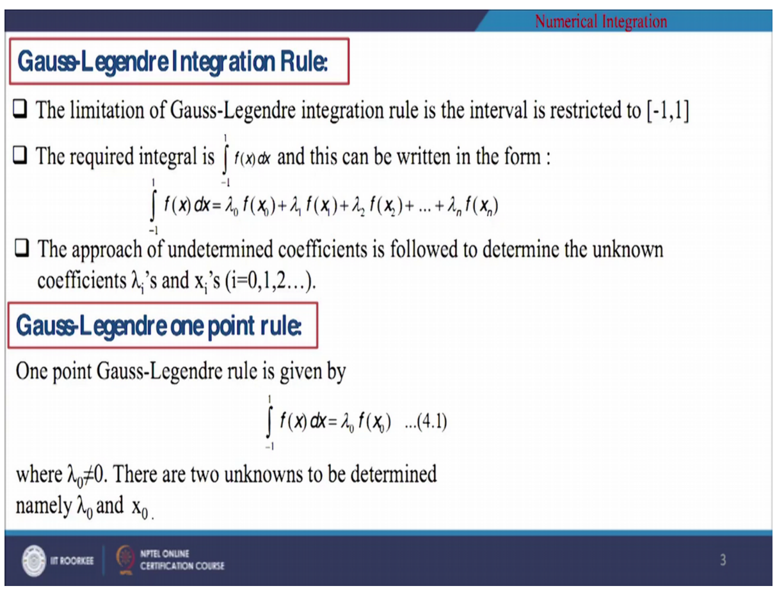
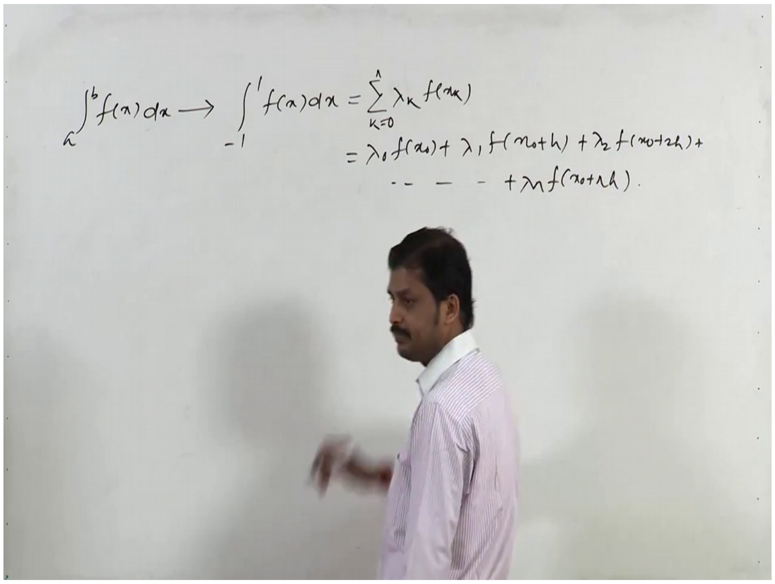
Numerical Methods
By Dr. Ameeya Kumar Nayak
Department of Mathematics
Indian Institute of Technology, Roorkee
Lecture 35
Numerical Integration Part 5

Welcome to the lecture series on numerical methods, currently we are discussing numerical integration. In the last lectures we have discussed about this numerical integration based on trapezoidal rule, 1 by 3 Simpson's rule, 3 by 8 Simpson's rule. And today we will just go for this numerical integration based on quarters rule here based on Gauss Legendre integration methods. So first we will just go for this Gauss Legendre integration rule for one point formula then for two point formula then for three point formula and then we will just go for error analysis of Gauss Legendre integration rule.

