Numerical Methods
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Lecture 36
Introduction to Ordinary Differential Equations

Hello everyone, so welcome to the fourth module of this course from my side and it is the last module of the overall course numerical methods and in this module we will learn numerical methods for solving ordinary differential equations. So in this lecture I will introduce ordinary differential equations to you what we mean by the solution of ordinary differential equations what type of ordinary differential equations we have, what is the conditions for existence and uniqueness of the solutions of a differential equation.

And finally we will talk about why we need numerical solution for ordinary differential equations.

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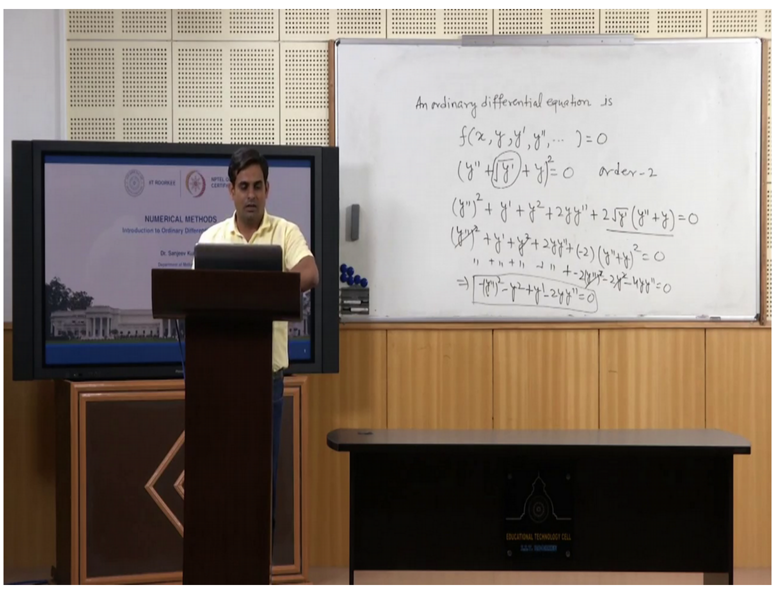

So in particular an ordinary differential equation is a relation between independent variable, dependent variable and derivatives of dependent variable with respect to independent variable like y prime here y prime is dy by dx that is the first derivative of y with respect to x y double prime and so on. So here x is independent variable as I told you and y is the dependent variable.

So the solution of this is a function y which is a function of x which satisfy this particular relation. So such a relation is called ordinary differential equation. Now we are having 2 terms for a given ordinary differential equation that is one is degree another one is order. So first what is order? An order of a differential equation is the order of the highest derivative in the relation, for example if I am having a differential equation so it is a differential equation of second order because here we are having second order derivative which is the highest order derivative in the given equation.

The if I am having this kind of thing let us say then here again the order is 3 because highest order derivative in this equation is 3. The other thing is degree the degree of a differential equation is the power of highest order term in the equation provided equation should not have any radical sign. So for example the degree of this differential equation is 1 because here highest degree order derivative is y double prime and there is no square root type of term in this equation on the dependent variable and hence the degree is 1.

The degree of this differential equation is again 1, if I am having an equation of the form let us say y double prime plus square root of y prime plus y equals to 0. Now, this differential equation is having order 2 but here I cannot say that degree is 1, why? Because we are have this square root term of the first order derivative of the dependent variable in the equation. First of all we have to remove this square root, so how can we remove? Let us make the square of the overall equation so it will become square of this equals to 0 so this will become y double prime whole square plus y prime from this term plus y square so square of each term plus 2 y y double prime plus 2.

So we will get this equation let us simplify it more so basically we need to simply this term so when I will simplify this term this will become y double prime square plus y prime plus y square plus twice y y double prime plus I can substitute this from the original equation, so this will become minus 2 y double dash plus y whole square equals to 0 or finally if I simplify it more here I will be having minus 2 y double prime square minus 2 y square and then finally minus 4 y y double prime equals to 0 and these terms as such or if I simplify it more minus y double prime square so this term cancel with this one minus y square and then I am having 2 y y double prime here and here I am having minus 4 y y double prime plus y dash is coming from here this term minus 2 y y double prime equals to 0.

And hence after simplification we get this equation and here highest order derivative is again means that is y double prime and degree of this is 2 hence the degree of this differential equation is 2. Now further more we can classify the ordinary differential equations into 2 classes, 1 is linear ODE and the nonlinear ODE.

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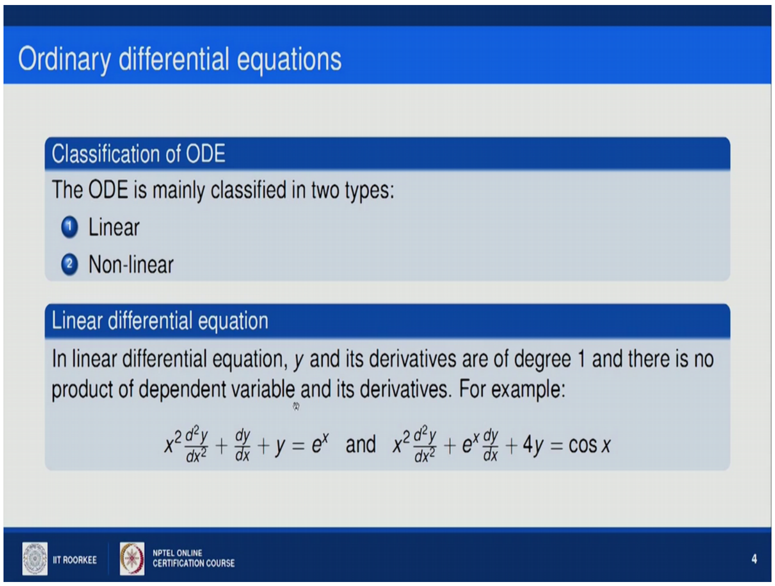

If in the differential equation y and its derivatives are of degree 1 and there is no product of dependent variable and its derivative terms then the equation is called linear differential equation.

