I take this term YN minus 3 inthe right hand side so I will get this expression and let us denote the this term which I have taken out as E1 that is the error in predictor.

And this is given as by this expression that is 14 H upon 45 del 4 Y prime N minus 3 and this I can say it is of order H5. That is by the Taylor series 14H rest to power 5 upon 45 into fifth order derivative of Y at some point xi1 where this point xi1 exist between XN minus 3 to XN plus 1 in this interval.

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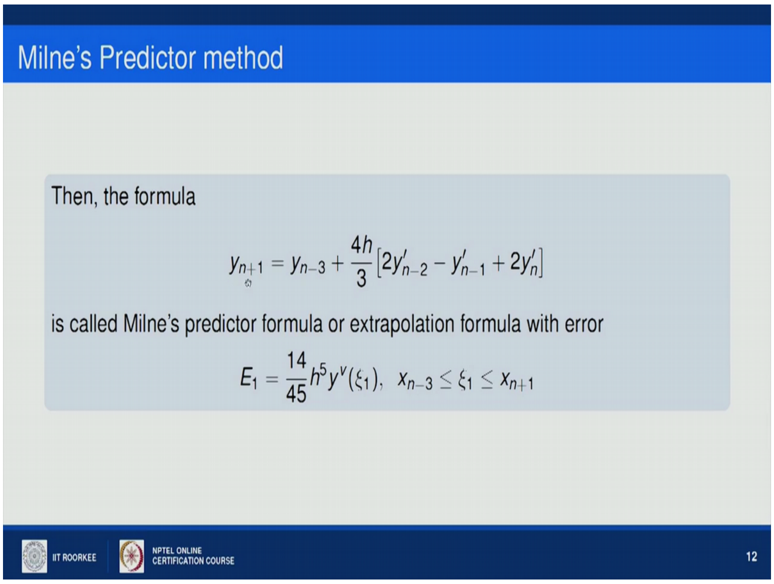

So thus the formula YN plus 1 equals to YN minus 3 plus 4H upon 3 2Y prime N minus 2 minus Y prime N minus 1 plus 2 times Y prime N is called the Milne's predictor formula or the extrapolation formula with error of order H rest to power 5 given by this expression. Now we need to derive corrector formula. So from here we will get the value of YN plus 1 but that will be the predict value. We have to correct this value and for this we need corrector formula.

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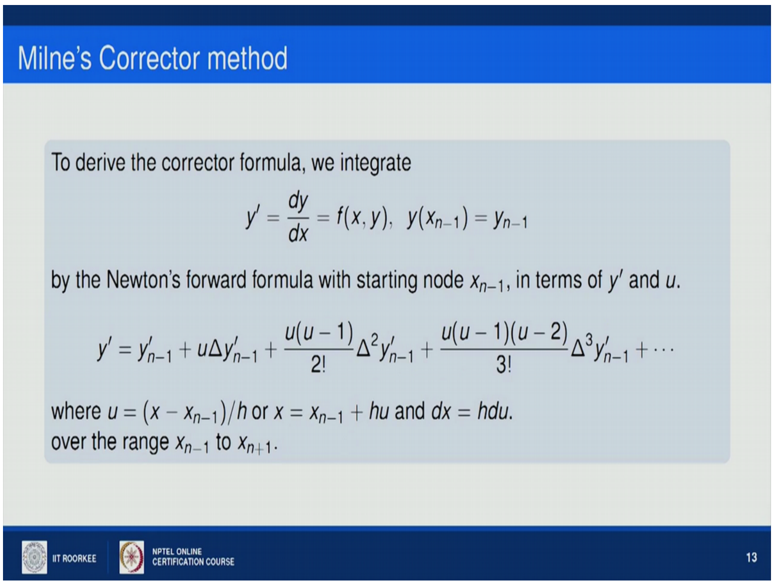

So for the corrector formula again we will consider the same initial value problem but with different initial point. So now we will consider initial point as X of N minus 1. So and we consider that Y at X equals to XN minus 1 equals to YN minus 1. Again we will use Newton's forward formula with starting node XN minus 1 in terms of Y dash and U.

So it is given by this expression Y dash equals to Y dash N minus 1 plus U times del Y dash N minus 1 plus U into U minus 1 upon factorial 2 del square Y dash N minus 1 and so on. So we have taken it up to del 3 and here U is X minus XN minus 1 upon H or I can write from this equation X equals to XN minus 1 plus H times U.

So it will become DX equals to H times DU which is the same as in predictor formula. However range is different in the predictor we were having the range from XN minus 3 to XN plus 1. But here we are using the range from XN minus 1 to XN plus 1. So we are using here only three steps wherever in predictor formula we were using total 4 steps.