(Refer Slide Time: 0:56)



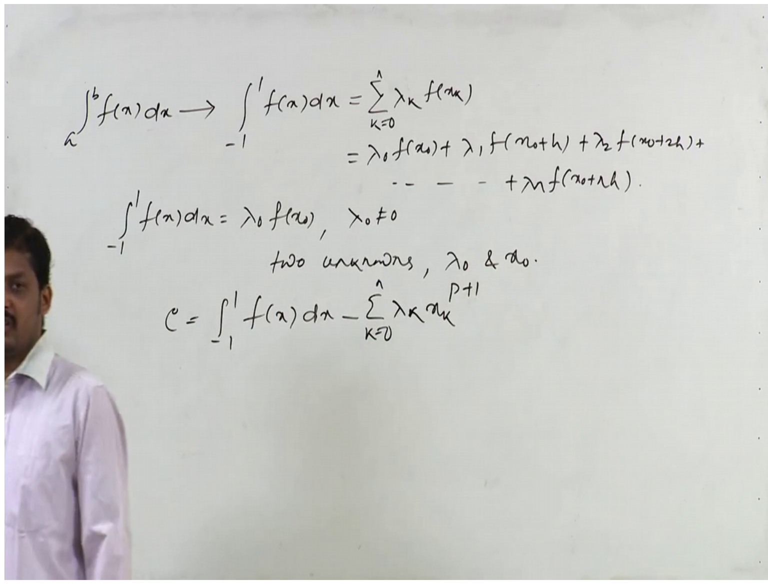


So if you just go for this Gauss Legendre integration methods basically in the beginning of this lectures we have discussed that whenever we are just representing an integration this integration is usually written in the form of like a to b f of x and this strict rule for this like Gauss Legendre integration method is that we have to transform this integration range a to b to minus 1 to 1 here. So first transformation we have to do that one and once this transformation has been made then we can just go for this like series expansion or this linear combination of this functional values at a different nodal points.

So this means that once we are just transforming this integration range a to b to minus 1 to 1 f of x dx then we can just represent this as a linear combination form that is k equals to 0 to n lambda k f of x k here or it can be represented in the form like lambda 0 f of x 0 plus lambda 1 f of x 0 plus h lambda 2 f of x 0 plus 2 h plus up to lambda n f of x 0 plus n h here. So the approach of this integration rule is that first we will just go for this undetermined coefficient form, undetermined coefficient form means lambda 0 is not known to us, lambda 1 is not known to us, lambda 2 is not known to us and x 0 is not known to us and x 0 plus h is not known to us.

Even if these points are not present there we can just consider any functional or any polynomial approximation to this integration here to get these nodal points here.

(Refer Slide Time: 3:17)



So first we will just go for this like Gauss Legendre one point rule here that is usually Gauss Legendre one point rule is written in the form of like minus 1 to 1 f of x dx this can be written as lambda 0 f of x 0 here since we are just considering for a polynomial of degree 1 here so that is why it is called like one point formula here.

And if you just see here lambda 0 is not equals to 0 so then we will have 2 unknowns here that are like lambda 0 and x 0 here. So if we want to determine these values for lambda 0 and x 0 then we will just go for this like approximation of this function with a exact polynomial order p and if this is approximated with a function or a polynomial of order p there then the error constant can be written in the form of like c equals to minus 1 to 1 f of x dx minus summation k equals to 0 to n here that is lambda n or lambda k x k to the power p plus 1 since we usually we are just considering this function or this polynomial which should be if it is order greater than p than it will just give a non-exact term then we will have a error is existing at that point.

(Refer Slide Time: 4:50)

Error in Gauss-Legendre Integration Rule:

Error constant is given by,

$$c = \int_{-1}^{1} f(x)\,dx - \sum_{k=0}^{n} \lambda_n x_k^{p+1}$$

Where p is the order of the method.

The error term is then given by

$$R_n(f) = \frac{c}{(p+1)!} f^{p+1}(\xi) \quad , a < \xi < b$$

Making the formula exact for $f(x)=1,x$ we get

$$f(x)=1: \quad \int_{-1}^{1} dx = 2 = \lambda_0$$

$$f(x)=x: \quad \int_{-1}^{1} x\,dx = 0 = \lambda_0\, x_0$$

Since, $\lambda_0 \neq 0$, we get $x_0=0$.