For example, x square d 2 y over dx square plus dy over dx plus y equals to e raise to power x. Here you can notice this the second order derivative term is having degree 1 the first order derivative term is having degree 1 y is also having degree 1. Hence and there is no cross product term of the dependent variable or its derivative. So hence it is an example of linear differential equation, similarly this is an example of linear differential equation if the differential equation is not linear than it is said to be nonlinear differential equation.

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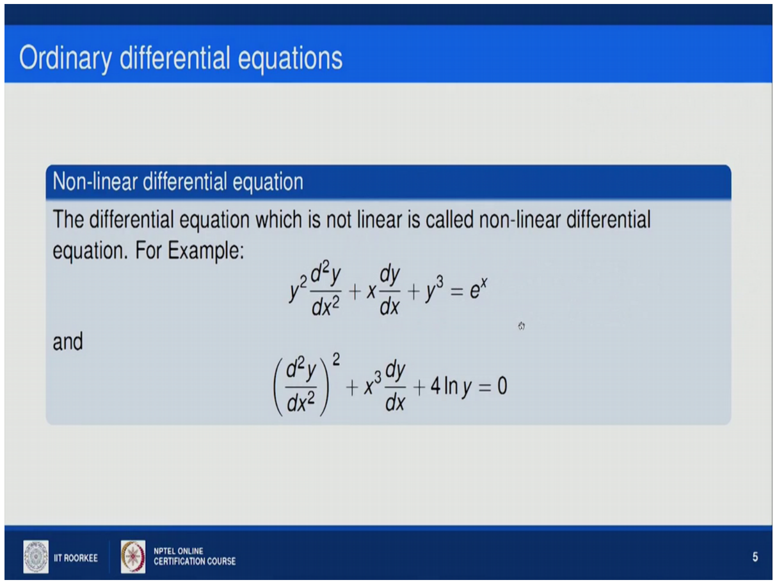

For example this difference equation here you can note down this term here it y cube so here I am having degree 3 of the dependent variable y so it is nonlinear due to this term more over the first term is the product of y square with second order derivative of y. So hence it is again nonlinear term in the differential equation and hence it is a nonlinear differential equation. Furthermore if I take this particular equation so here I am having y double prime square and due to this term it is nonlinear because here I am having square term and here 4 of log y again it is not a linear term of the depended variable and hence this is a nonlinear term.

So due to these 2 terms it is nonlinear. The linear differential equations are easy to solve and we are having several analytical method to find out the solutions of linear differential equation. However, in case of nonlinear differential equation very few methods are there for solving them analytically and that is if the given equation is having in a specified form then only we can solve nonlinear differential equation using analytical method.

Moreover whenever we solve nonlinear differential equation we get a close form solution, close form means something solution in terms of an integral and hence they are not quite useful in real life application scenario. So what we need to do? We need it is better to find out some alternative solution for them instead of such an analytical solution and there we need numerical solutions.

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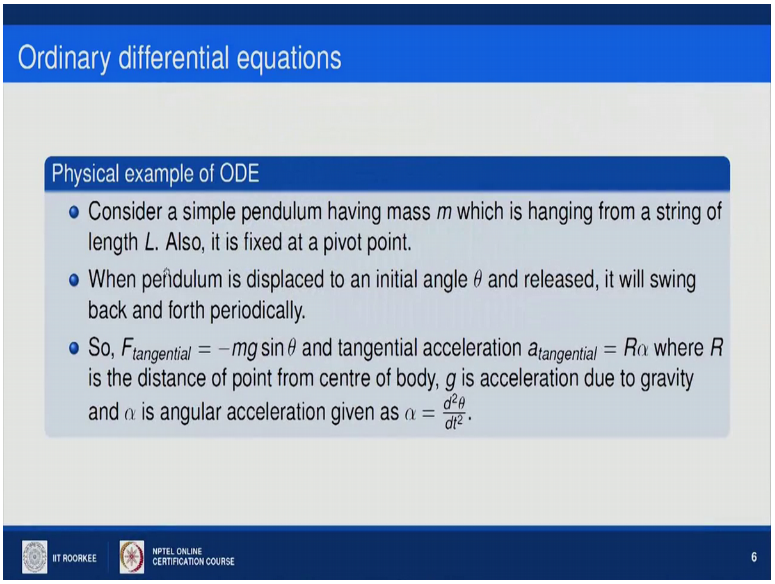

However before going to detail about numerical solution let me give some example of ordinary differential equations the first example is very simple it is coming from the motion of simple pendulum.

So consider a simple pendulum having mass m which is hanging from a string of length l. Also it is fixed at a pivot point. When pendulum is displaced to an initial angle theta and released, it will swing back and forth periodically. So, here tangential force is negative of m into g into sin theta that is mass where g is the acceleration due to gravity and tangential acceleration is given by R into alpha, where alpha is second order derivative of the angle with respect to time that is d 2 theta over dt square and R is length of the string.

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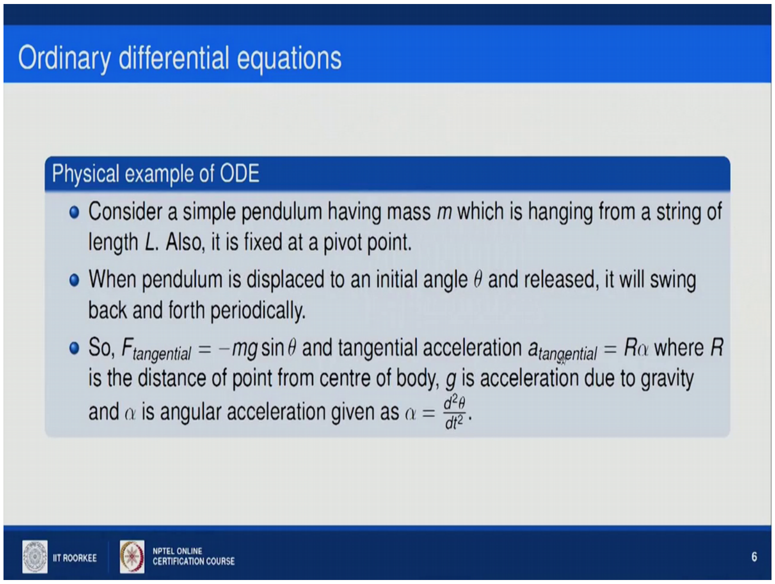

So I can write it tangential acceleration is l into d 2 theta over dt square.

