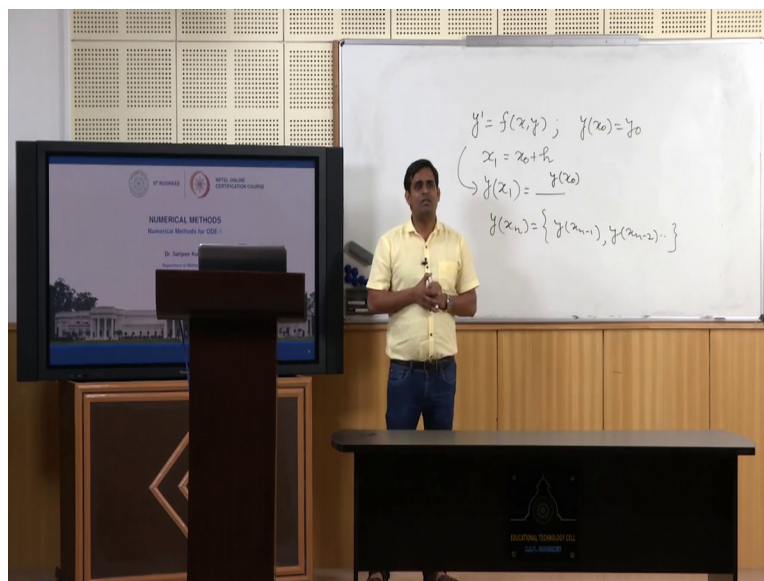


Numerical Methods
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Lecture 37
Numerical Methods for ODE-1

Hello everyone, so as I told you in the previous lecture that most of the differential equations those are associated with or coming from the real life phenomena or some from the some real life process do not possess analytical solution. So hence we require numerical methods to solve such differential equations. So numerical methods give an approximate solution that is it will not give a close form type of a solution that y is a function of some x , x , x , okay, no it will give you the value of y for the given x or where you want to obtain your solution.

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Suppose I am having an initial value problem y' of f of x y and $y(0)$ equals to y_0 so I am having like this y' equals to f of x , y together with a condition that y at x_0 equals to y_0 . Suppose you want to calculate a solution at x equals to x_1 and which is your x_0 plus h , where h is a small number. Now so it means by a numerical method I will calculate the value of y at x equals to x_1 and that will be a number, okay.

Here we are having two types of numerical methods, one is called single step and the another one is multi step numerical methods. In single step numerical methods we use just previous the value for calculating the value at next step just the previous step only, okay. However in case of multi-step we use many previous step more than one previous steps. So for example suppose I want to calculate the value of y at x equals to x 1, so here I will use value of y of x not here.

So if I am using only value at x not than it is a single step method. However suppose I want to find out y at x equals to x n and I am using y at x n minus 1 then it is a multi-step method. In the next including this and next couple of lectures we will talk about we will do some single step methods and in the final lecture we will take a multi-step method. Now in this module we will take few numerical methods for example Euler's method, Runge-Kutta method and then a multi-step to solve ordinary differential equations.

So in this lecture I will talk about Euler's method but before going to Euler's method let me introduce a semi analytical method which is called Picard's. So what is Picard method?

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Ordinary differential equations

Picard's Method of Successive approximation

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0, x_0 > 0 \quad (1)$$

Integrating the differential equation (1) from x_0 to a general point x and using the initial condition $y = y_0$ at $x = x_0$ as

$$y(x) - y(x_0) = \int_{x_0}^x f(x, y) dx \quad (2)$$

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Ordinary differential equations

Picard's Method of Successive approximation

The solution of (2) is obtained in an iterative manner according to the scheme

$$y^{n+1}(x) = y_0 + \int_{x_0}^x f[x, y^n(x)] dx, n = 0, 1, 2, \dots \quad (3)$$

where $y^n(x)$ denotes the n th iteration and $y^0(x) = y_0$

- Since the solution is obtained as a function of x , the difference between two successive approximation will also be a function of x .
- Hence, the accuracy of the solution will be dependent on the values of x , and the solution obtained will be valid for certain range of x , for the prescribed accuracy.



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Ordinary differential equations

Picard's Method of Successive approximation

The equation (3) is known as Picard's iteration formula. This method is not a numerical method. It may be called semi-analytical or approximate analytical method.