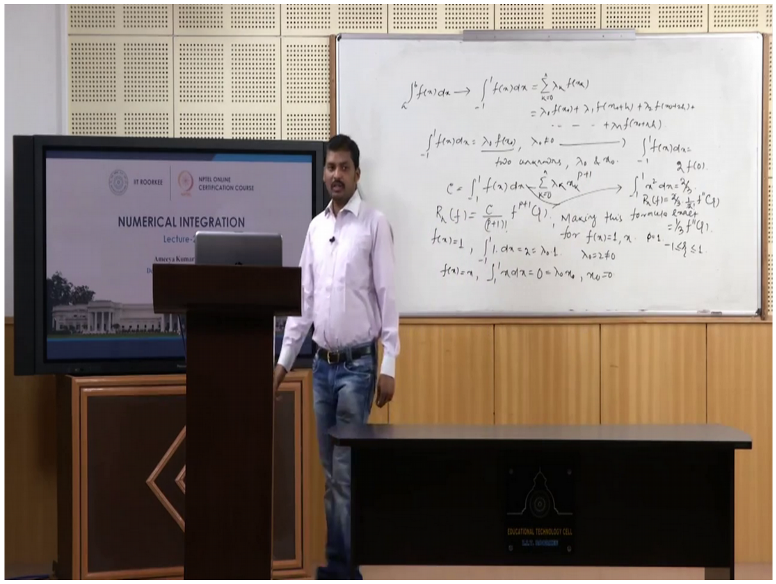
Therefore, the one point Gauss-Legendre integration formula takes the form:

$$I = \int_{-1}^{1} f(x)\,dx = 2\,f(0) \qquad \text{...(4.2)}$$

The error constant is given by, $c = \int_{-1}^{1} x^2 dx - 0 = \frac{2}{3}$

The error term is given by, $R_n(f) = \frac{c}{2!} f''(\xi)$

$$= \frac{1}{3} f''(\xi), \quad -1 < \xi < 1$$



And this error term that is R n of f here that is usually represented in the form of like c by p plus 1 factorial f to the power p plus 1 zeta, where zeta should be lies between a and b and if you just go for this like calculation of this coefficients here like lambda 0 and x 0 here then making this formula exact this means that error term should be 0 if you just consider here making this formula exact since already I have explained that we have to consider this formula is exact for polynomial of order p if p equals to 0, 1, 2 up to p it will just give you the exact solution here this means that 0 value we can just obtain here.

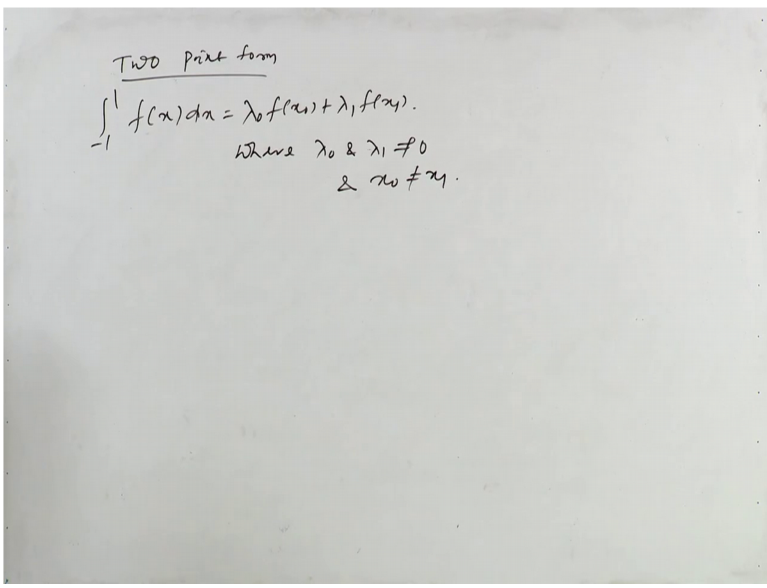
So if you just use this making this formula exact for f of x equals to like 1 and x since we can just consider that as 1, x, x square up to x to the power p since already we have considered that this is for p equals to here function is approximated or the polynomial is approximated with a polynomial of degree 1 here. So that is why we can just consider this one I equals to 1 here or p equals to 1 here so that is why two functions we can just consider first function is x to the power 0 that is as 1 here, second function is x to the power 1 as x here.

