Numerical Methods

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Lecture 37

Numerical Methods for ODE-1

Hello everyone, so as I told you in the previous lecture that most of the differential equations those are associated with or coming from the real life phenomena or some from the some real life process do not possess analytical solution. So hence we require numerical methods to solve such differential equations. So numerical methods give an approximate solution that is it will not give a close form type of a solution that y is a function of some x, x, x, okay, no it will give you the value of y for the given x or where you want to obtain your solution.

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Suppose I am having an initial value problem y prime of f of x y and y 0 equals to x 0 so I am having like this y prime equals to f of x, y together with a condition that y at x not equals to y not. Suppose you want to calculate a solution at x equals to x 1 and which is your x not plus h, where h is a small number. Now so it means by a numerical method I will calculate the value of y at x equals to x 1 and that will be a number, okay.

Here we are having two types of numerical methods, one is called single step and the another one is multi step numerical methods. In single step numerical methods we use just previous the value for calculating the value at next step just the previous step only, okay. However in case of multi-step we use many previous step more than one previous steps. So for example suppose I want to calculate the value of y at x equals to x 1, so here I will use value of y of x not here.

So if I am using only value at x not than it is a single step method. However suppose I want to find out y at x equals to x n and I am using y at x n minus 1 then it is a multi-step method. In the next including this and next couple of lectures we will talk about we will do some single step methods and in the final lecture we will take a multi-step method. Now in this module we will take few numerical methods for example Euler's method, Runge-Kutta method and then a multi-step to solve ordinary differential equations.

So in this lecture I will talk about Euler's method but before going to Euler's method let me introduce a semi analytical method which is called Picard's. So what is Picard method?

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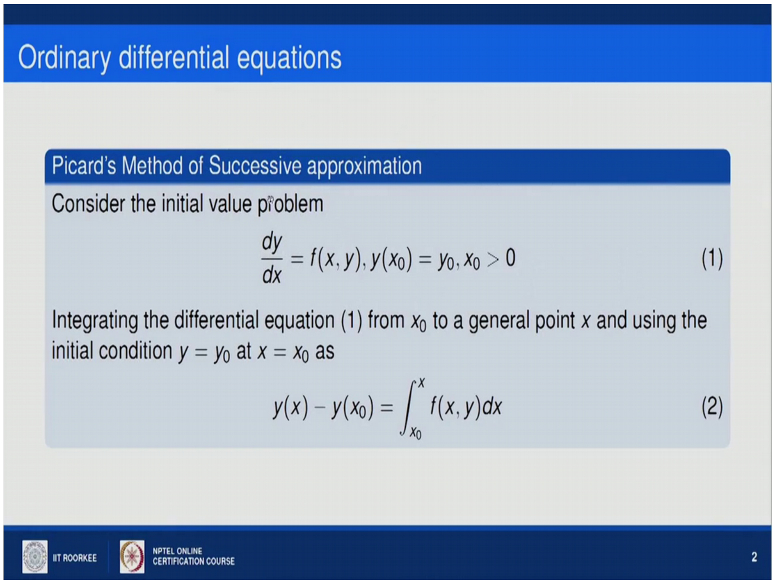

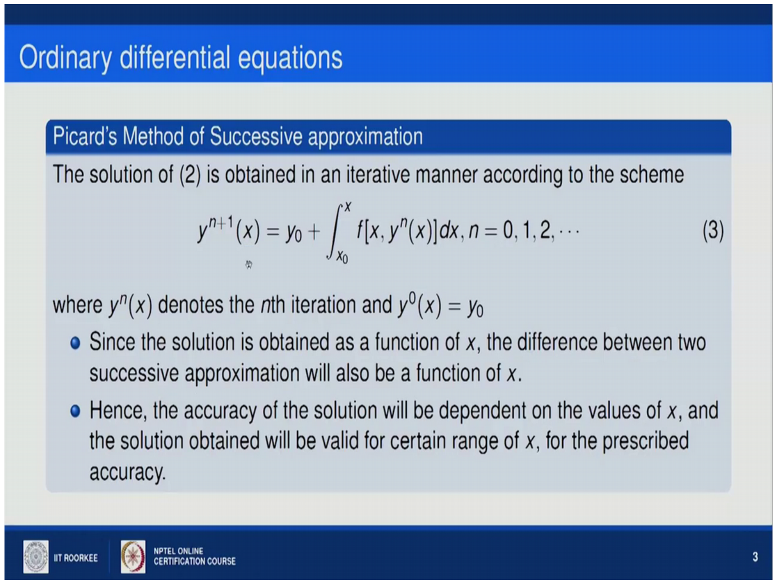

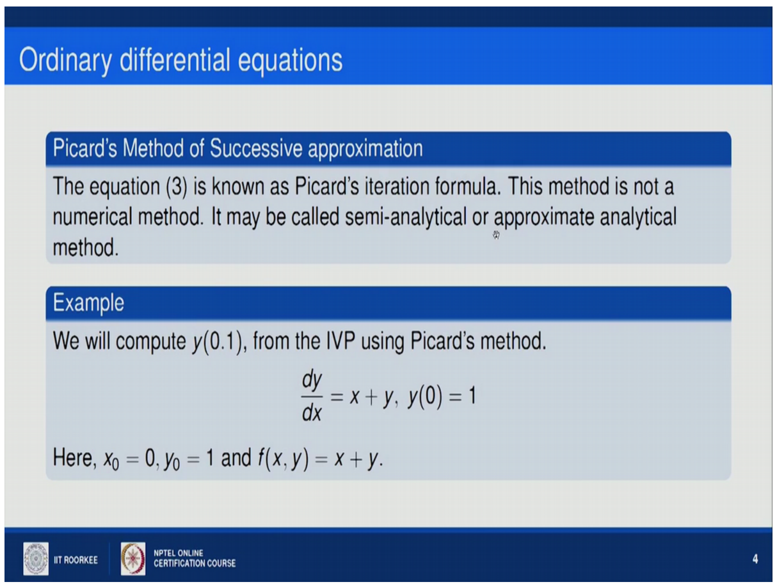

So Picard's method of successive approximation can be given like this consider the initial value problem dy over dx that is f of x, y and y at x 0 is y not and the domain of x is x greater than x not. So integrating the differential equation 1 from x 0 to a general point x I can I am using the initial condition y at x not equals to y not I can write y x minus y at x not equals to integral x not to x f of x, y dx.

So the solution of this particular equation is obtained in a an iterative manner according to the scheme that y at n plus 1 iteration or the n plus 1 approximation of y will be y not plus x not to x f of x and y, x in nth iteration into dx where n equals to 0, 1, 2. So for example you want to calculate the approximation of y in first iteration so it will become y 0 plus x not to x f x y 0 dx you want to get second iteration of this or second approximation then it will become y 0 plus x not 2 x f x y 1 dx and so on.