Example

We will compute $y(0.1)$, from the IVP using Picard's method.

$$\frac{dy}{dx} = x + y, y(0) = 1$$

Here, $x_0 = 0$, $y_0 = 1$ and $f(x, y) = x + y$.



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So Picard's method of successive approximation can be given like this consider the initial value problem dy over dx that is f of x, y and y at x_0 is y_0 and the domain of x is $x \geq x_0$. So integrating the differential equation 1 from x_0 to a general point x I can use the initial condition y at x_0 equals to y_0 . I can write $y(x) - y_0 = \int_{x_0}^x f(x, y) dx$.

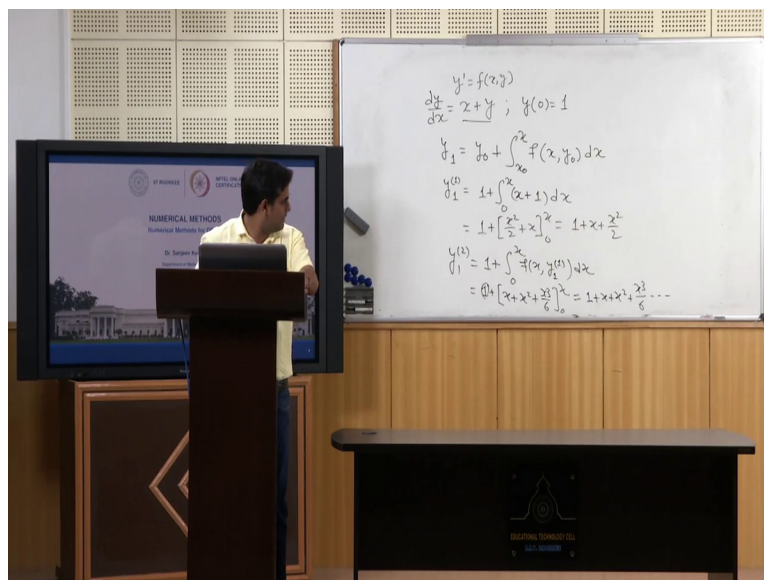
So the solution of this particular equation is obtained in an iterative manner according to the scheme that y at $n+1$ iteration or the $n+1$ approximation of y will be $y_0 + \int_{x_0}^x f(x, y^n) dx$ where n equals to $0, 1, 2$. So for example you want to calculate the approximation of y in first iteration so it will become $y_0 + \int_{x_0}^x f(x, y_0) dx$.

you want to get second iteration of this or second approximation then it will become y_0 plus x not $2 \times f(x, y_1) dx$ and so on.

So since the solution is obtained as a function of x the difference between 2 successive approximation will also be a function of x because in each successive approximation you will get a function of x hence the accuracy of the solution will depend on the value of x and the solution obtained will be valid for certain range of x , okay if you want to obtain n accuracy up to let us say δ number δ order of the order of δ it will be valid for a particular domain of x it may happen that this particular solution is not valid for some other domain for up to given accuracy. So this particular equation is called Picard's iteration formula and as I told you it is not a numerical method it is something like between numerical and analytical methods so I can say it semi analytic or approximate analytical method.

So let us consider this particular problem and we will apply the Picard's method on to this example.

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So I am having example as dy/dx equals to x plus y so it is a quite popular example and you will find this example in many books so here y_0 equals to 1 that is initially x is 0 and the value initial value of y at x equals to 0 or value of y at x equals to 0 is 1. Now y_1 is given at y_0 plus x 0 to x f of so here f of x, y is x plus y this particular number because my original problem is y' equals to f of x, y .

So y_1 equals to y_0 , y_0 is 1 plus x_0 is 0, 0 to x f of x, y so initially so x plus initially y at y equals to 0 is 1 so it is 1 dx and let me write it 1. So it is 1 plus x square will become x square plus 2 into x_0 to x and it is coming 1 plus x plus x square upon 2. Now the next iterate of y is given as again 1 plus 0 to x f of x, y dx so this will be 1 plus now it will become x plus 1 plus x plus x square by 2, so 1 will become x plus $2x$ so x square plus so it will be when I am putting x this and when I am putting 0 so it will become 2 plus x plus x square plus x^3 by 6, so we will move in this way, sorry, it will be 1 because 1 is outside of this.