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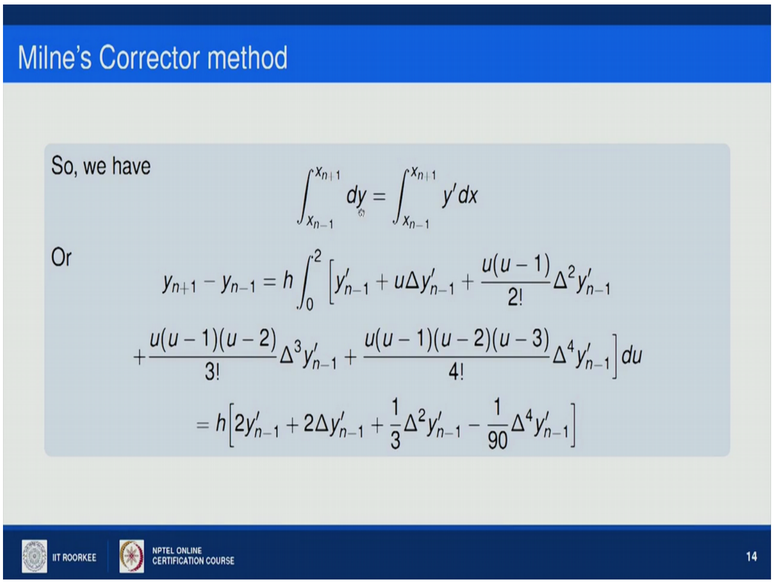

So we have again the integral over XN minus 1 to XN plus 1 of DY equal to integral of Y prime DX over XN minus 1 to XN plus 1. So in the same way as we did in predictor formula we can take this range as YN plus 1 minus YN minus 1 and this is equals to H times integral 0 to 2. Because you can see earlier it was from 0 to 4 in the predictor.

Y prime N minus 1 plus U del of Y prime N minus 1 plus U into U minus 1 upon factorial 2 del square Y prime N minus 1 and plus third order and fourth order term and U of this. So finally after integrating it over U with respect to U and putting the upper limit of U as 2 and lower limit as 0 it is coming out as H times 2 times Y prime N minus 1 plus 2 delta Y prime N minus 1 plus 1 by 3 del square Y prime N minus 1 minus 1 upon 90 del rest to del 4 Y prime N minus 1.

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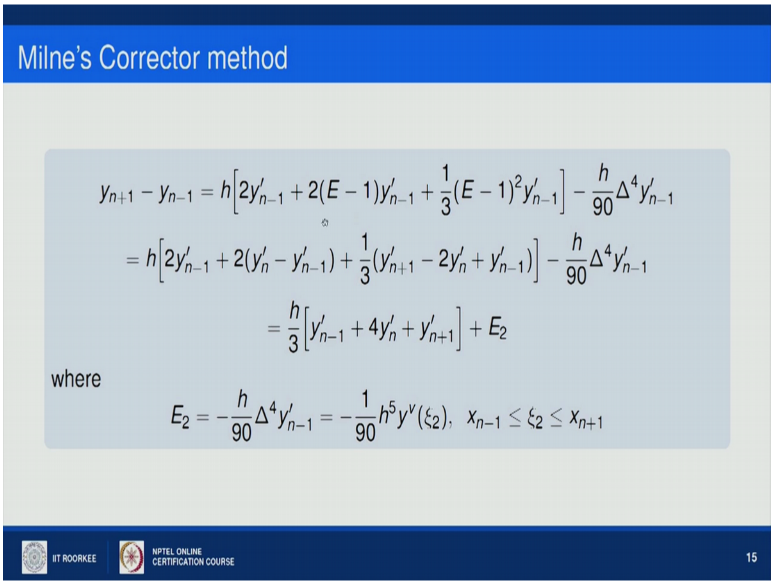

So again we will replace this forward difference operator by the shift operator that is we replace del by del equals to E minus 1 and we will get this expression.

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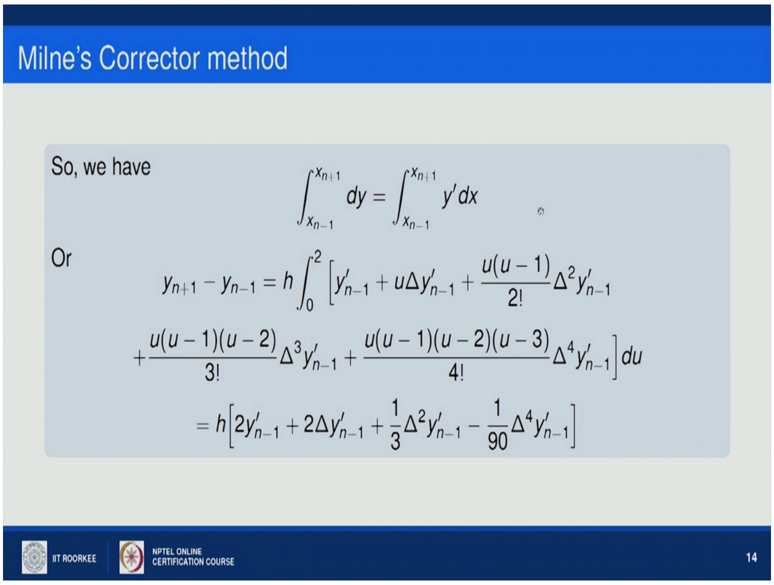

And here again like we have done in the earlier step we will take this 1 upon 90 del rest to power 4 Y prime N minus 1 outside the bracket.