So if you will just put these functional values for this integration here, then we can just obtain that as like f of x equals to 1 suppose then we will have this minus 1 to 1 1 into dx so this can be written as like 2 this equals to if you just put these functional values that as lambda 0 into 1 here. So that is why lambda 0 can be taken as 2 here and for the second one f of x equals to x here minus 1 to 1 x dx this is equals to 0 here and it can be written as lambda 0 into next point f of x 0 equals to x 0 here so x 0.

So that is why we can just consider that since lambda 0 equals to 2 here that is not equals to 0 we can consider x 0 equals to 0 here. So the total formulation for this one point formula it can be represented as minus 1 to 1 f of x dx this equals to 2 f of 0 and this error constant c can be written as since it is just represented in this form here so c can be represented as like minus 1 to 1 so immediate next term if you just consider here that as x to the power p plus 1 so x square into dx here.

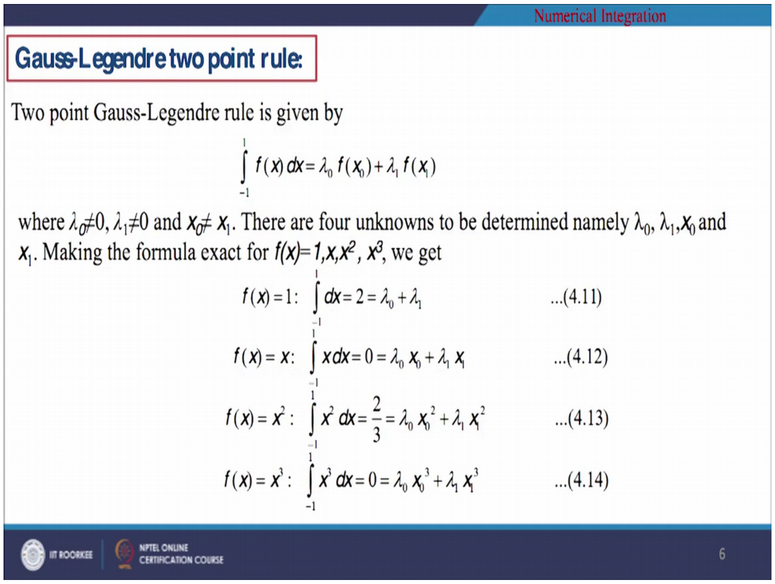
So that can just give you like 2 by 3 here, so 2 by 3 into this is like a c by p plus 1 factorial so that is why this error term or R n of f it can be written as 2 by 3 into 1 by 2 factorial f to the power p plus 1, so f double dash of zeta here 2, 2 cancel it out so this can be written in the form of 1 by 3 f double dash of zeta. So zeta should be lies between like minus 1 to 1 here, so this is the error term for one point Gauss Legendre integration method.

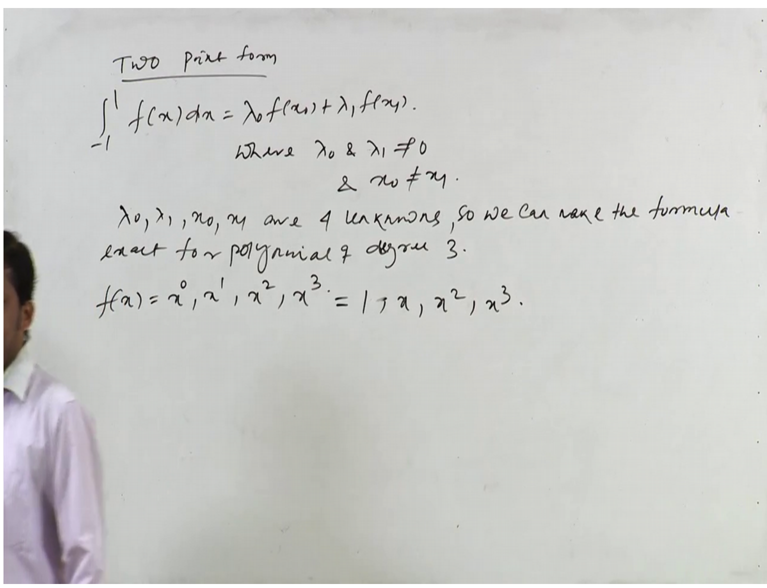
(Refer Slide Time: 9:19)



So then we will just go for Gauss Legendre 2 point rule here, if you just discuss about Gauss Legendre rule for 2 point form here then we can just write minus 1 to 1 f of x dx this equals to lambda 0 f of x 0 plus lambda 1 f of x 1 here. So 2 point means we can just consider 2 points here so that is why lambda 0 is the first coefficient here and lambda 1 is the second coefficient here where we can just consider lambda 0 and lambda 1 are not equals to 0 and x 0 is not equals to x 1 here.