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Ordinary differential equations

Example cont...

The third iteration gives

$$y^3(x) = 1 + \int_0^x f[x, y^2(x)] dx = 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{6}\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

and $y^3(0.1) = 1.110337$

The fourth approximation gives

$$y^4(x) = 1 + \int_0^x f[x, y^3(x)] dx = 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$$

and $y^4(0.1) = 1.1103417$. Hence, $y(0.1) = 1.11034$.

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So if I put the value 0.1 for x y_1 is coming 1.105 while y_2 is coming 1.110167 then I calculate y_3 so y_3 will again 1 plus 0 to x f of x, y_2 dx so after putting all this I am getting 1 plus x plus x square plus x cube upon 3 plus x raise to power upon 24. So when I put x equals to 0.1 I am getting y_3 as that is the third approximation of y at x equals to 0.1, 1.110337 while it was in the second approximation second iteration it was 1.110167. So it is correct up to third decimal third place after decimal, if I take one more iteration or one more approximation of y then I am getting this particular expression for y at fourth iteration 1 plus x plus x square and so on.

And at x equals to 0.1 it is coming as 1.1103417, so up to 5 decimal places it is correct and it is given as 1.11034. So this was a semi analytical method but now we will move to pure numerical methods and in the category of pure numerical methods the very first method we will take a very simple method that is called Euler's method.

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Euler's method

- Euler's method is one of the simplest numerical method.
- Consider the initial value problem $y' = f(x, y), y(x_0) = y_0$ (4)
- Using this method, we will determine the numerical approximate values of the unknown function y at points x that are larger than x_0 .
- Let h be the step size i.e. distance between two consecutive points. So, the discrete points be $x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, x_3 = x_0 + 3h$, etc.

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So consider again the same initial value problem y' equals to f of x, y , y, x not equals to y not so using Euler's method we will determine the numerical approximate value of the unknown function y at point x those are larger than x not means after the initial point.

So if h be the step size distance between two consecutive points that is h and R points are uniform we are assuming here. So my x_1 will become x not plus h , x_2 will become x_1 plus h that is x not plus 2 times h and so on.

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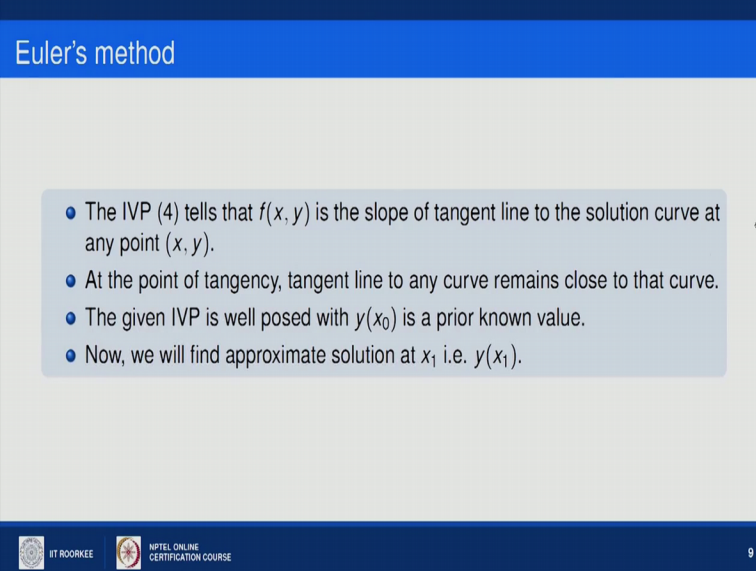
Euler's method

- We will find approximations $y(x_n)$ for each x_n .
- So, we start with mesh point x_0 at which initial value of y i.e. y_0 is given.
- Successively, we will calculate y_1 from y_0 , y_2 from y_1 and continue until sufficiently many values are not calculated.

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So by Euler's method we will find approximations of y at all those x_i like x_0 is given so x_1, x_2, x_n like that. So in that way we can find the value of y in a given interval at some uniform points. So we start with x_0 and then we will calculate y_1 then from using y_1 we will calculate y_2 using y_2 we will calculate y_3 and so on. So the initial value problem that is a prime y' is equal to $f(x, y)$ at x_0 equals to y_0 tells us that $f(x, y)$ is basically nothing but the slope of tangent line to the solution curve at point x, y because it is dy/dx equals to $f(x, y)$.