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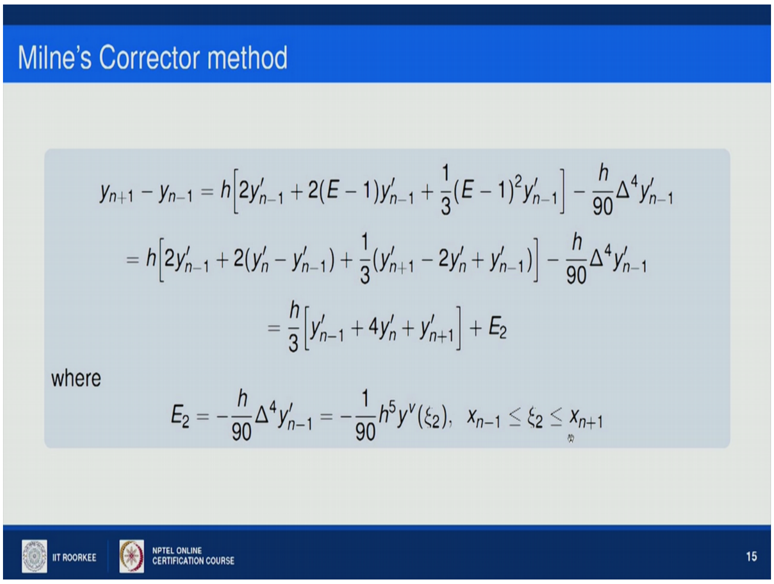

So we have done it here. After simplification it is coming out as H by 3 Y prime N minus 1 plus 4 times Y prime N plus Y prime N plus 1 plus E2. Where E2 is the error and it is of the order H rest to power 5 which is similar to the predictor one.

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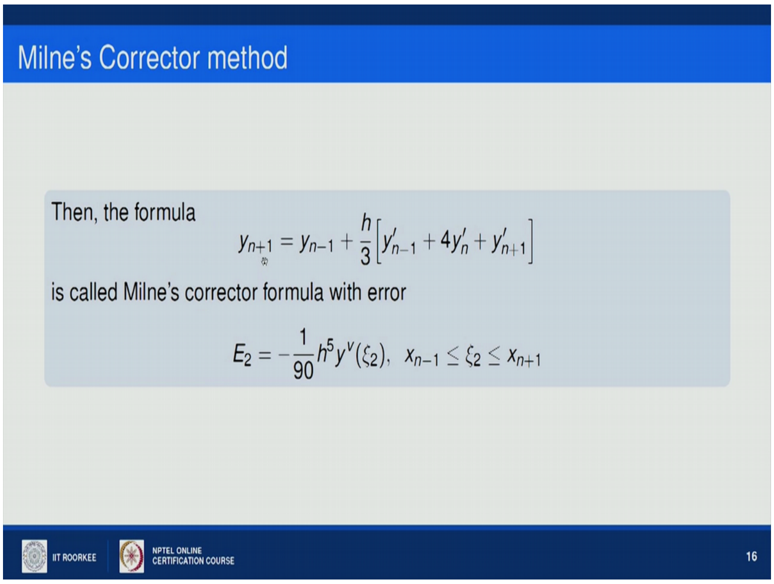

So the formula YN plus 1 equals to YN minus 1 plus H by 3 into Y prime N minus 1 plus 4 times Y prime N plus Y prime N plus 1 is called the Milne's corrector formula with error an error of order H rest to power 5.

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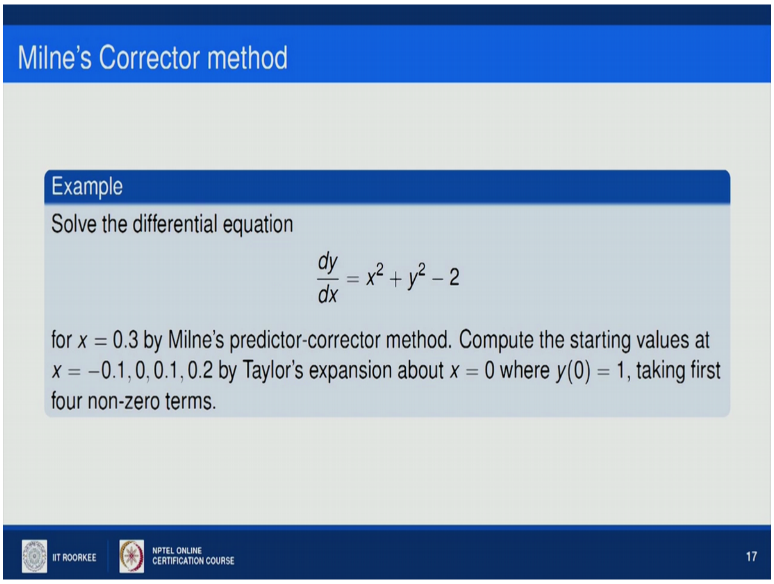

So hence we are having corrector formula as this one.