(Refer Slide Time: 10:09)





And this should be determined to making this formula exact then we can just consider this polynomials of order like 0, 1, 2, 3 since we will have here 4 unknowns if you just see here lambda 0, lambda 1, x 0, x 1 are 4 unknowns here. So we can just make the formula exact for polynomial of degree 3 so that is why this function can be written as in the form of f of x equals to x to the power 0, x to the power 1, x to the power 2, x to the power 3 here or we can just write this as 1, x, x square, x to the power 3 here for which exactly this function or this polynomial approximation just give the exact value here.

(Refer Slide Time: 11:13)

Numerical Integration

Gauss-Legendre two point rule:

Two point Gauss-Legendre rule is given by

$$\int_{-1}^{1} f(x)\,dx = \lambda_0 f(x_0) + \lambda_1 f(x_1)$$

where $\lambda_0 \neq 0$, $\lambda_1 \neq 0$ and $x_0 \neq x_1$. There are four unknowns to be determined namely λ_0, λ_1, x_0 and x_1. Making the formula exact for $f(x) = 1, x, x^2, x^3$, we get

$$f(x) = 1: \quad \int_{-1}^{1} dx = 2 = \lambda_0 + \lambda_1 \qquad \text{...(4.11)}$$

$$f(x) = x: \quad \int_{-1}^{1} x\,dx = 0 = \lambda_0 x_0 + \lambda_1 x_1 \qquad \text{...(4.12)}$$

$$f(x) = x^2: \quad \int_{-1}^{1} x^2\,dx = \frac{2}{3} = \lambda_0 x_0^2 + \lambda_1 x_1^2 \qquad \text{...(4.13)}$$

$$f(x) = x^3: \quad \int_{-1}^{1} x^3\,dx = 0 = \lambda_0 x_0^3 + \lambda_1 x_1^3 \qquad \text{...(4.14)}$$

IIT ROORKEE | NPTEL ONLINE CERTIFICATION COURSE 6

Numerical Integration

Eliminating λ_0 from (4.12) and (4.14), we get

$$\lambda_1 x_1^3 - \lambda_1 x_1 x_0^2 = 0, \quad or \quad \lambda_1 x_1 (x_1 - x_0)(x_1 + x_0) = 0$$

Now, $\lambda_1 \neq 0$ *and* $x_0 \neq x_1$. *Hence*, $x_1 = 0$ *or* $x_1 = -x_0$.

If $x_1 = 0$, (4.12) *gives* $x_0 = 0$, *which is not possible*

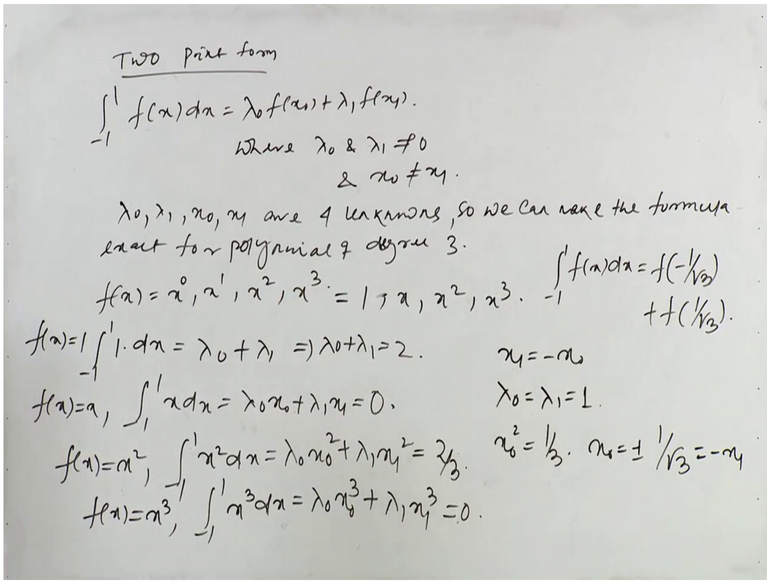
Therefore, $x_1 = -x_0$.