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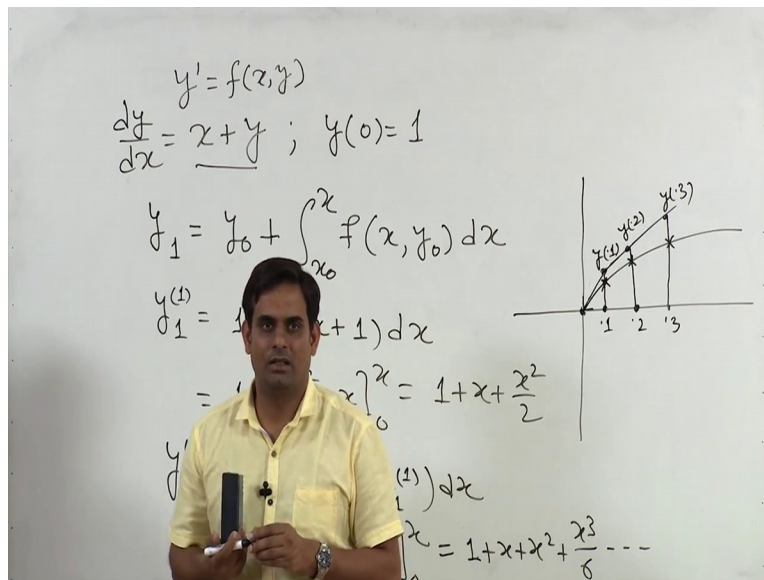
Euler's method

- The IVP (4) tells that $f(x, y)$ is the slope of tangent line to the solution curve at any point (x, y) .
- At the point of tangency, tangent line to any curve remains close to that curve.
- The given IVP is well posed with $y(x_0)$ is a prior known value.
- Now, we will find approximate solution at x_1 i.e. $y(x_1)$.

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So at the point of tangency tangent line to any curve remain close to the curve. So the given initial value problem is well posed with initial value of y at x_0 and which is known to us, so now we will find approximate x_1 at y approximation of y at x equals to x_1 .

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So the equation of basically what we are having suppose and this is the solution curve so let us say the value at x not equals to 0 it is given I want to find out it at point 1, at point 2 so at this point I will let me make it more smooth.

So this is the tangent line at x not y not. So at this point this will be my numerical solution y at 0.1 where the exact solution is this one. Now again at this point I will try to approximate the value of y at x equals to 0.2 so it will go like this and so while the exact values are these values.

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Euler's method

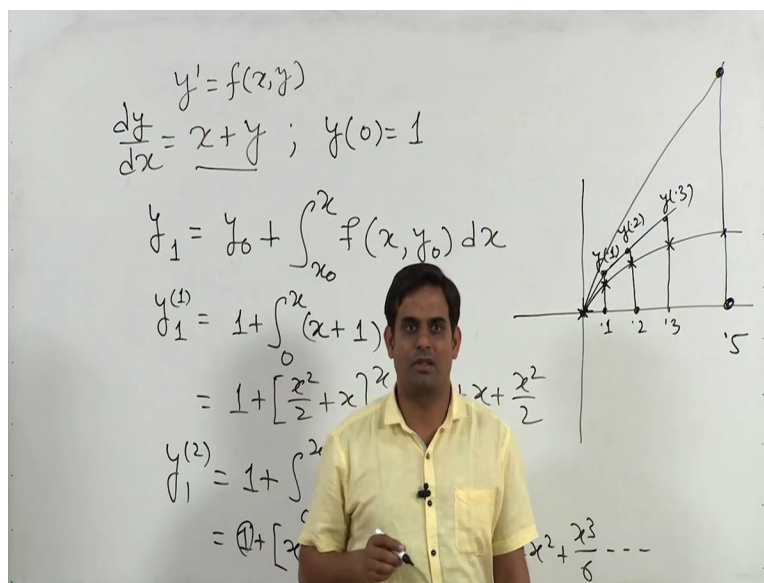
- From equation (6), we have
$$y(x_1) \approx y_1 = y_0 + (x_1 - x_0)y'(x_0) = y_0 + hf(x_0, y_0)$$
- Now we will approximate the value of $y(x_1)$.
- For this, we can use the tangent line at $(x_1, y(x_1))$ but we don't have the value of $y(x_1)$.
- However, we have the approximation of $y(x_1)$ i.e. y_1 .