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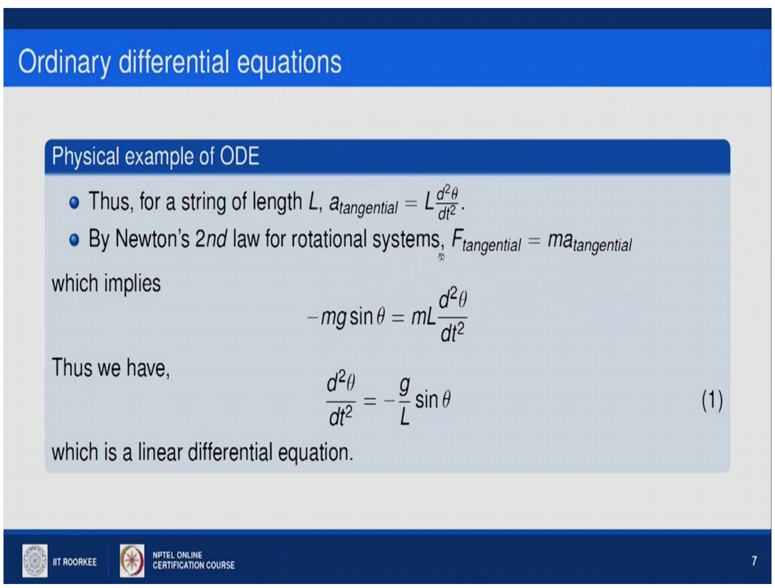

So by Newton second law for rotational systems the tangential force should be equals to m into tangential acceleration so I can write minus mg sin theta equals to m times l d 2 theta over dt square, so m will be cancel so we can have a second order linear differential equation which is d 2 theta over dt square equals to minus g upon l into sin theta.

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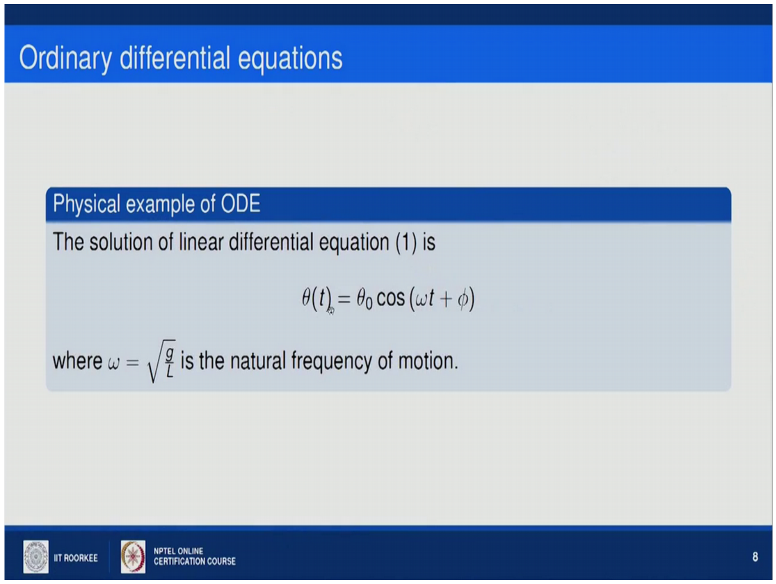

We can solve this differential equation analytically and solution of this equation is given as theta at any time t equals to theta 0 which is the initial angle cos omega t plus 5 where omega is square root upon g upon l and it is the natural frequency of the motion of the simple pendulum.

Similarly we can have another physical example of ordinary differential equation and it is coming from exponential growth and decay.

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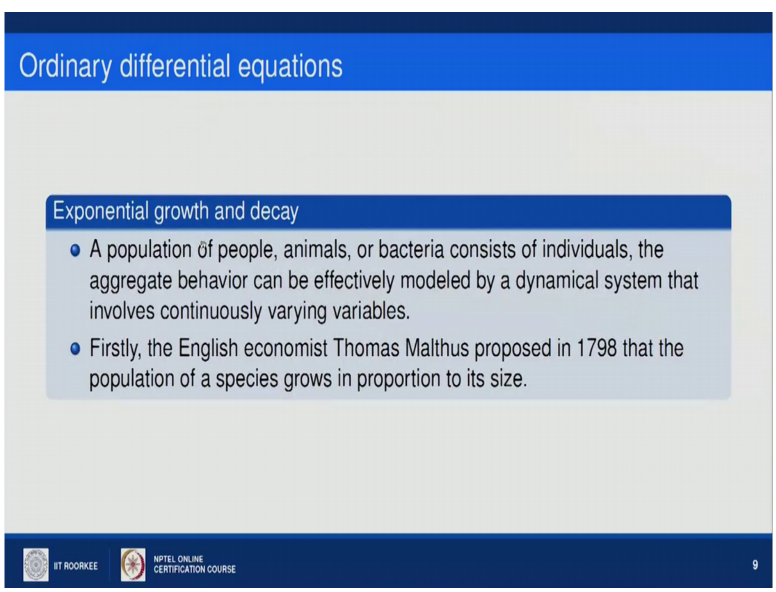

So it is taken from the mathematical biology or biomathematics and basically the population of human animals or bacteria consist of individual the aggregate behavior can be effectively modeled by a dynamical system that involves continuous varying variables.