Substituting in (4.12) *we have*, $\lambda_0 - \lambda_1 = 0$ *or* $\lambda_0 = \lambda_1$

Substituting in (4.11) *we have*, $\lambda_0 = \lambda_1 = 1$

Substituting in (4.13) *we get*, $x_0^2 = \frac{1}{3}$, *i.e.* $x_0 = \pm\frac{1}{\sqrt{3}} = -x_1$

IIT ROORKEE | NPTEL ONLINE CERTIFICATION COURSE 7



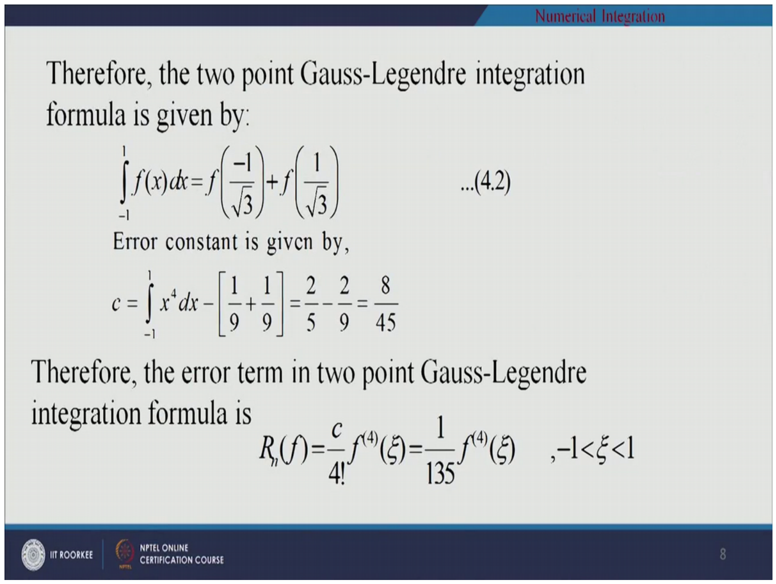
So if you just use these values then we can just obtain here like minus 1 to 1 first point if f of x equals to 1 if you just consider 1 into dx this equals to lambda 0 plus lambda 1 here.

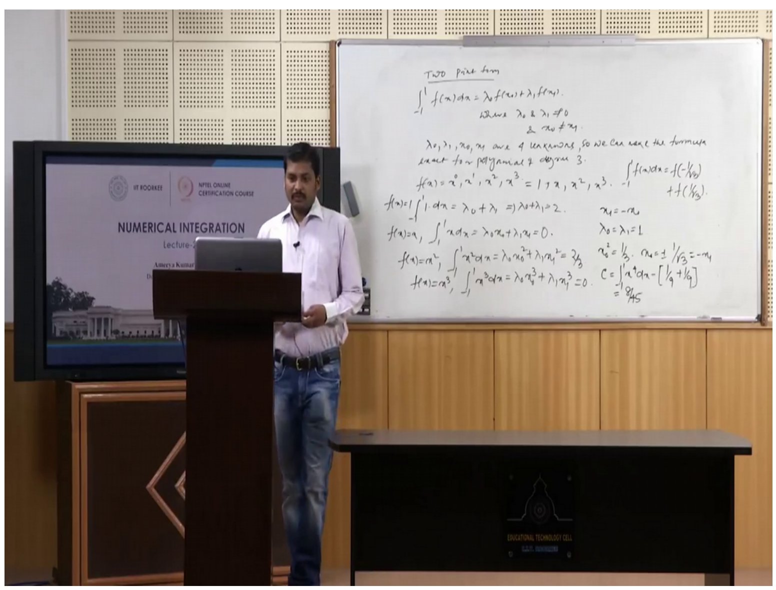
This implies that lambda 0 plus lambda 1 this equals to 2 if you just use f of x equals to x then we can just write minus 1 to 1 x dx this equals to lambda 0 x 0 plus lambda 1 x 1 this equals to 0 here. Similarly if you just use f of x equals to x square here then you can just write minus 1 to 1 x square dx this equals to lambda 0 x 0 square lambda 1 x 1 square here and this is nothing but 2 by 3 here. Similarly if you just use f of x equals to x to the power 3 here this can be written as minus 1 to 1 x to the power 3 dx this equals to lambda 0 x 0 to the power 3 lambda 1 x 1 to the power 3 this equals to 0 here again.