So since the solution is obtained as a function of x the difference between 2 successive approximation will also be a function of x because in each successive approximation you will get a function of x hence the accuracy of the solution will depend on the value of x and the solution obtain will be valid for certain range of x, okay if you want to obtain n accuracy up to let us say delta number delta order of the order of delta it will be valid for a particular domain of x it may happen that this particular solution is not valid for some other domain for up to given accuracy. So this particular equation is called Picard's iteration formula and as I told you it is not a numerical method it is something like between numerical and analytical methods so I can say it semi analytic or approximate analytical method.

So let us consider this particular problem and we will apply the Picard's method on to this example.

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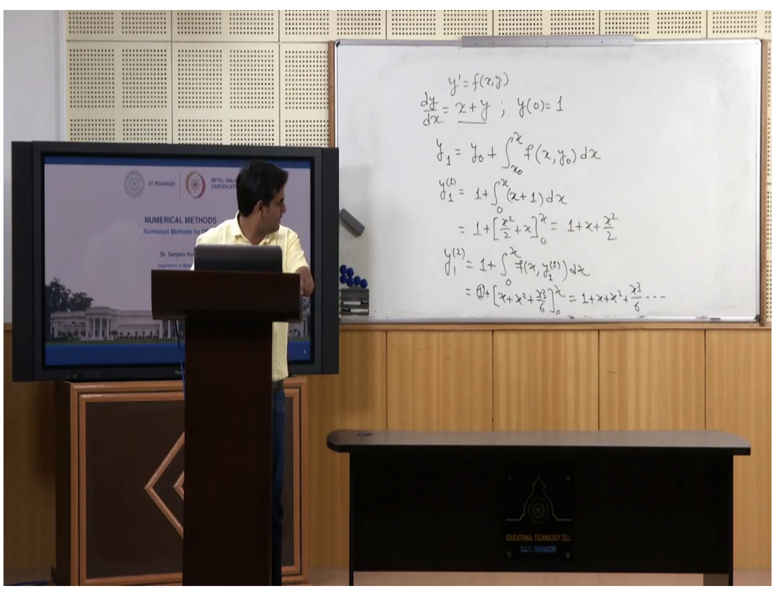

So I am having example as dy over dx equals to x plus y so it is a quite popular example and you will find this example in many books so here y 0 equals to 1 that is initially x is 0 and the value initial value of y at x equals to 0 or value of y at x equals to 0 is 1. Now y 1 is given at y 0 plus x 0 to x f of so here f of x, y is x plus y this particular number because my original problem is y prime equals to f of x, y.

So y 1 equals to y 0, y 0 is 1 plus x 0 is 0, 0 to x f of x, y so initially so x plus initially y at y equals to 0 is 1 so it is 1 dx and let me write it 1. So it is 1 plus x square will become x square plus 2 into x 0 to x and it is coming 1 plus x plus x square upon 2. Now the next iterate of y is given as again 1 plus 0 to x f of x dx so this will be 1 plus now it will become x plus 1 plus x plus x square by 2, so 1 will become x plus 2x so x square plus so it will be when I am putting x this and when I am putting 0 so it will become 2 plus x plus x square plus x 3 by 6, so we will move in this way, sorry, it will be 1 because 1 is outside of this.

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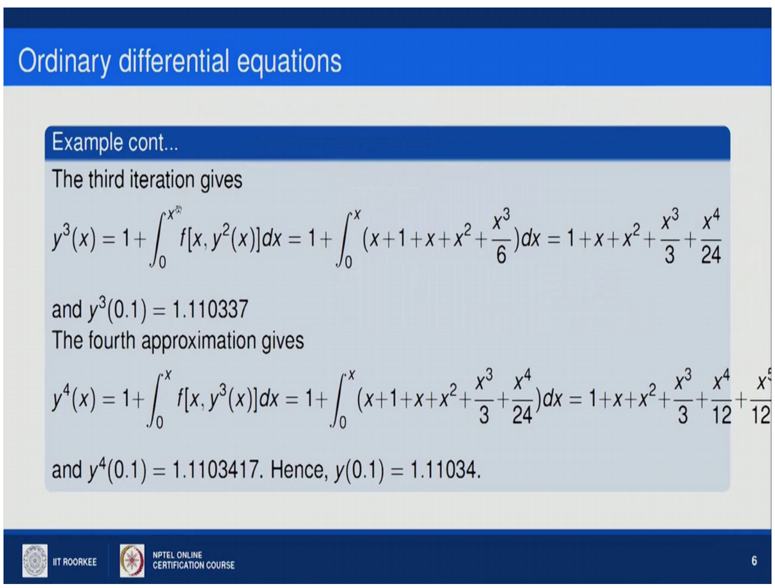

So if I put the value 0.1 for x y 1 is coming 1.105 while y 2 is coming 1.110167 then I calculate y 3 so y 3 will again 1 plus 0 to x f x y 2x dx so after putting all this I am getting 1 plus x plus x square plus x cube upon 3 plus x raise to power upon 24. So when I put x equals to 0.1 I am getting y 3 as that is the third approximation of y at x equals to 0.1, 1.110337 while it was in the second approximation second iteration it was 1.110167. So it is correct up to third decimal third place after decimal, if I take one more iteration or one more approximation of y then I am getting this particular expression for y at fourth iteration 1 plus x plus x square and so on.

And at x equals to 0.1 it is coming as 1.1103417, so up to 5 decimal places it is correct and it is given as 1.11034. So this was a semi analytical method but now we will move to pure numerical methods and in the category of pure numerical methods the very first method we will take a very simple method that is called Euler’s method.

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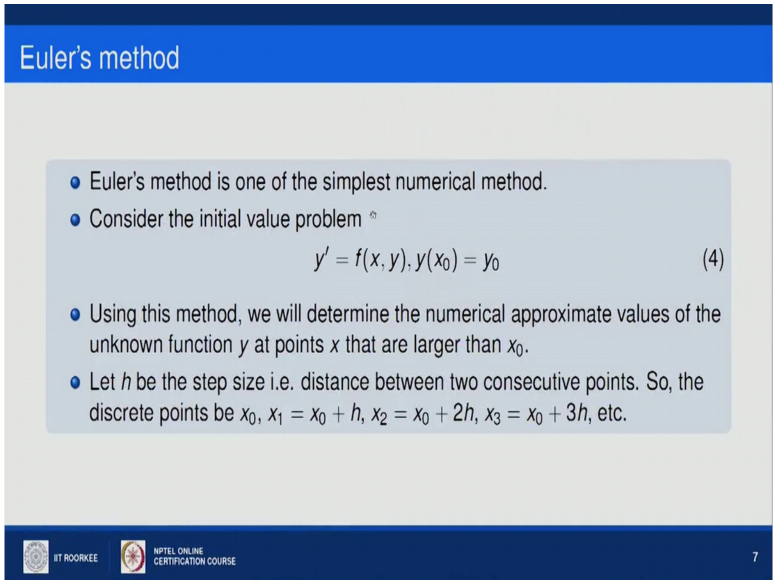

So consider again the same initial value problem y prime equals to f of x, y y, x not equals to y not so using Euler's method we will determine the numerical approximate value of the unknown function y at point x those are larger than x not means after the initial point.