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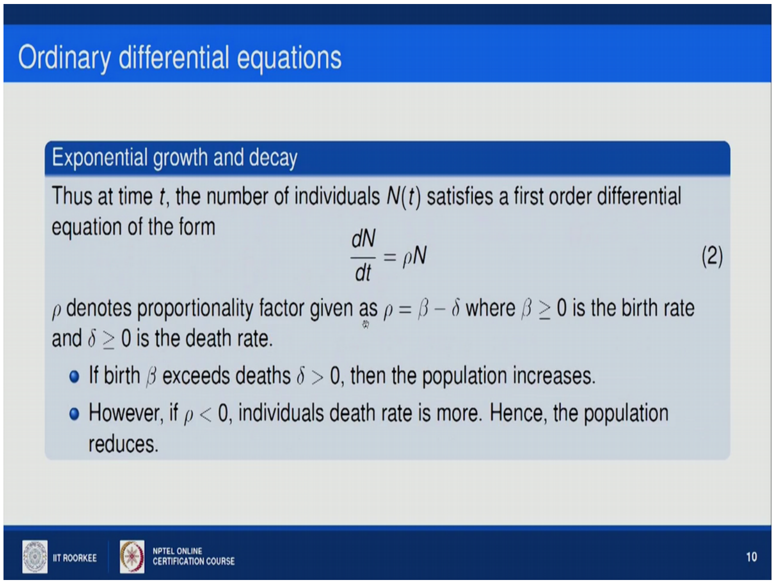

So for example at if we assumed that at any time t the number of persons or number of people is Nt which satisfy a first order differential equation of the form dN over dt equals to rho N, here rho denotes a proportionality factor given as beta minus delta where beta is non-negative number and it is giving the birth rate while delta again non-negative number is the death rate. So if beta exceeds delta means rho is positive and hence population is increasing if rho is negative then rate is more compare to the birth rate and hence population is decreasing.

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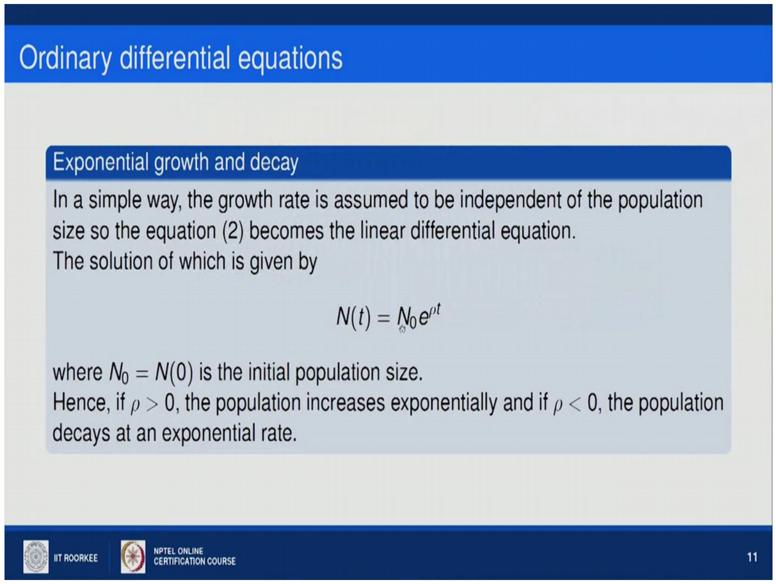

The solution of this is given by Nt equals to N not e raise to power rho t again it is explicit analytical solution that we can obtain using variable separable method for the differential equation and here N not is the initial population that is at time t equals to 0. And from this solution we can see that if rho is positive the population increase exponentially while if rho is negative the population decrease exponentially again.

Now whenever we solve the differential equation analytically like I have taken two examples, one is of population model and here solution is coming Not equals to N not t raise to power rho t. While in the other example that was from the motion of simple pendulum the solution was coming theta equals to theta not cos omega t plus 5.

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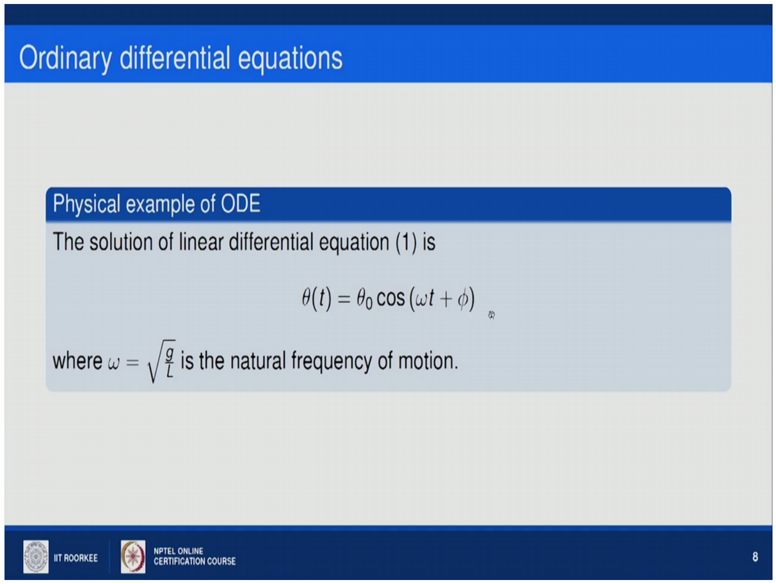

In both of the cases what we are having we are having some constant in the solution how to find out the values of those constant because these are the general solutions and for a particular problem we need particular solution and for finding the particular solutions we need conditions.