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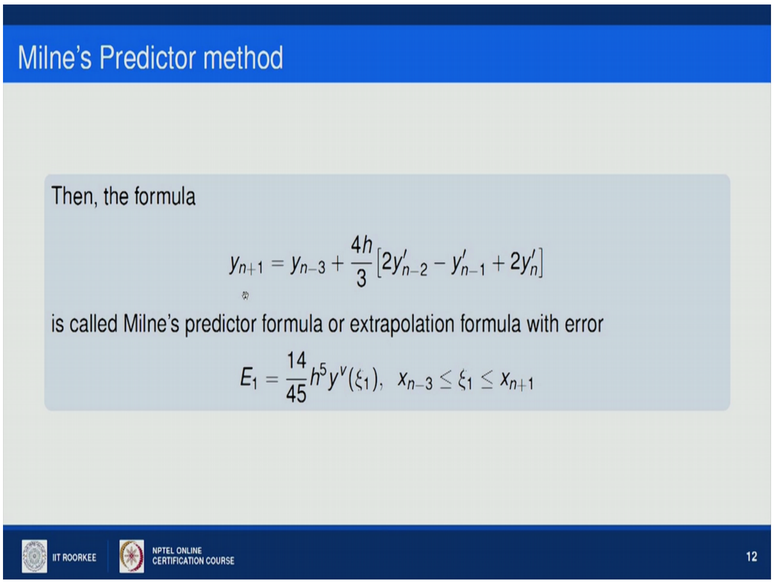

From here we will get an initial value or a predicted value of YN plus 1 which will be corrected by the corrected corrector formula this one which we have just derive. So combination of these two formulas are called Milne's PC formula predictor corrector method. So let us take an example.

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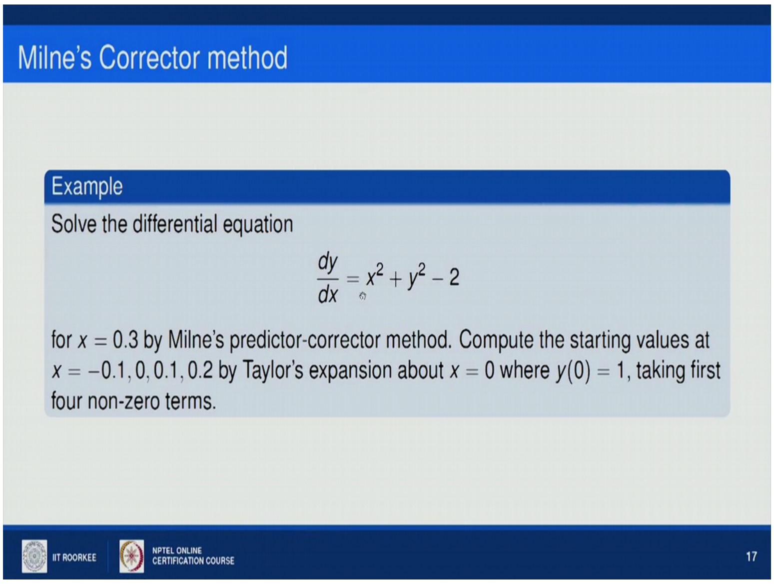

So we need to solve the differential equation DY over DX equals to X square plus Y square minus 2. For X equals to 0 point 3 by Milne predictor corrector method compute the starting values at X equals to minus point 1, 0, point 1 and point 2 by the Taylor's expansion about X equals to 0 where Y0 is 1 taking first non zero terms 4 non zero terms in the Taylor's series expansion.

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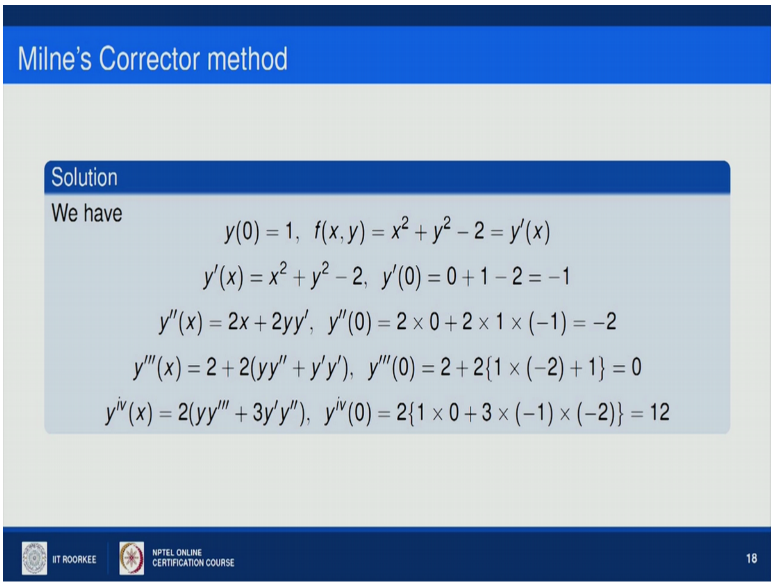

So here Y0 is 1. F of XY is X square plus Y square minus 2 that is my Y prime X. So basically what I am having?