So if h be the step size distance between two consecutive points that is h and R points are uniform we are assuming here. So my x 1 will become x not plus h, x 2 will become x 1 plus h that is x not plus 2 times h and so on.

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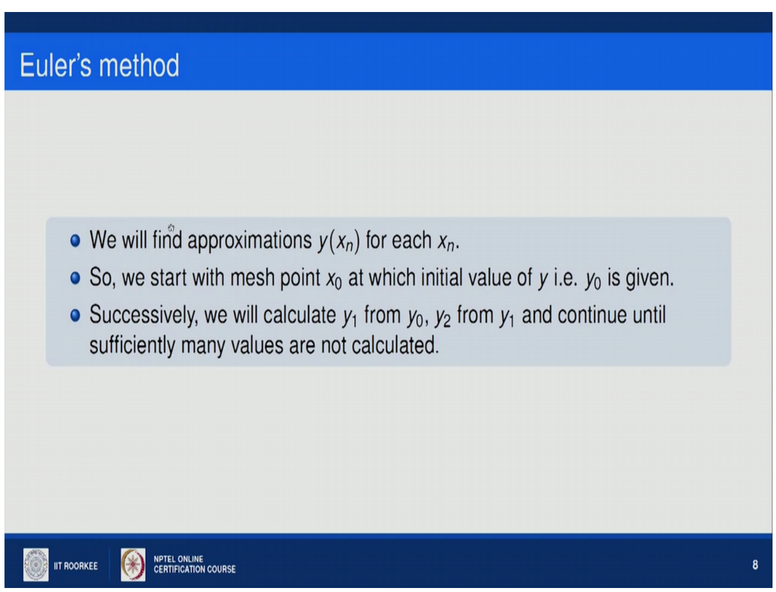

So by Euler's method we will find approximations of y at all those x i like x not is given so x 1, x 2, x n like that. So in that way we can find the value of y in a given interval at some uniform points. So we start with x not and then we will calculate y 1 then from using y 1 we will calculate y 2 using y 2 we will calculate y 3 and so on. So the initial value problem that is a prime y prime is equals to f of x, y y at x not equals to y not tells us that f of x, y is basically nothing but the slope of tangent line to the solution curve at point x, y because it is dy over dx equals to f of x, y.

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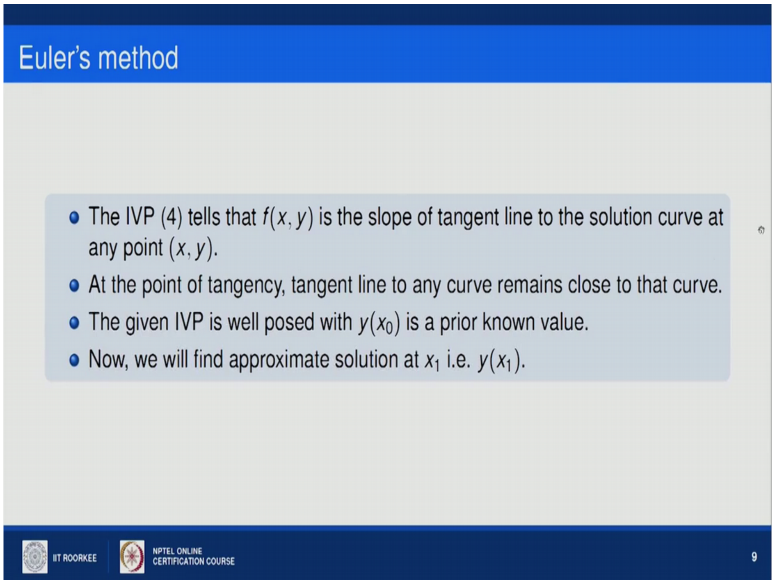

So at the point of tangency tangent like to any curve remain close to the at curve. So the given initial value problem is well posed with initial value of y at with at x not and which is known to us, so now we will find approximate x 1 at y approximation of y at x equals to x 1.

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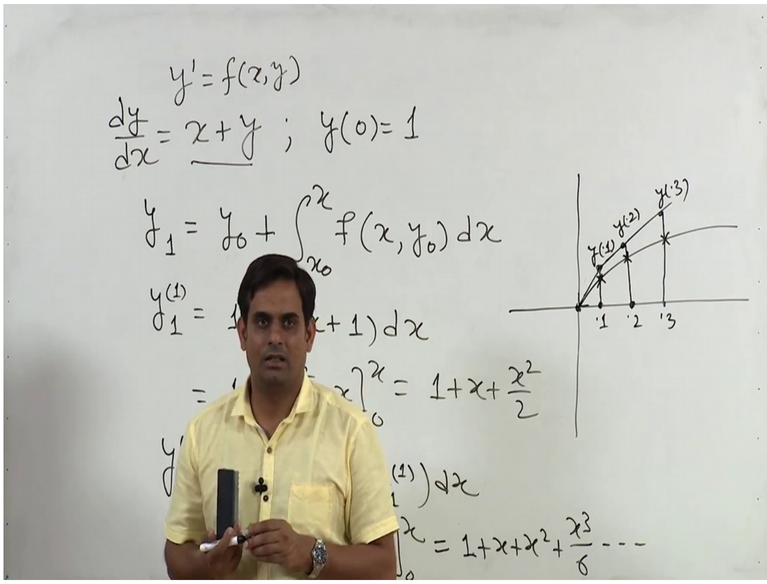

So the equation of basically what we are having suppose and this is the solution curve so let us say the value at x not equals to 0 it is given I want to find out it at point 1, at point 2 so at this point I will let me make it more smooth.

So this is the tangent line at x not y not. So at this point this will be my numerical solution y at 0.1 where the exact solution is this one. Now again at this point I will try to approximate the value of y at x equals to 0.2 so it will go like this and so while the exact values are these values.