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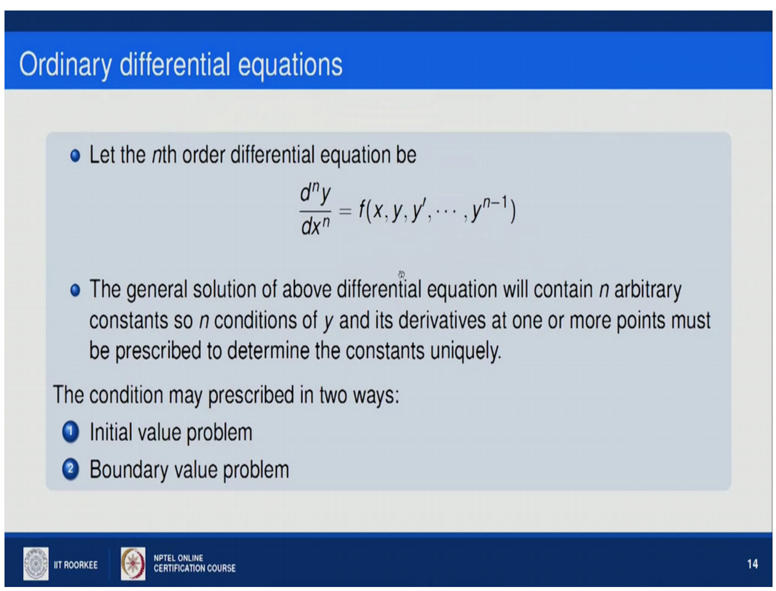

So the solution of the any nth order differential equation will contain an arbitrary constant, if you are having like we are having population model equation they are we were having only first order equation first order ODE and we are having only one constant N not. Similarly if you are having second order equation we will be having two arbitrary constant in the solution basically in the general solution. So if you are having nth degree or sorry nth order ODE then you will be having n number of arbitrary constant in the general solution.

And to eliminate both constant for getting a particular solution we need some conditions.

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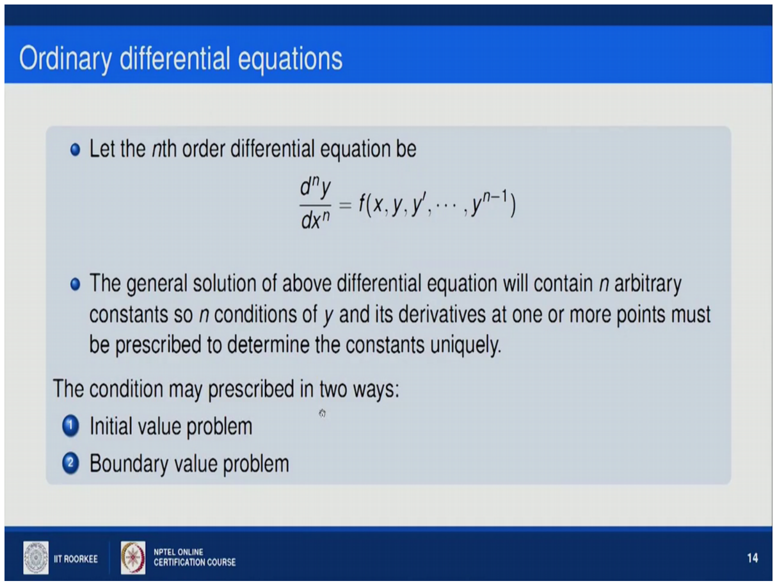

So there are two types of conditions in case of ordinary differential equations one is called initial value another one is boundary value means if initial condition is given to you the differential equation together with initial condition is called initial value problem. However if boundary conditions are given then the differential equation with those boundary conditions is called boundary value problem.

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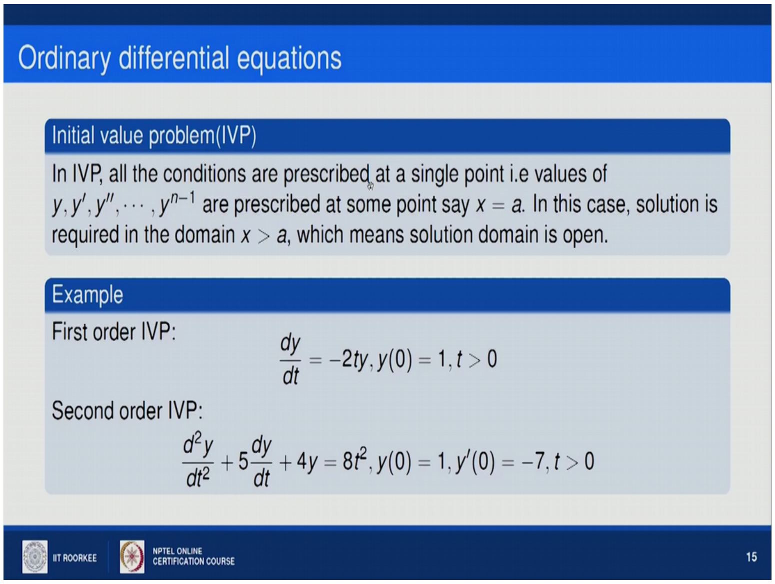

So let us take some example of this, so initial value problem all the conditions are prescribed at a single point that is value of y, y prime, y double prime up to if differential equation of nth order then we will be having initial conditions up to n minus 1 derivative order derivative. And those in the initial conditions are given at some point x equals to a and in this case we can find out or we require the solution in the domain x greater than a because x equals to a is the initial point so we have to move in the right side of that particular point.

And it means the solution domain is open after a it may be any point. So this is an example of first order initial value problem here differential equation is dy over dt so y is a function of t and it is equals to minus twice ty and y 0 equals to 1 and t is greater than 0. So initial condition is y at t equals to 0 is 1 this is the differential equation and hence together it is called an initial value problem. Similarly it is an example of second order initial value problem and here d 2 y over dt square plus 5 dy over dt plus 4 y equals to 8 t square. The two initial conditions are given as y at t equals to 0 is 1 and y prime at t equals to 0 is minus 7 and t is greater than 0 is the domain of the t.