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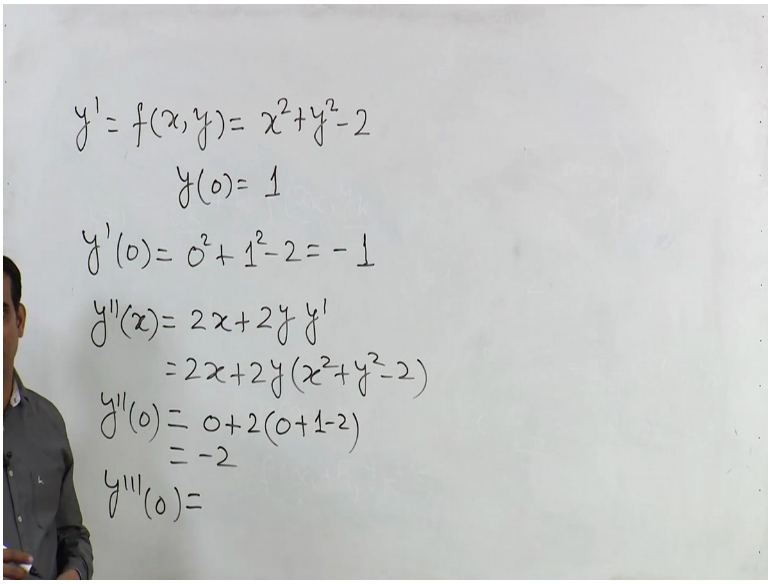

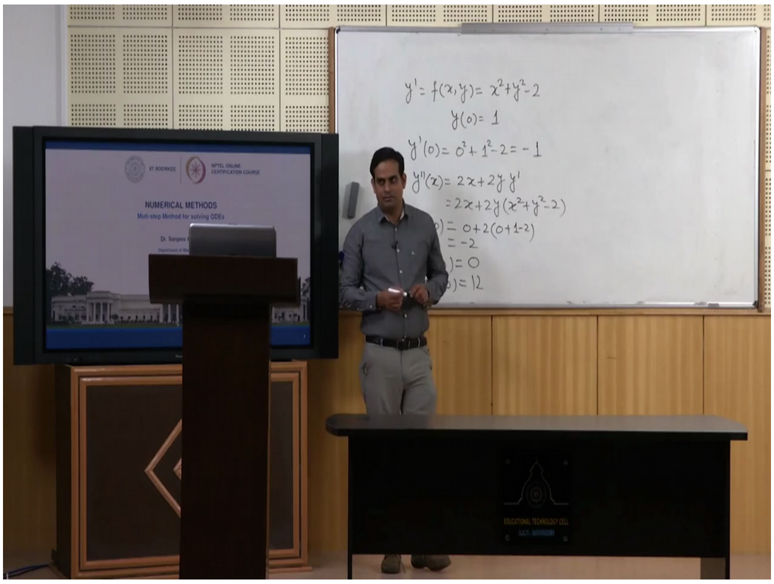

So Y prime is F of XY and it is X square plus Y square minus 2 and it is also given that Y0 equals to 1. So now Y prime 0 will become 0 square plus 1 square minus 2 because it is F of XY and it is coming out as minus 1. Now I will calculate Y double prime Y double prime X will become twice of X plus twice of Y into Y prime. So it will become twice of X plus twice of Y into X square plus Y square minus 2.

And when I will calculate the value of Y Prime at double prime at 0 it will be 0 plus 2 times Y at 0 will become 2 0 plus 1 minus 2. So minus 2 similarly I will calculate Y triple prime at 0 and again I will differentiate this with respect to X and I will substitute the value of y prime from here in the expression and it will come out finally as 0. I will also calculate as I told you in the Taylor series expansion we will use first four terms. So I will calculate this value and this will come out as 12.

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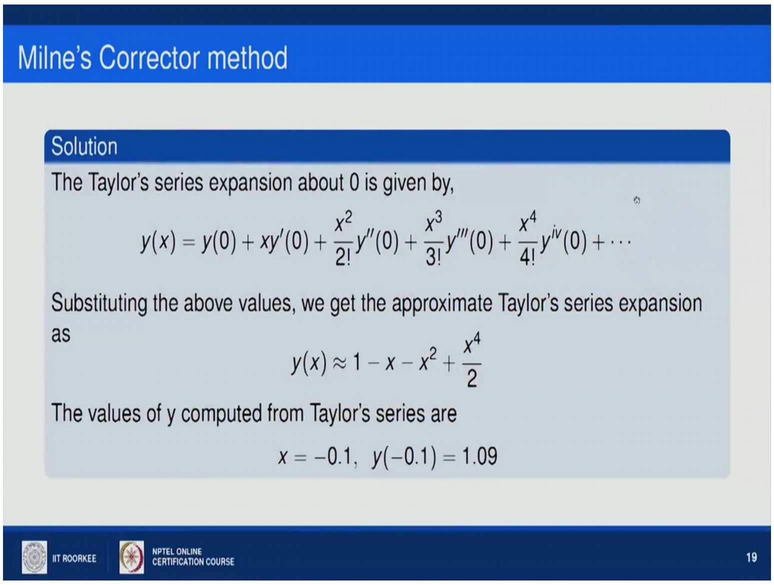

Now after calculating these values I will use the Taylor series expansion of Y about X equals to 0. So YX can be written as Y0 plus X times Y prime 0 plus X square upon factorial 2 Y double prime 0 and so on. So substituting all these values I will get YX which is approximately equal to 1 minus X minus X square plus X rest to power 4 upon 2.