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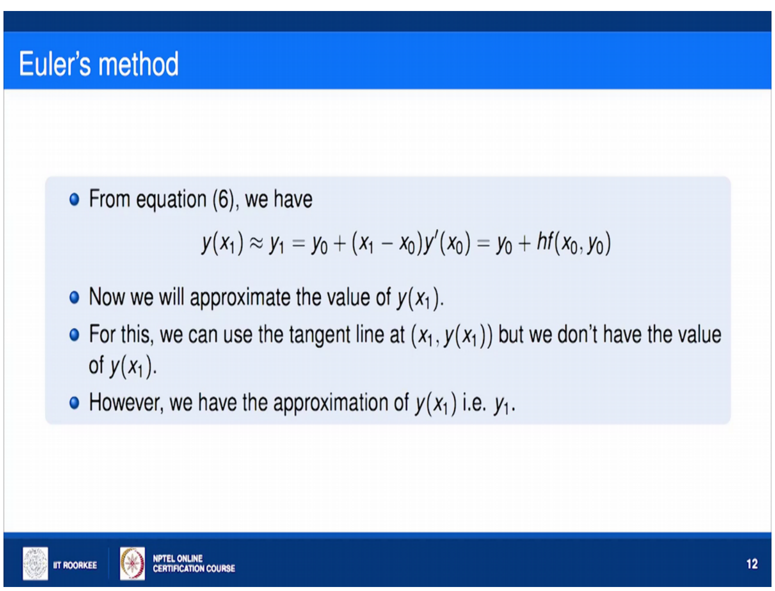

So this is the idea of Euler's method, so mathematically we can write that the tangent line at x equals to x not is given as y equals to y not plus x minus x not y prime x not thus approximation to the value of the solution at x equals to x 1 is the y coordinate that is x equals to x 1 on the tangent line.

So y at x 1 is given that is approximated value of y 1 is given by y 0 plus x 1 minus x not y dash x not and as I told you this particular approximation is valid only when x 1 is very close to x not means I am having a very small h very small step size.

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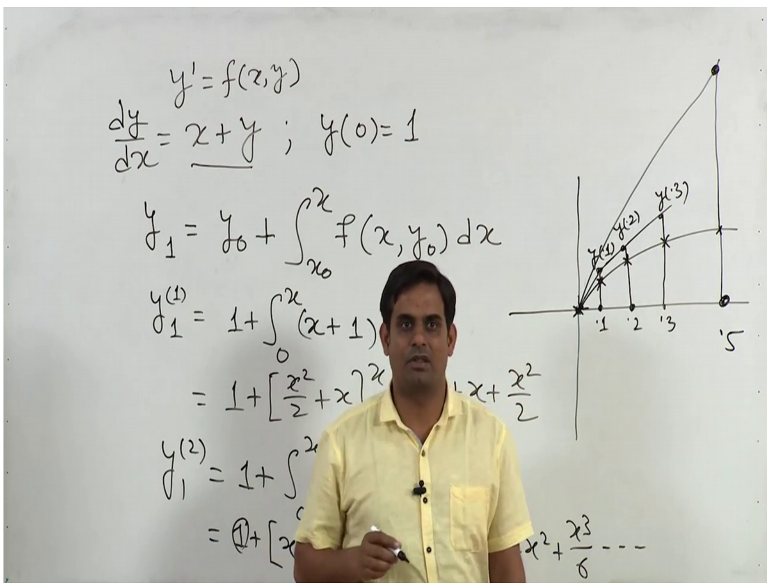

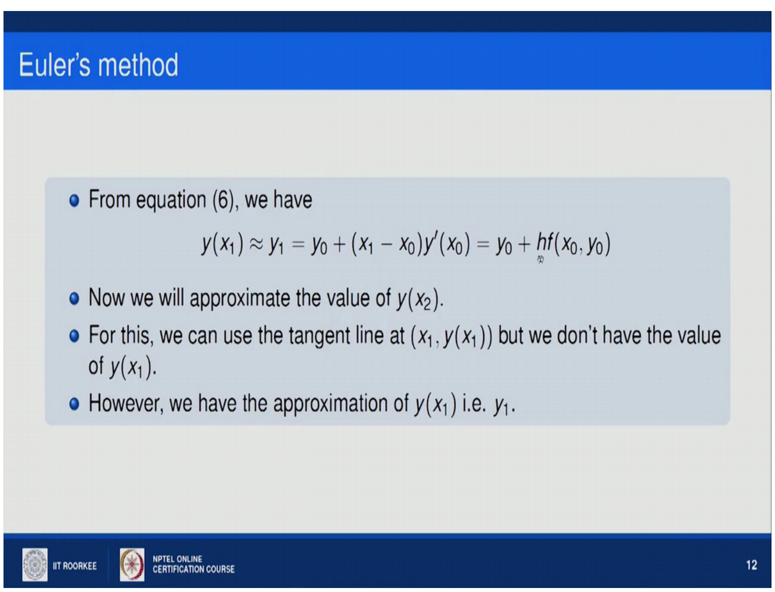

If I take a large step size let us say I am having value at 0 and I want to find out at let us say at 0.5 so I will take a tangent here and tangent line will go there so my original solution is this one but I will get this one as the approximation, so I will get more error if the step size is large.

So once we calculate y 1 that is basically y not plus h times f of x not y not from the tangent line we will calculate y 2. So y 2 will become basically y 1 plus h into f x 1, y 1, similarly y 3 will become y 2 plus h times f x 2, y 2 so I will get a sequence of iterations.

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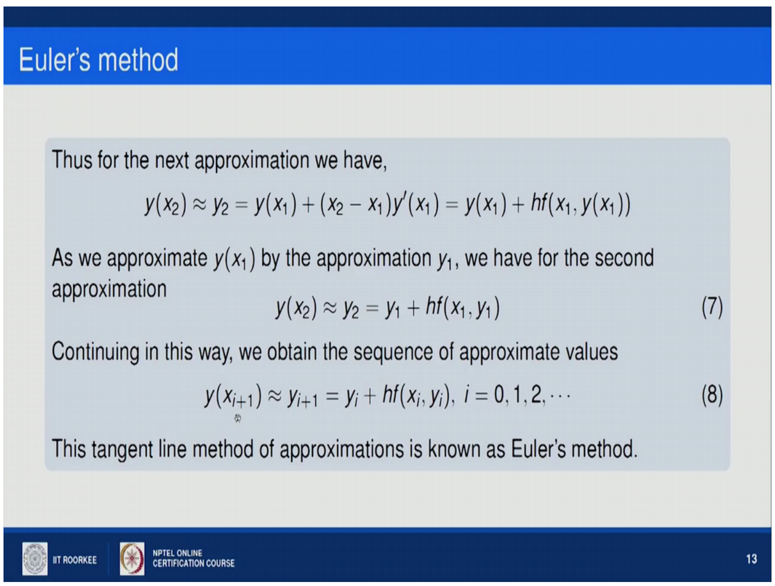

In general y at any point x i plus 1 is given as y i plus h times f x i, y i. Now what is the error in this approximation because from this figure itself you can see this is the error in my this approximation if I take the largest this is the error this is the error at second point that is when x is 0.2 and so on.