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So this is the idea of Euler's method, so mathematically we can write that the tangent line at x equals to x not is given as y equals to y not plus x minus x not y prime x not thus approximation to the value of the solution at x equals to x 1 is the y coordinate that is x equals to x 1 on the tangent line.

So y at x 1 is given that is approximated value of y 1 is given by y 0 plus x 1 minus x not y dash x not and as I told you this particular approximation is valid only when x 1 is very close to x not means I am having a very small h very small step size.

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Euler's method

- From equation (6), we have

$$y(x_1) \approx y_1 = y_0 + (x_1 - x_0)y'(x_0) = y_0 + hf(x_0, y_0)$$

- Now we will approximate the value of $y(x_2)$.
- For this, we can use the tangent line at $(x_1, y(x_1))$ but we don't have the value of $y(x_1)$.
- However, we have the approximation of $y(x_1)$ i.e. y_1 .



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If I take a large step size let us say I am having value at 0 and I want to find out at let us say at 0.5 so I will take a tangent here and tangent line will go there so my original solution is this one but I will get this one as the approximation, so I will get more error if the step size is large.

So once we calculate y_1 that is basically y not plus h times f of x not y not from the tangent line we will calculate y_2 . So y_2 will become basically y_1 plus h into $f(x_1, y_1)$, similarly y_3 will become y_2 plus h times $f(x_2, y_2)$ so I will get a sequence of iterations.

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Euler's method

Thus for the next approximation we have,

$$y(x_2) \approx y_2 = y(x_1) + (x_2 - x_1)y'(x_1) = y(x_1) + hf(x_1, y(x_1))$$



As we approximate $y(x_1)$ by the approximation y_1 , we have for the second approximation

$$y(x_2) \approx y_2 = y_1 + hf(x_1, y_1) \quad (7)$$

Continuing in this way, we obtain the sequence of approximate values

$$y(x_{i+1}) \approx y_{i+1} = y_i + hf(x_i, y_i), \quad i = 0, 1, 2, \dots \quad (8)$$

This tangent line method of approximations is known as Euler's method.



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In general y at any point x_{i+1} is given as y_i plus h times $f(x_i, y_i)$. Now what is the error in this approximation because from this figure itself you can see this is the error in my this approximation if I take the largest this is the error this is the error at second point that is when x is 0.2 and so on.

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Euler's method

Error in the Euler's Method

The Euler's method may be obtained on expanding $y(x_0 + h)$ by Taylor's series about x_0 as

$$y(x_0 + h) = y(x_0) + hy'(x_0) + R$$

where R is the remainder term given by

$$R = \frac{h^2}{2} y''(\xi), \quad x_0 \leq \xi \leq x_1$$

Neglecting R in the above expansion gives the Euler's formula

$$y(x_1) \approx y_1 = y_0 + hy'(x_0) = y(0) + hf(x_0, y_0)$$

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So, how to calculate a bound on this error? So for this we will use Taylor series method so the value of y at x not plus h , so x not plus h basically x_1 so the value of y at x_1 can be obtained by the Taylor series approximation of y about x_0 so $y(x_0 + h)$ is $y(x_0) + h y'(x_0)$ plus second order and higher order terms. So let me because if you see this particular thing first three terms the left hand side term and two right hand side terms of this expression this equation it is $y(x_1)$ and it is $y_0 + h f(x_0, y_0)$ that is the Euler's formula for approximating y_1 from the x_0, y_0 .

So it means this is the error term R . So here R is the remainder term and it is given $\frac{h^2}{2} y''(\xi)$ where ξ is somewhere between x_0 and x_1 .