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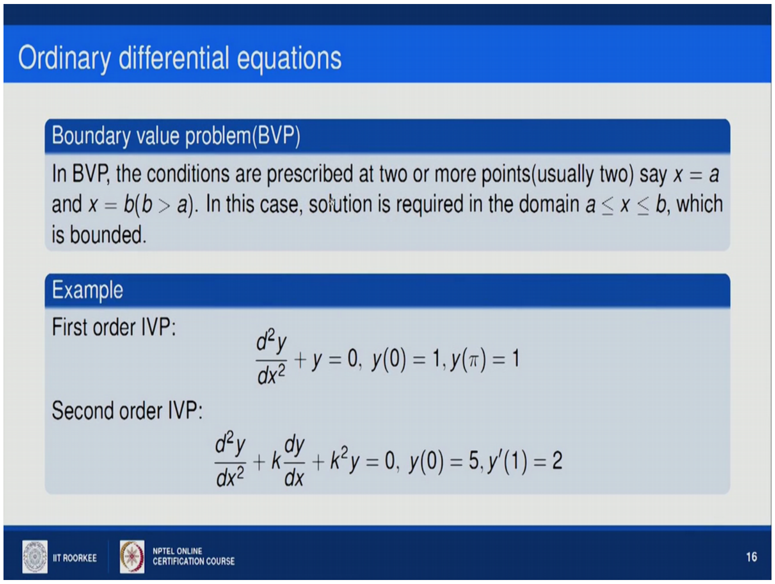

On the other hand in boundary value of problems the conditions are prescribed at two or more points usually at two say x equals to a and x equals to b where b is always greater than a. In this case solution is require in the domain x belongs to close interval a to b which is bounded. So this is if I take this first order differential equation d 2 y over dx square plus y equals to 0 and the here domain is the close interval from 0 to 5 then y 0 equals to 1 and y pi equals to 1.

Now we will talk the existence and uniqueness of the solution for a given differential equation in theoretical sense means when the solution exist and if it is exist whether it is unique or not and if it is exist or unique in which interval it is means what is the interval of the existence or uniqueness of the solution.

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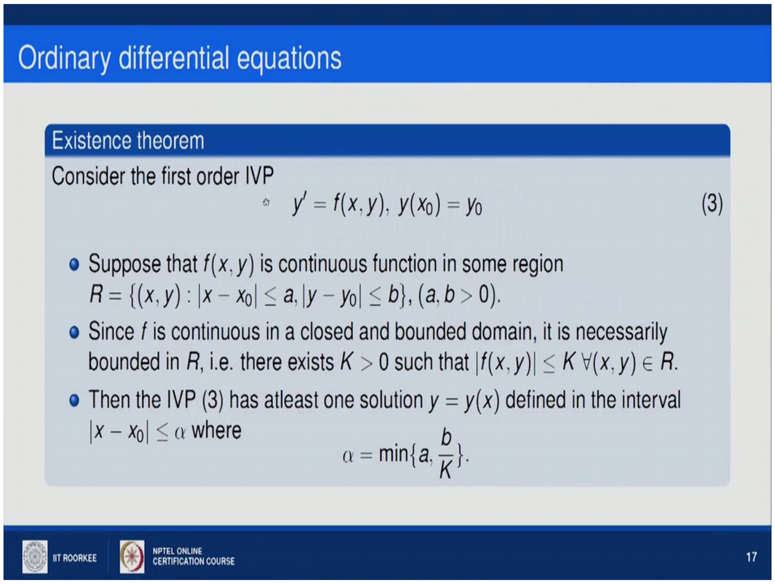

So again we will take a first order initial value problem y prime equals to f of x, y y at x not is y not.

Now suppose this f is continuous function in some region that is the region in two dimensional domain R which is having all the points x, y where x is means for x we are having the interval that is x minus x not absolute value is less than equals to a and for y we are having y minus y not less than equals to b and a, b are positive. So the center of the rectangle at x not n y not so since f is continuous and our interval is a close interval so it will be bounded because function is continuous interval is closed so it is bounded in R bounded means there exist a K such that the absolute value of f of x, y will be less than equals to K for all x, y belonging to R.

If this is true, then we will say that the initial value problem that is y prime equals to f x y together with condition y x not equals to y not has at least one solution y equals to y x define in the interval x minus x not less than equals to alpha and where alpha is given by minimum of a b up on K. So please note that here interval will become smaller for the existence of solution.

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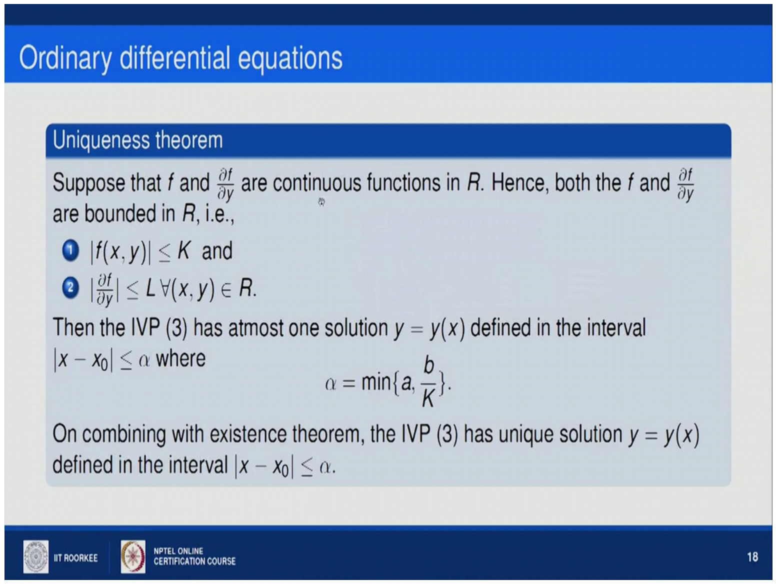

So for uniqueness suppose that (())(22:53) over del y are continuous function in R hence both f and del f over del y are bounded in R this means f of x, y is bounded by K del f or del y is bounded by L in R.