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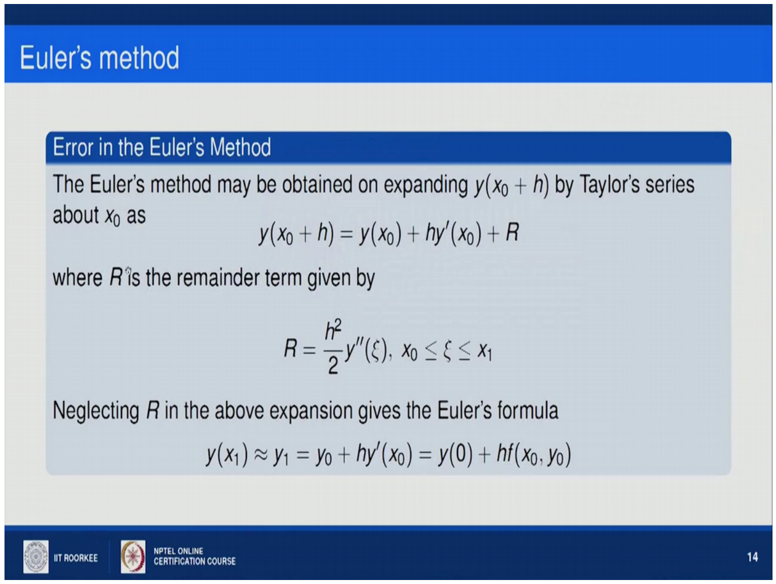

So, how to calculate a bound on this error? So for this we will use Tailor series method so the value of y at x not plus h, so x not plus h basically x 1 so the value of y at x 1 can be obtained by the Tailor series approximation of y about x 0 so y f 0 plus h is y x 0 plus h y prime x 0 plus second order and higher order terms. So let me because if you see this particular thing first three terms the left hand side term and two right hand side terms of this expression this equation it is my y 1 and it is y 0 plus h f of x y that is the this the Euler's formula for approximating y 1 from the x 0, y 0.

So it means this is the error term R. So here R is the remainder term and it is given h square by 2 y double prime Xi where Xi is somewhere between x not and x 1.

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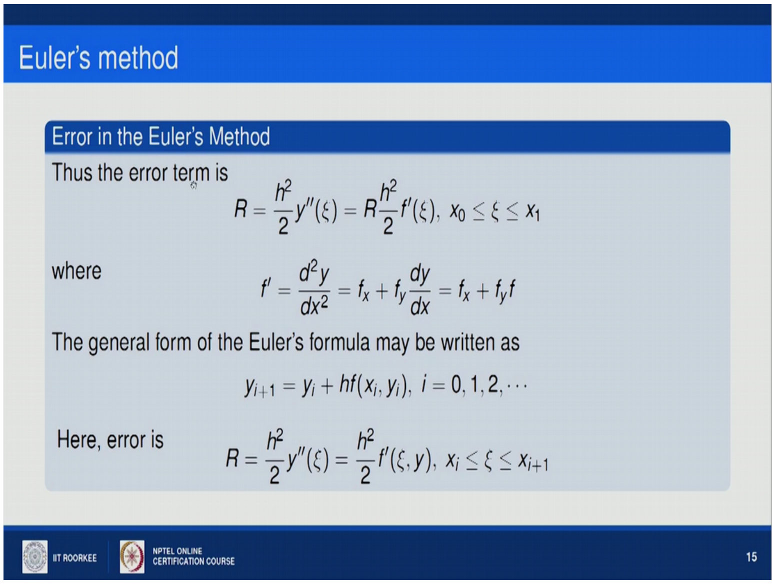

So the error term is R it is h square by 2 y double prime Xi that is R times h square by 2 because we know that dy by dx equals to f so d2y over dx square means y double prime will be f dash so I can write it as f dash Xi. So, how to calculate f dash? Because f dash f is a function of x and y and y is a function of again x, so f dash will become f of x plus f, y into dy upon dx so I am writing and again dy by dx will be f.

So f prime will be f x plus f y into dy over dx so f x plus f y into f so the error term as I told you will be h square upon 2 f prime Xi y, finally this particular thing.

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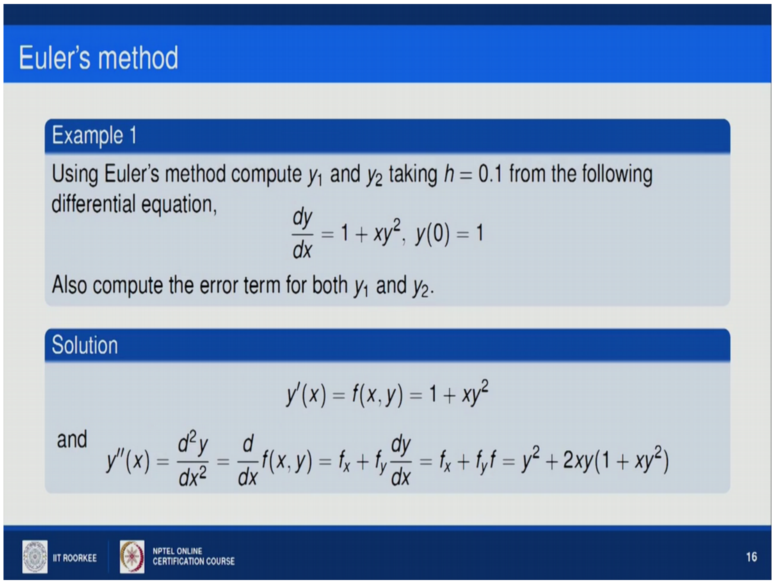

So let us take an example and we will solve this example using the Euler's method. So here we are having our initial value problem as dy over dx equals to 1 plus x into y square, so here my function small f of x y is 1 plus x, y square. The given initial condition is y at x equals to 0 is 1 that is why 0 is 1, now I need to take a step size h equals to 0.1 and I need to calculate the values of y 1 and y 2.

The value of y 1 is the value of y at x equals to 0.1 and the value of y 2 is the value of y at x equals to 0.2. So if we compare this given problem with the standard form of initial value problem then my f of x, y is 1 plus x, y square I need to calculate y double prime x that will be d2y over dx square that is d by dx of f of x, y and it becomes del f over del x plus del f over del y into dy over dx del f over del x is given as y square and del f over del y is 2x, y into y dash and y dash again 1 plus x, y square.

So y double dash x is y square plus 2 x, y into 1 plus x, y square. Why I am calculating this? Because in the question they are asking to compute error terms also and for that I need y double prime x.