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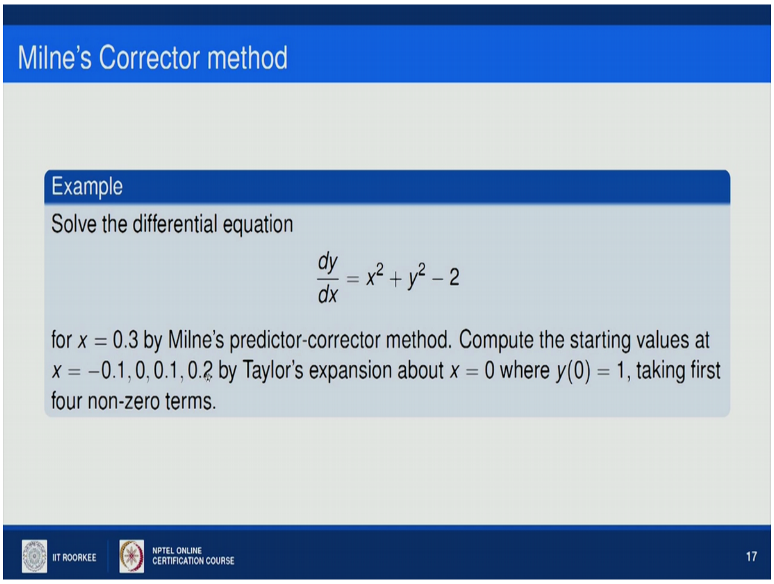

After getting this particular expression I will calculate the value of Y at minus 0 point 1, 0, point 1 and point 2.

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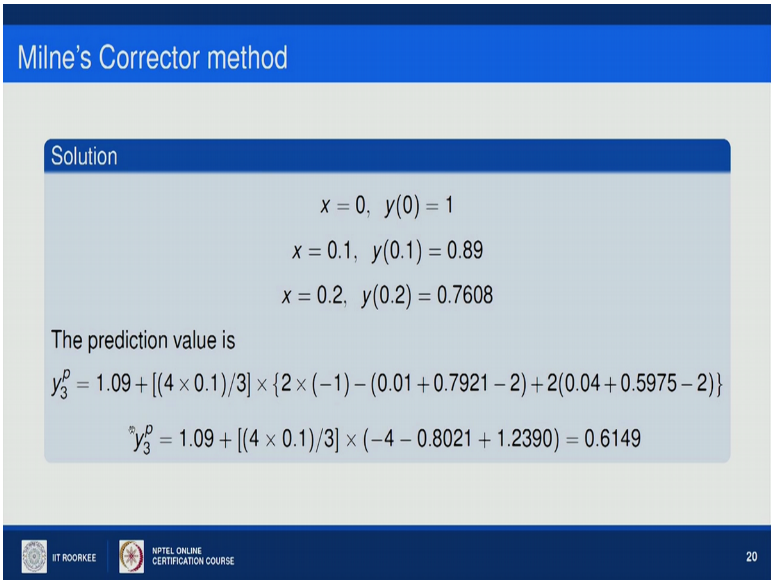

So once I will calculate it.

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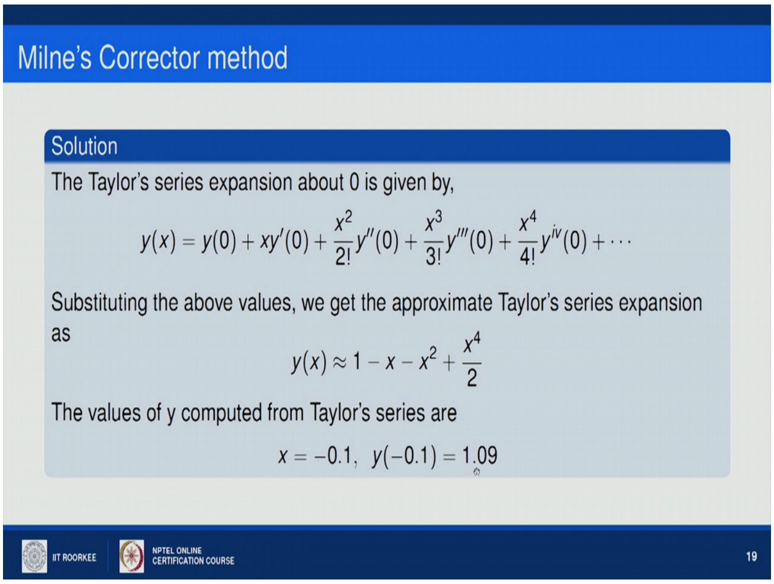

I will get Y at the point X equal to minus point 1 equals to 1 point 09.