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Euler's method

Error in the Euler's Method

Thus the error term is

$$R = \frac{h^2}{2} y''(\xi) = R \frac{h^2}{2} f'(\xi), \quad x_0 \leq \xi \leq x_1$$

where

$$f' = \frac{d^2 y}{dx^2} = f_x + f_y \frac{dy}{dx} = f_x + f_y f$$

The general form of the Euler's formula may be written as

$$y_{i+1} = y_i + hf(x_i, y_i), \quad i = 0, 1, 2, \dots$$

Here, error is

$$R = \frac{h^2}{2} y''(\xi) = \frac{h^2}{2} f'(\xi, y), \quad x_i \leq \xi \leq x_{i+1}$$

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So the error term is R it is h^2 by 2 $y''(\xi)$ that is R times h^2 by 2 because we know that dy/dx equals to f so d^2y/dx^2 means y'' will be f' so I can write it as $f'(\xi)$. So, how to calculate f' ? Because f is a function of x and y and y is a function of again x , so f' will become $f_x + f_y \frac{dy}{dx}$ so I am writing and again dy/dx will be f .

So f' will be $f_x + f_y \frac{dy}{dx}$ so $f_x + f_y f$ so the error term as I told you will be h^2 upon 2 $f'(\xi, y)$, finally this particular thing.

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Euler's method

Example 1

Using Euler's method compute y_1 and y_2 taking $h = 0.1$ from the following differential equation,

$$\frac{dy}{dx} = 1 + xy^2, \quad y(0) = 1$$



Also compute the error term for both y_1 and y_2 .

Solution

$$y'(x) = f(x, y) = 1 + xy^2$$

and

$$y''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}f(x, y) = f_x + f_y \frac{dy}{dx} = f_x + f_y f = y^2 + 2xy(1 + xy^2)$$

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So let us take an example and we will solve this example using the Euler's method. So here we are having our initial value problem as $\frac{dy}{dx}$ equals to 1 plus x into y square, so here my function small f of x, y is 1 plus x, y square. The given initial condition is y at x equals to 0 is 1 that is why 0 is 1, now I need to take a step size h equals to 0.1 and I need to calculate the values of y_1 and y_2 .

The value of y_1 is the value of y at x equals to 0.1 and the value of y_2 is the value of y at x equals to 0.2. So if we compare this given problem with the standard form of initial value problem then my f of x, y is 1 plus x, y square I need to calculate y double prime x that will be $\frac{d^2y}{dx^2}$ over dx square that is $\frac{d}{dx}$ of f of x, y and it becomes $\frac{\partial f}{\partial x}$ plus $\frac{\partial f}{\partial y}$ into $\frac{dy}{dx}$ $\frac{\partial f}{\partial x}$ is given as y square and $\frac{\partial f}{\partial y}$ is $2x, y$ into y dash and y dash again 1 plus x, y square.

So y double dash x is y square plus $2x, y$ into 1 plus x, y square. Why I am calculating this? Because in the question they are asking to compute error terms also and for that I need y double prime x .

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Euler's method

Example 1 cont...

For $x = 0.1$, $y_1 = y_0 + hf(x_0, y_0) = 1 + 0.1(1 + 0 \times 1) = 1.1$
and the error is given by

$$\epsilon_1 = \frac{h^2}{2} y''(\xi) = \frac{h^2}{2} f'(\xi), 0 \leq \xi \leq 0.1$$
$$= \frac{0.01}{2} (y^2 + 2xy(1 + xy^2))$$

Maximum truncation error is

$$\epsilon_1 = \frac{0.01}{2} (1.1^2 + 2 \times 0.1 \times 1.1(1 + 0.1 \times 1.1^2)) = 0.005 \times 1.45662 = 0.00728$$

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Now, let us take the first step and calculate the value of y_1 , so here x is 0.1 so y_1 is y_0 plus h times f of x_0, y_0 by the Euler's formula y_0 is given as 1 plus h is 0.1 and f of x, y is $1 + x, y$ square so it will become $1 + x$ not into y not square x not is 0 so this term become 0 so $1 + 0.1$ it is 1.1.

Now if I compute the error ϵ_1 is given as $\frac{h^2}{2} y''(\xi)$ where ξ is somewhere between 0 to 0.1. So $y''(\xi)$ is $f'(\xi)$ and I have already calculated this so this is 0.01 upon 2 into this is the expression for $y''(\xi)$. So maximum truncation error is 0.01 upon 2 into 1.1 whole square because this is the value of y_1 plus 2 times 0.1 into 1.1 into 1 plus 0.1 into 1.1 square.

So after simplifying this we are getting the maximum truncation error in y_1 that is given by ϵ_1 and the value numerical value of this is 0.00728.