So if you just use these 4 equations for 4 unknowns then if you just do some calculations obviously we can just obtain this values that is in the form of like a x 1 equals to minus of x 0 here and lambda 0 this equals to lambda 1 this will just give you 1 here. So this is a just a simple calculation if you just do this means that eliminating lambda 0 from 4.1 to and 4.14 here we get that lambda 1 x 1 to the power 3 minus lambda 1 x 1 x 0 square this equals to 0 and from there itself we can just get x 1 equals to minus x 0 here and since lambda 0 equals to lambda 1 equals to 1 here than we can just write here x 0 square this equals to 1 by 3 here.

So that is why x 0 can be written as like plus or minus 1 lambda by square root 3 and since x 0 equals to minus of x 1 here we can just consider this formula as in the form of like minus 1 to 1 f of x dx this equals to f of minus 1 by root 3 plus f of 1 by root 3 here since lambda 0 and lambda 1 are 1 one there itself.

(Refer Slide Time: 13:55)

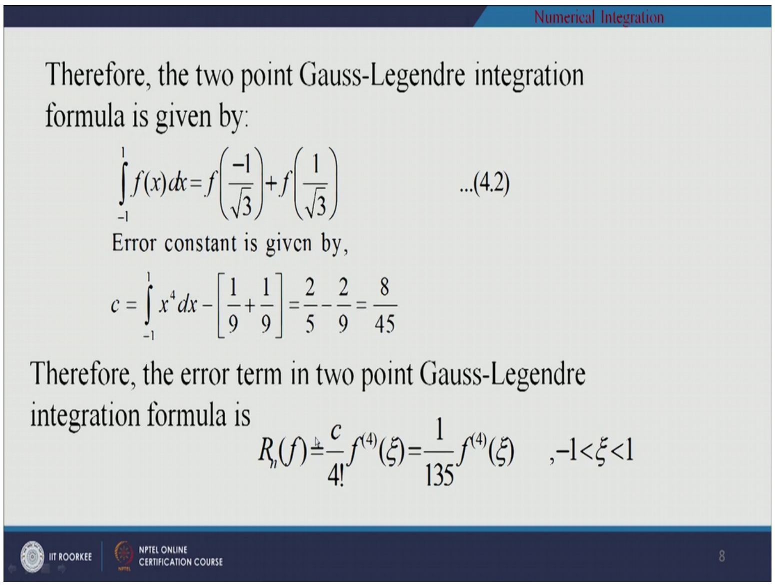


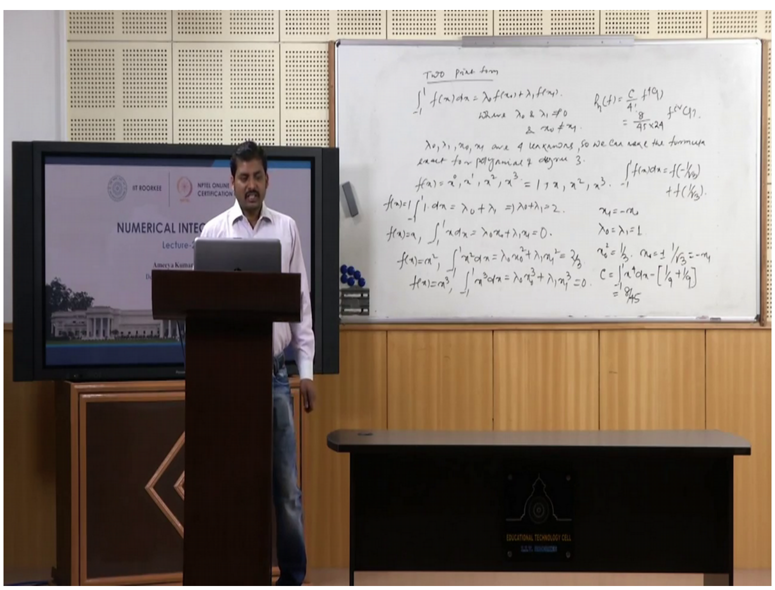


So if you just go for this error constant here so the error can be written in the form of like C equals to like first function minus 1 to 1 x to the power 4 dx minus if you just consider these functional values that will just give you 1 by 3 plus 1 by 3 here, sorry this is whole to the power 4 this is fourth power here so that is why this will be 9 this is will be 9 here.

Since 1 by root 3 whole to the power power 4 since x to the power 4 we have just considered there so that is why minus of 1 by root 3 whole to the power that both will just give you here 1 by 9 plus 1 by 9 here and final value if you will just go for this calculation so that will just give you 8 by 45.

(Refer Slide Time: 14:56)

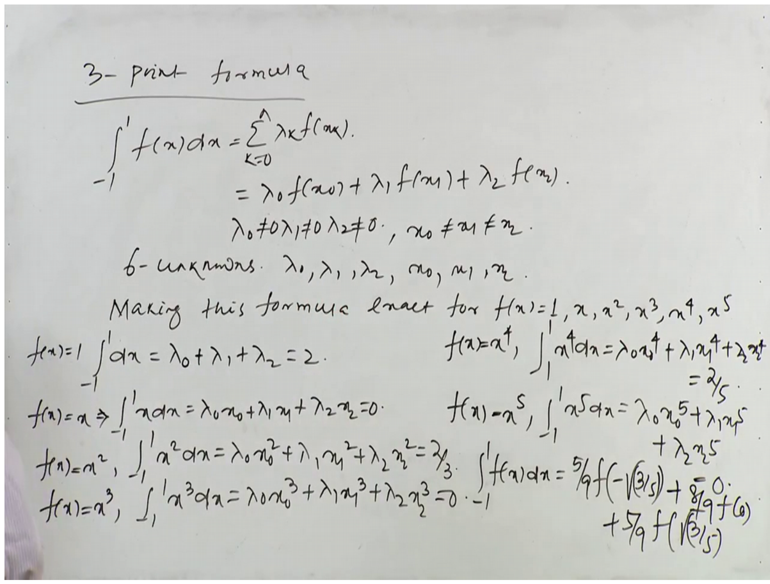