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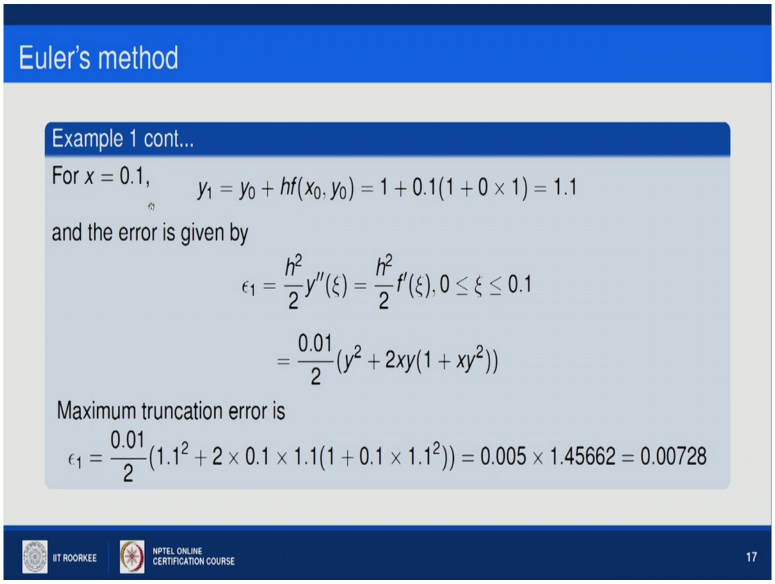

Now, let us take the first step and calculate the value of y 1, so here x is 0.1 so y 1 is y 0 plus h times f of x 0, y 0 by the Euler's formula y 0 is given as 1 plus h is 0.1 and f of x, y is 1 plus x, y square so it will become 1 plus x not into y not square x not is 0 so this term become 0 so 1 plus 0.1 it is 1.1.

Now if I compute the error epsilon 1 is given as h square upon 2 into y double prime Xi where Xi is somewhere between 0 to 0.1. So y double prime Xi is f prime Xi and I have already calculated this so this is 0.01 upon 2 into this is the expression for y double prime. So maximum truncation error is 0.01 upon 2 into 1.1 whole square because this is the value of y 1 plus 2 times 0.1 into 1.1 into 1 plus 0.1 into 1.1 square.

So after simplifying this we are getting the maximum truncation error in y 1 that is given by epsilon 1 and the value numerical value of this is 0.00728.

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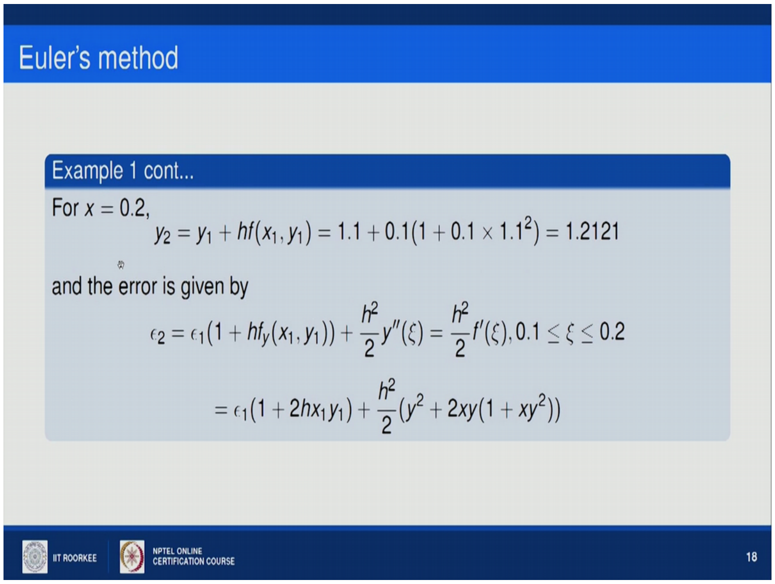

Now I will calculate y at x equals to 0.2 that is my y 2 so y 2 is y 1 plus h time f of x 1, y 1 that is 1.1 plus 0.1 into 1 plus 0.1 into 1.1 whole square, so please note that for calculating y 2 I am taking the values from the previous iteration or previous step so my x 1 is now 0.1 and y 1 is 1.1 which we have calculated from x 0, y 0.

So after simplifying this I am getting this value as 1.2121. So my y at x equals to 0.2 is 1.2121, the truncation error in this step is given by epsilon 2 that is epsilon 1 times 1 plus h f y x 1, y 1 plus h square upon 2 into y double prime Xi, where Xi is now between 0.1 to 0.2.

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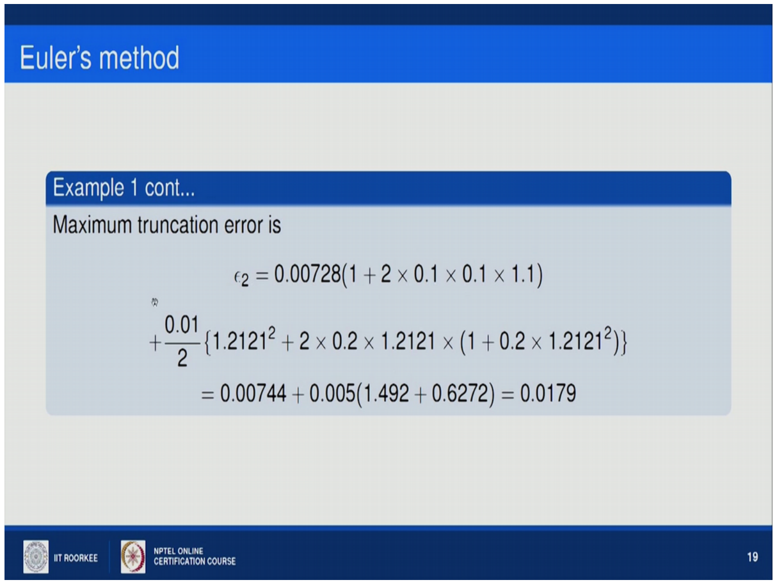

So I have substituted all these values here and then this term so maximum truncation error will be given by this particular expression plus this is for the second term of the epsilon 2 and after simplifying it I am getting this value as 0.0179.

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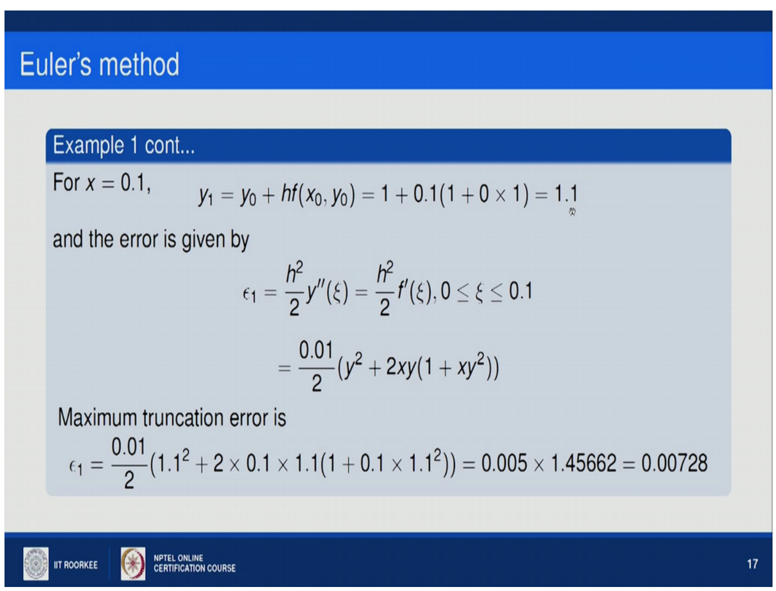