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Euler's method

Example 1 cont...

For $x = 0.2$,

$$y_2 = y_1 + hf(x_1, y_1) = 1.1 + 0.1(1 + 0.1 \times 1.1^2) = 1.2121$$

and the error is given by

$$\epsilon_2 = \epsilon_1(1 + hf_y(x_1, y_1)) + \frac{h^2}{2}y''(\xi) = \frac{h^2}{2}f'(\xi), 0.1 \leq \xi \leq 0.2$$
$$= \epsilon_1(1 + 2hx_1y_1) + \frac{h^2}{2}(y^2 + 2xy(1 + xy^2))$$

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Now I will calculate y at x equals to 0.2 that is my y_2 so y_2 is y_1 plus h time f of x_1, y_1 that is 1.1 plus 0.1 into 1 plus 0.1 into 1.1 whole square, so please note that for calculating y_2 I am taking the values from the previous iteration or previous step so my x_1 is now 0.1 and y_1 is 1.1 which we have calculated from x_0, y_0 .

So after simplifying this I am getting this value as 1.2121. So my y at x equals to 0.2 is 1.2121, the truncation error in this step is given by ϵ_2 that is ϵ_1 times 1 plus $h f_y(x_1, y_1)$ plus $\frac{h^2}{2} y''(\xi)$, where ξ is now between 0.1 to 0.2.

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Euler's method

Example 1 cont...

Maximum truncation error is

$$\epsilon_2 = 0.00728(1 + 2 \times 0.1 \times 0.1 \times 1.1) + \frac{0.01}{2} \{1.2121^2 + 2 \times 0.2 \times 1.2121 \times (1 + 0.2 \times 1.2121^2)\}$$

$$= 0.00744 + 0.005(1.492 + 0.6272) = 0.0179$$

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So I have substituted all these values here and then this term so maximum truncation error will be given by this particular expression plus this is for the second term of the epsilon 2 and after simplifying it I am getting this value as 0.0179.

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Euler's method

Example 1 cont...

For $x = 0.1$, $y_1 = y_0 + hf(x_0, y_0) = 1 + 0.1(1 + 0 \times 1) = 1.1$

and the error is given by

$$\epsilon_1 = \frac{h^2}{2} y''(\xi) = \frac{h^2}{2} f'(\xi), 0 \leq \xi \leq 0.1$$

$$= \frac{0.01}{2} (y^2 + 2xy(1 + xy^2))$$

Maximum truncation error is

$$\epsilon_1 = \frac{0.01}{2} (1.1^2 + 2 \times 0.1 \times 1.1(1 + 0.1 \times 1.1^2)) = 0.005 \times 1.45662 = 0.00728$$

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Euler's method

Example 1 cont...

For $x = 0.2$,

$$y_2 = y_1 + hf(x_1, y_1) = 1.1 + 0.1(1 + 0.1 \times 1.1^2) = 1.2121$$

and the error is given by

$$\begin{aligned}\epsilon_2 &= \epsilon_1(1 + hf_y(x_1, y_1)) + \frac{h^2}{2}y''(\xi) = \frac{h^2}{2}f'(\xi), 0.1 \leq \xi \leq 0.2 \\ &= \epsilon_1(1 + 2hx_1y_1) + \frac{h^2}{2}(y^2 + 2xy(1 + xy^2))\end{aligned}$$



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Euler's method

Example 1 cont...

Maximum truncation error is

$$\begin{aligned}\epsilon_2 &= 0.00728(1 + 2 \times 0.1 \times 0.1 \times 1.1) \\ &+ \frac{0.01}{2}\{1.2121^2 + 2 \times 0.2 \times 1.2121 \times (1 + 0.2 \times 1.2121^2)\} \\ &= 0.00744 + 0.005(1.492 + 0.6272) = 0.0179\end{aligned}$$



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So here my y_1 is 1.1 the maximum truncation error in y_1 is 0.00728 and then y_2 is coming out as 1.2121 and the maximum truncation error for this x_2 is 0.0179.

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Euler's method

Example 2

Suppose the interval in which we have to find the solution is $[0, 1]$. Also,

$$f(x, y) = x + 2y, \quad y(0) = 0$$

So, we have $x_0 = 0$ and $y_0 = 0$.