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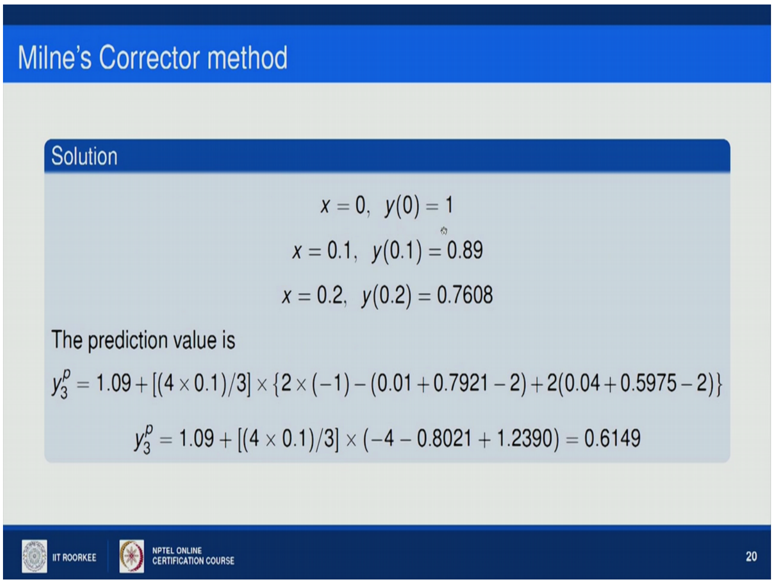

Y at X equals to 0 again coming out as 1 which is also given to us as an initial condition. Then Y at 0 point 1 is 0 point 89 and Y at 0 point 2 is 0 point 7608. Now I will use I am having all these values of Y at these points these 4 points so I will get the I will calculate the value of Y at 0 point 3 using the Milne predictor formula. So I will put the value here in this formula and after simplifying it I am getting this value as 0 point 6149. Once I will get this value that is the predicted value of Y3. No I need to correct this value.

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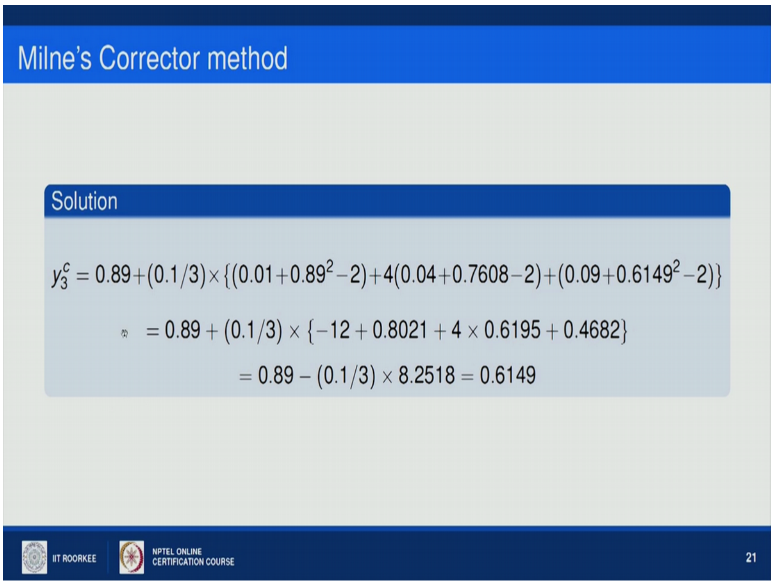

So for this I will use corrector formula which is given by this particular equation.

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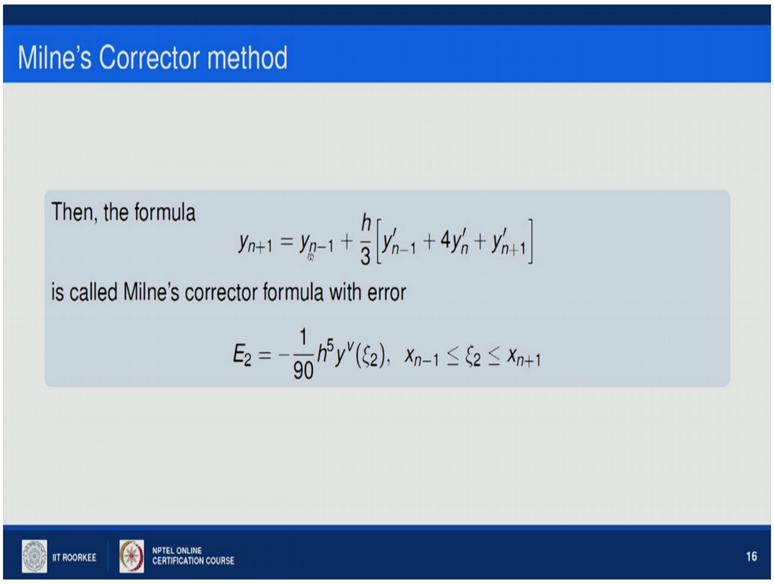

That is YN plus 1 equals to 2 YN minus 1 plus H by 3 Y prime N minus 1 plus 4 times Y prime N plus Y prime N plus 1.

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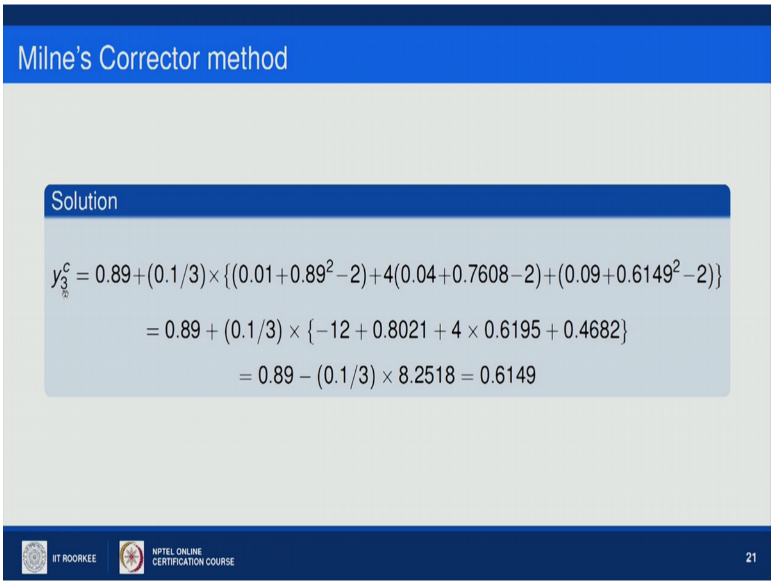