Then initial value problem has at most one solution, so this condition 1 is giving existence of solution the second is given uniqueness because it is saying at most one solution y equals to y, x in the interval again x minus x 0 less than equals to alpha where alpha is minimum of a and b upon K. So it means as I told you if I combine it with the existence theorem the solution will be unique there exist a unique solution. If f is continuous del f over del y is continuous in a close rectangle.

However we can replace the condition that is del f over del y is continuous and bounded in the given rectangle rectangular domain by a weaker condition we can replace this condition.

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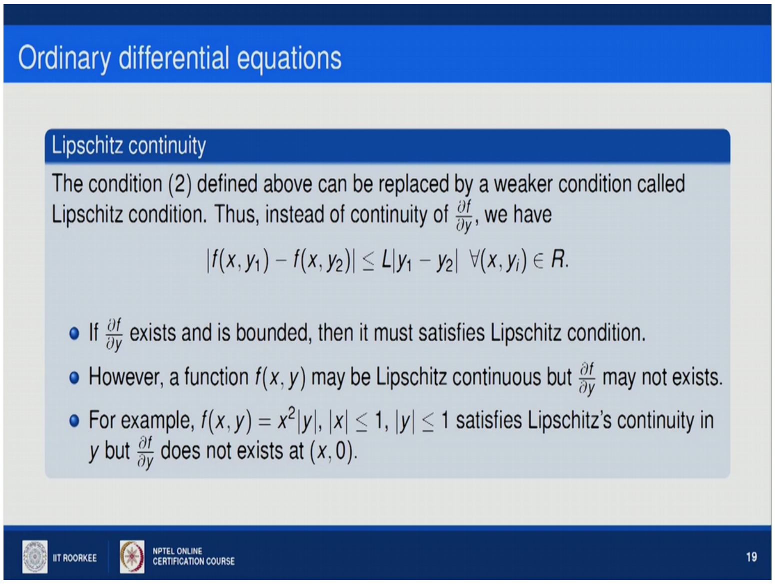

And the weaker condition is called Lipschitz condition. So thus instead of continuity of del f over del y we can have if f is Lipschitz continuous that is f of x, y 1 minus f of x, y 2 is less than equals to L times y 1 minus y 2 for all x, y i belongs to R belong to R so then the solution is unique.

So basically this condition is coming if del f over del y is continuous this implies this Lipschitz condition will always hold for a continuous partial derivative. However (())(24:47) is not true for example if you take f equals to x square into mod y where mod of x is less than equals to 1 mod of y is less than equals to 1 is the domain this particular function satisfy Lipschitz continuity or Lipschitz condition in this particular rectangular domain, however the function does not have partial derivate with respect to y that is f del f or del y does not exist at x 0.

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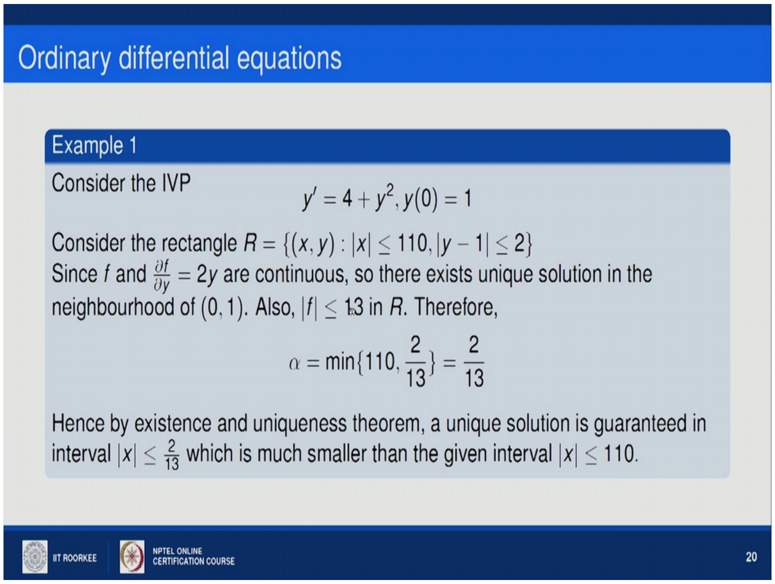

Again take this simple example so here y prime equals to 4 plus y square y 0 equals to 1 and our rectangle is given by R where mod of x is less than equals to 110 mod of y minus 1 is less than equals to 2. So since f and del f over del y that is del f over f is for plus y square here, so del f over del y is 2 y both are continuous f is bounded because the maximum value of f in this R can be 13, so therefore the solution exists for the interval mod x less than equals to 2 up on 13 and why I am saying 2 up on 13 because here according to first existence theorem minimum of 110 upon and 2 upon 13 that is beta upon K so 2 upon 13.

And the solution will be unique because del f over del y is also continuous in this rectangular domain. Similarly we can say for this however this particular initial value problem one solution is given by y equals to x plus c into 5 raise to power 5 upon 3. And another solution is y equals to 0, so the solution exists but for this particular example it is not unique. Now take this particular example y prime equals to 2 y upon x here y of x 0 equals to y 0 so here f of x, y is given by 2 y upon x and del f over del y is 2 upon x both of these functions are define for all x except the y axis.

So by uniqueness theorem for all x, excluding the y axis there exists a unique solution define in an open interval around x not. Also exist solution of this initial value problem is y equals to c times x square where c is any arbitrary constant. So you can check that all these solutions when x is 0, y is 0 so all these solutions will pass through the point 0, 0 but none of them will pass through any point at y axis except the origin.