By Euler's method we will find values for x_1 and y_1 . Take $h = 0.25$

$$x_1 = x_0 + h = 0 + 0.25 = 0.25 \text{ So,}$$
$$x_1 = 0.25$$
$$y_1 = y_0 + hf(x_0, y_0) = 0 + 0.25(0 + 2 \times 0) = 0$$
$$\Rightarrow y_1 = 0$$

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Now if we take another example that is if f is x plus $2y$ the problem is same y prime equals to f of x, y and y_0 is 0 here so if I apply Euler's method with step size 0.25 suppose I want to calculate value at 4 points between 0 to 1 that is at 0.25 at 0.5 at 0.75 and then finally at 1 .

So value at x equals to 0.25 that is y_1 is given by this one and it is coming out to be 0 .

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Euler's method

Example 2 cont...

Thus we have $x_1 = 0.25, y_1 = 0$

Now we move on to find (x_2, y_2) .

$$x_2 = x_1 + h = 0.25 + 0.25 = 0.5$$
$$y_2 = y_1 + hf(x_1, y_1) = y_1 + h(x_1 + 2y_1) = 0 + 0.25(0.25 + 2 \times 0) = 0.0625$$
$$\Rightarrow y_2 = 0.0625$$

So we have $x_2 = 0.5$ and $y_2 = 0.0625$

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Euler's method

Example 2 cont...

Now we will find (x_4, y_4) .

$$x_4 = x_3 + h = 0.75 + 0.25 = 1$$

$$y_4 = y_3 + hf(x_3, y_3) = y_3 + h(x_3 + 2y_3) = 0.21875 + 0.25(0.75 + 2 \times 0.21875) = 0.515625$$

$$\Rightarrow y_4 = 0.515625$$

We have $x_4 = 1$ and $y_4 = 0.515625$



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At x equals to 0.5 y is coming using the Euler's process as 0.179570. Similarly when x is 0.75 y is coming at 0.21875 and finally at x equals to 1 y equals to 0.515625.

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Euler's method

Example 2 cont...

Again, if we find the true solution of the given IVP, it is given as

$$y(x) = 0.25e^{2x} - 0.5x - 0.25$$

So for this above true solution, we will find the iterative values as calculated using Euler's formula. We have

x	y
0.00	0.000000
0.25	0.037180
0.50	0.179570
0.75	0.495422
1.00	1.097264



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Euler's method

Example 2 cont...

Thus we have $x_1 = 0.25$, $y_1 = 0$

Now we move on to find (x_2, y_2) .

$$x_2 = x_1 + h = 0.25 + 0.25 = 0.5$$

$$y_2 = y_1 + hf(x_1, y_1) = y_1 + h(x_1 + 2y_1) = 0 + 0.25(0.25 + 2 \times 0) = 0.0625$$

$$\Rightarrow y_2 = 0.0625$$

So we have $x_2 = 0.5$ and $y_2 = 0.0625$



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So the true solution of the initial value problem is given by this particular expression and if I calculate the values at 0.25 by the Euler's method we are getting 0 but from the exact solution it is 0.037180.

By that numeric Euler's method at 0.5 I am getting y as 0.0625 while the exact value at 0.5 is 0.179570 and similarly we are having big errors at x equals to 0.75 and x equals to 1 also. So it means the Euler's is having large error when compare to the exact solution, and what is the reason behind this? The reason is step size because here we are taking step size as 0.25 if you decrease the step size error will decrease and it happens always in Euler's method however you have to do more calculations.

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Euler's method

Example 2 cont...

- So from the exact solution of the given IVP, we can conclude that the numerical solution differs a lot from the true solution.
- The reason behind this inaccuracy is that we have taken a very large step size.
- Thus we should shrink the step size to get the more accurate solution.

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So if we shrink the step size we will get more accurate solution by using the Euler's method. So with this I will stop this lecture I will end this lecture so today we have learnt Picard's method and Euler's method, in the next lecture we will talk more accurate method when compare to the Euler's means one more (28:00) on the Euler's method and then Tailor series method means we will consider more means like in Euler's method we are taking only term up to first order in Tailor series approximation of y at about x_0 , however there we will take more like second order term also so that error will reduce. So we will talk about those methods in next lecture, thank you very much.