So after using this the corrected value of Y at X equals to 0 point 3 can be obtained as 0 point 6149. So in this way we can apply the Milne predictor corrector method for solving ordinary differential equations and here as I told you we should know the value of Y at more than 1 points in this Milne method we should know this value at least at 4 points.

For which either given to us or we need to calculate using Taylor series method or Euler's method or any other method. So this method is multistep method and it is more accurate compare to the single step method since the accuracy in this method of order H rest to power 5 in predictor as well as corrector formula. How and hence we can use larger step size for computation when compared to the Euler's method where for get getting a better accuracy we need to use smaller step size. So with this I will stop the discussion about this method.

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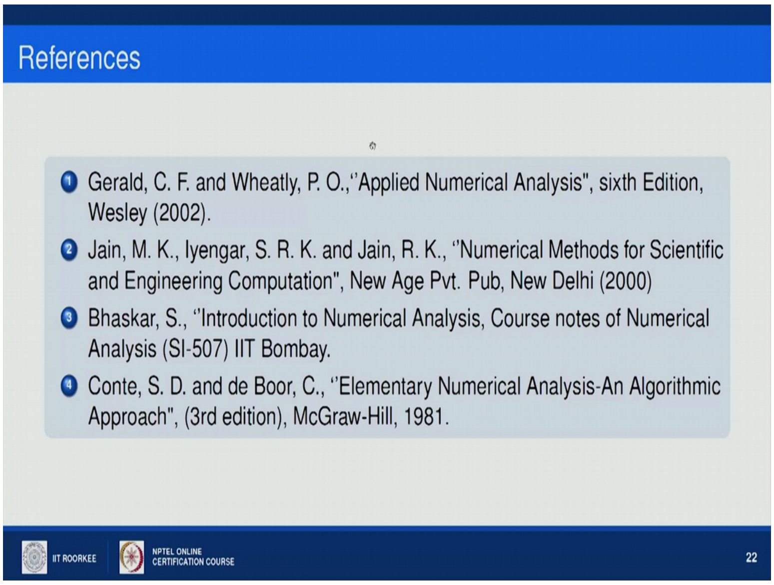

Now since it is the last lecture I would like to tell you about few references which I have used for making all these lectures. So the first one is the book Applied numerical analysis by Gerald and Wheatly and it is I have used the sixth edition of this book. The other book is numerical methods for scientific and engineer computation by Jain, Iyengar and Jain.

Moreover I have taken some of the notes of professor S. Bhaskar from IIT Bombay and notes of his course introduction to numerical analysis which is online at IIT Bombay website and the last reference which I have used the book elementary numerical analysis and algorithmic approach by Conte and de Boor that I have taken the third edition of this book which is published by mcgraw-Hill.

So these are the references which I have followed in this course. Apart from that I would like to acknowledge few people.

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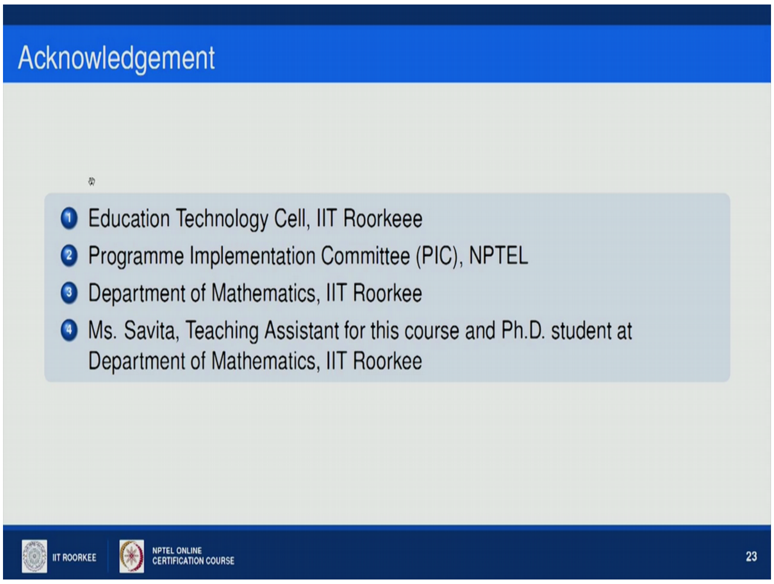

The first of all I would like to acknowledge education technology cell, IIT Roorkee specially professor B. K. Gandhi, the coordinator of ET cell at IIT Roorkee along with his team Dr Nivedita, Sharad, Mohan and other people who have along me during the shooting of this course. I am also thankful to program implementation committee NPTEL.

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