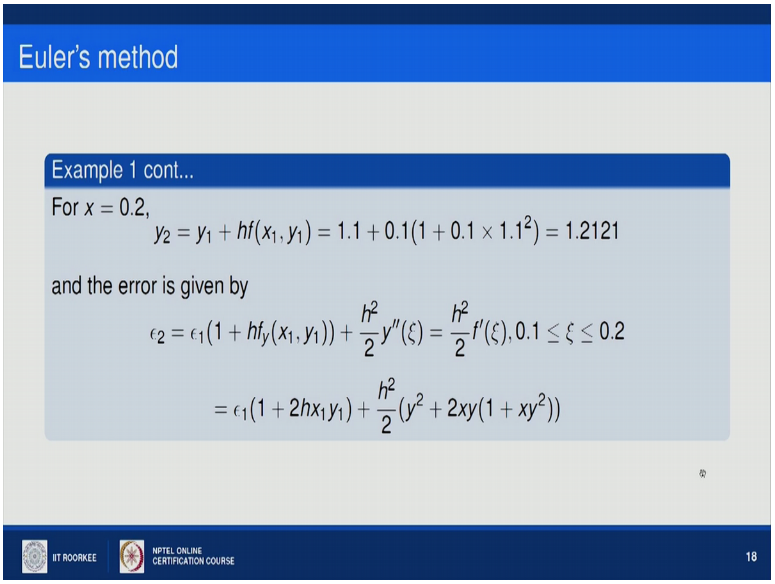

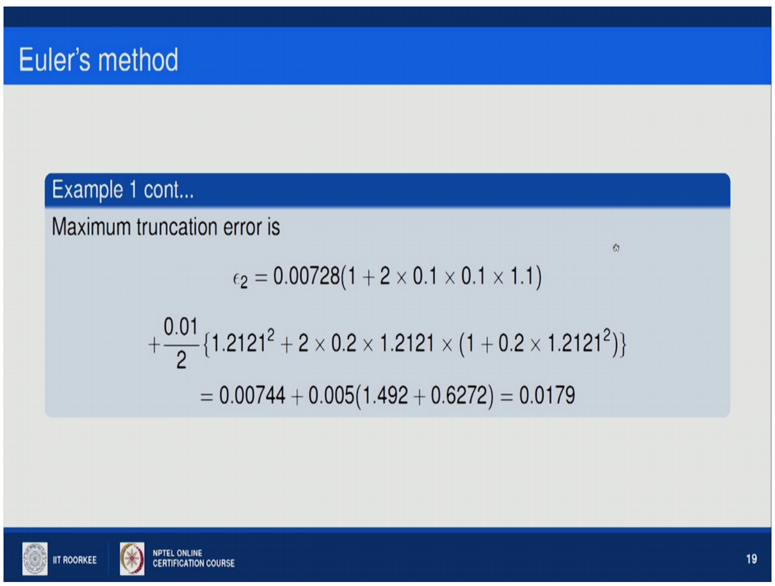

So here my y 1 is 1.1 the maximum truncation error in y 1 is 0.00728 and then y 2 is coming out as 1.2121 and the maximum truncation error for this x 2 is 0.0179.

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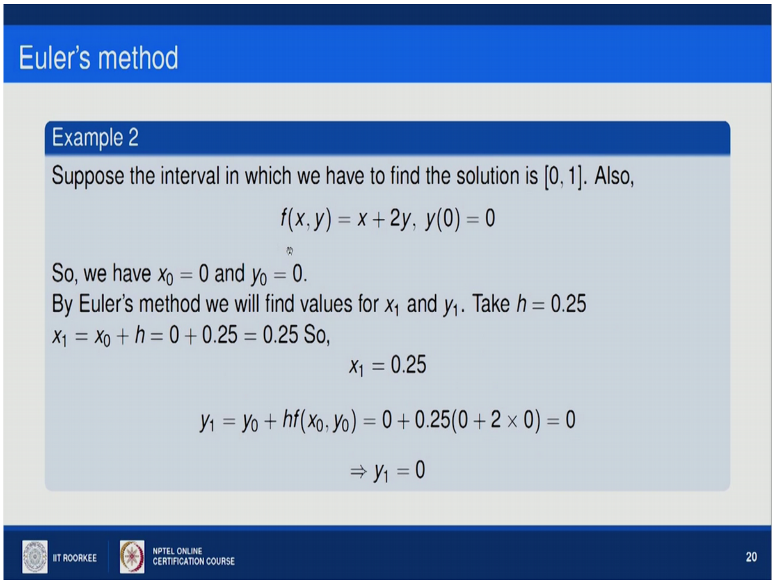

Now if we take another example that is if f is x plus 2 y the problem is same y prime equals to f of x, y and y 0 is 0 here so if I apply Euler's method with step size 0.25 suppose I want to calculate value at 4 points between 0 to 1 that is at 0.25 at 0.5 at 0.75 and then finally at 1.

So value at x equals to 0.25 that is y 1 is given by this one and it is coming out to be 0.

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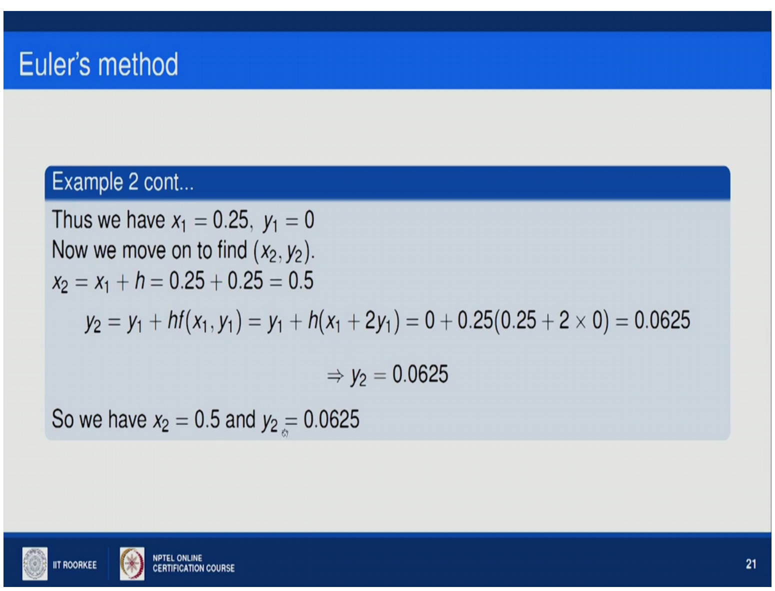