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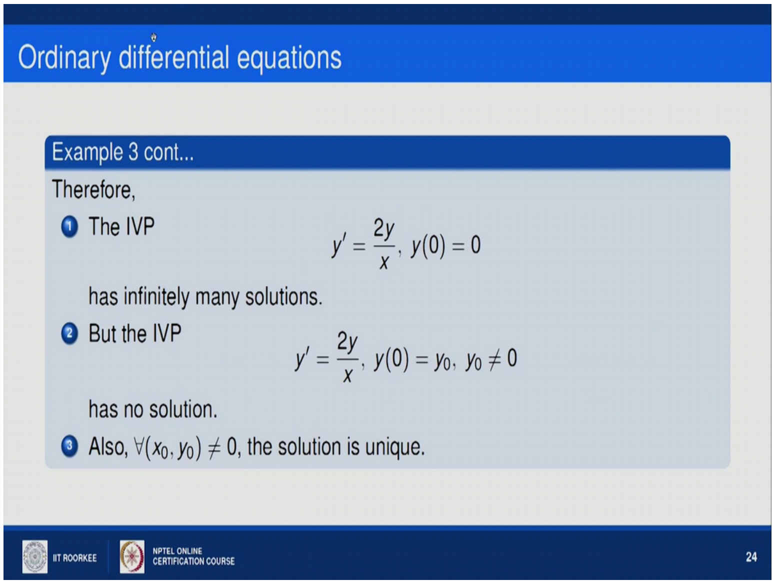

So the initial value problem as infinitely many solutions but the initial value problem if I replaced this 0 by some y not which is non-zero number so any point on the y axis which is not except the origin. Then, the initial value problem does not have any solution also for all x not y not when both are not 0 the solution is unique.

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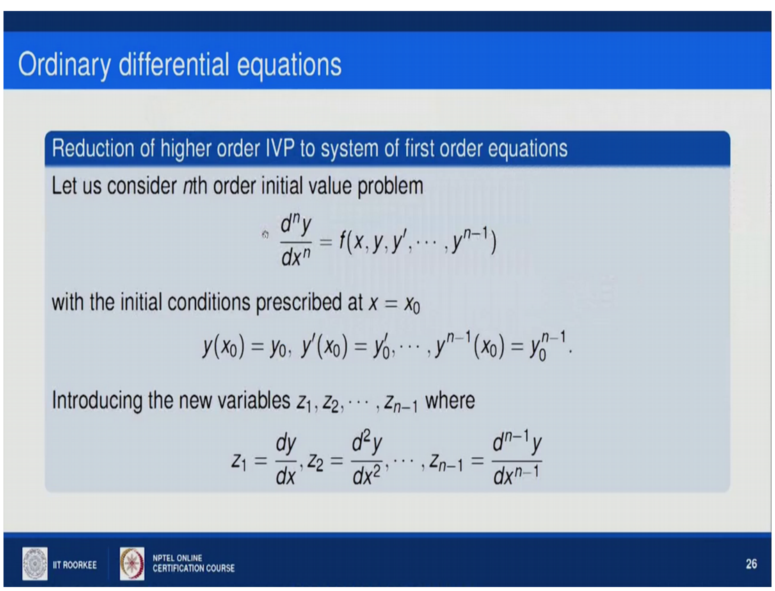

Now in the final thing I want to tell you about ordinary differential equation that is how to reduce nth order differential equation into a system of n linear differential equations or in first order equations.

So this is like I am having a nth order differential equation and I want to write it as a system of first order differential equation so that we can do very easily and then whatever technique we will learn for a single equation numerical technique that will be applicable to a system of linear equation ODE also first order ODE.

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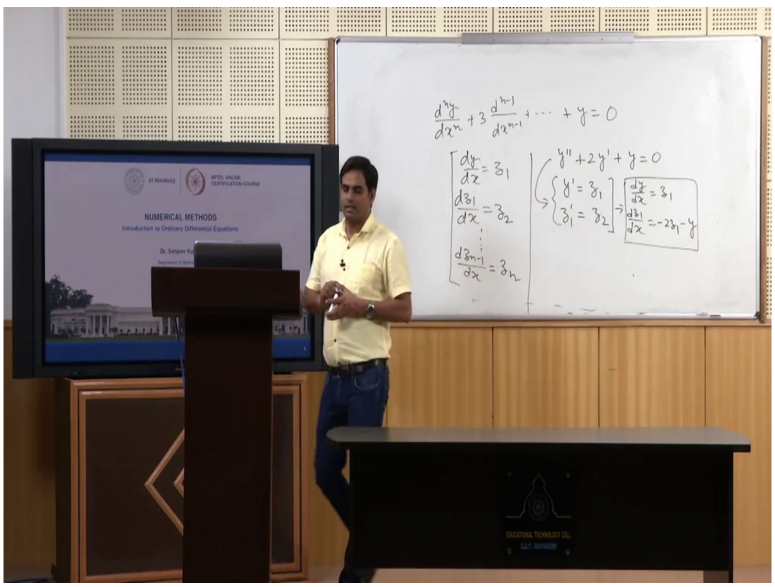

So for example if you are having a differential equation 0 or something else now it is a nth order differential equation, so if I want to reduce it into n first as a system of an first order differential equation then what I will do, first of all I will take dy over dx and I will say it z 1. Then what I will take? I will take d z 1 over dx that is basically d2y over dx square and this I will write z 2 and so on. So similarly dz2 over dx I will write z 3 so finally dz n minus 1 upon dx will become z n and hence I will be having these an first order differential equations and this dz n minus 1 over dx is nothing just dny over dxn and that I can write in terms of y z 1, z 2 up to z n minus 1 and hence I will be having a system of first order ODE.

For example if I take a equation let us say second order linear differential equation, now I want to write it as a system of 2 first order differential equations, so what is will take? I will write that y prime equals to z or let us say z 1, then I will write z 1 prime equals to z 2. So basically what I am having dy over dx equals to z 1 and d z 1 over dx that is basically d2y over dx square y double prime so this will become 2 minus 2 y minus y, so minus 2 y prime is minus 2 z 1 minus y.

So here I am having a system of two first order ODE in z 1 and y which is equivalent to a second order differential equation, similarly third order differential equation can be reduced into a system of three first order differential equations and so on. So in this lecture we learn few basic things about differential equations, in the next lecture we will talk about numerical methods about ordinary differential equations, thank you very much.