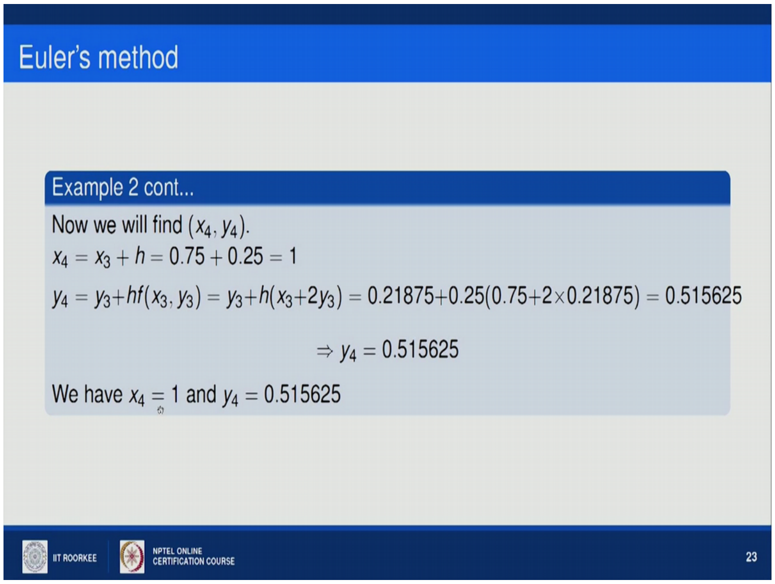

At y x equals to 0.5 y 21is coming using the Euler's process as 0.0625. Similarly when x 3 is 0.75 y 3 is coming at 0.21875 and finally at x 4 equals to 1 x equals to 1 y 4 that is we are y is coming out 0.515625.

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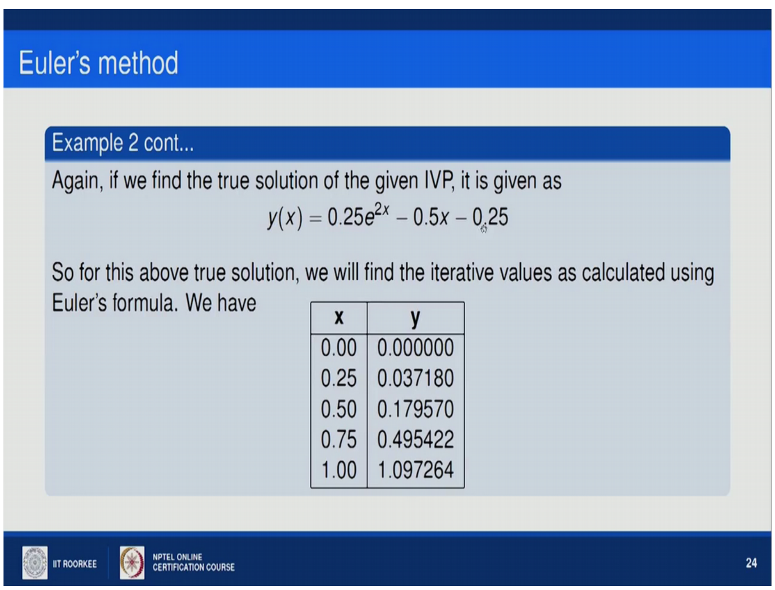

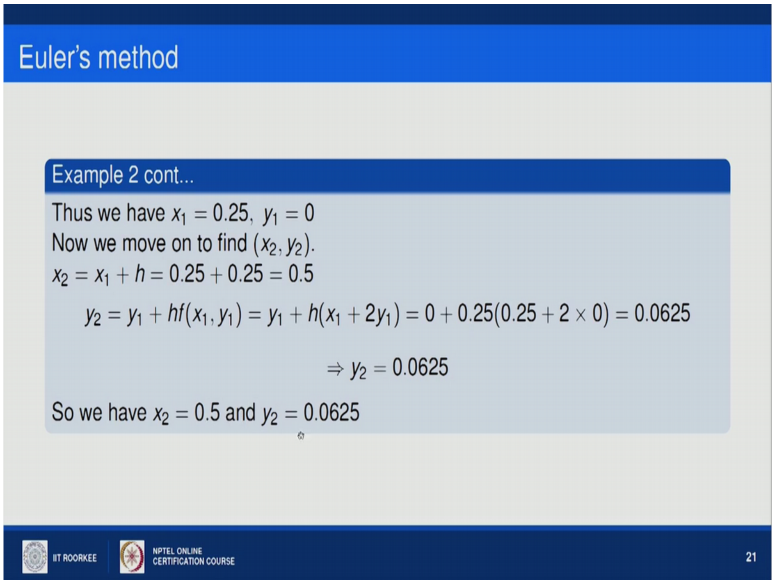

So the true solution of the initial value problem is given by this particular expression and if I calculate the values at 0.25 by the Euler's method we are getting 0 but from the exact solution it is 0.037180.

By that numeric Euler's method at 0.5 I am getting y as 0.0625 while the exact value at 0.5 is 0.179570 and similarly we are having big errors at x equals to 0.75 and x equals to 1 also. So it means the Euler's is having large error when compare to the exact solution, and what is the reason behind this? The reason is step size because here we are taking step size as 0.25 if you decrease the step size error will decrease and it happens always in Euler's method however you have to do more calculations.

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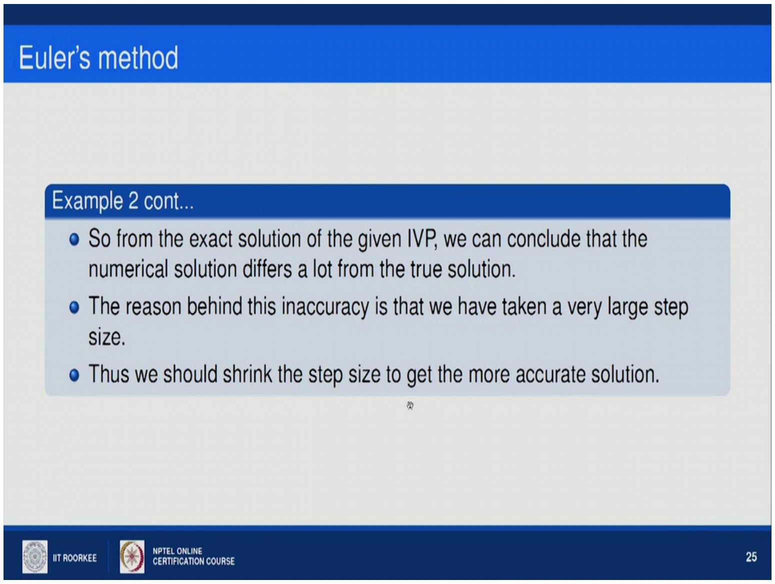

So if we shrink the step size we will get more accurate solution by using the Euler's method. So with this I will stop this lecture I will end this lecture so today we have learnt Picard's method and Euler's method, in the next lecture we will talk more accurate method when compare to the Euler's means one more (())(28:00) on the Euler's method and then Tailor series method means we will consider more means like in Euler's method we are taking only term up to first order in Tailor series approximation of y at about x 0, however there we will take more like second order term also so that error will reduce. So we will talk about those methods in next lecture, thank you very much.