Therefore if you just calculate this error here so error term R n of f that can be written as C by 4 factorial from this like earlier formulation if you just see here R n of f this can be written as C by 4 factorial f to the power 4 of zeta here.

So C is already calculated here 8 by 45 so 8 by 45 into 4 factorial so that can be written as like 24 sorry 24 f to the power 4 of zeta where zeta should be lies between minus 1 to 1 here.

(Refer Slide Time: 15:46)



Then we will just go for 3 point formula here, so like Gauss Legendre 2 point formula here also if you just write this formula as in the form 3 point formula for Gauss Legendre integration method so likewise we will just consider first one as like we will just transform this interval minus 1 to 1 f of x dx this can be written as like summation of k equals to 0 to n lambda k f of x k.

So that is why if you just consider 3 points so 3 points means we can just write this one as lambda 0 f of x 0 plus lambda 1 f of x 1 plus lambda 2 f of x 2 here. And if you just see this equation here that is basically lambda 0 is not equals to lambda 1 is not equals to lambda 2 or we can just consider, sorry this should be not be equals to 0, this should not be equals to 0, this should not be equals to 0 here but all of these coefficients like x 0 is not equals to x 1 is not equals to x 2 here.

So that is why we will have like a 6 unknowns we have here like lambda 0, lambda 1, lambda 2, x 0, x 1, x 2 and for 6 unknowns we have to like implement this polynomial approximation up to degree 5 to get 6 equations this means that up to degree 5 this will just provide exact solution. So if you just consider that making this formula exact for f of x equals to 1, x, x square, x cube, x to the power 4, x to the power 5 here since 6 different values we have to consider to get this formula exact here.

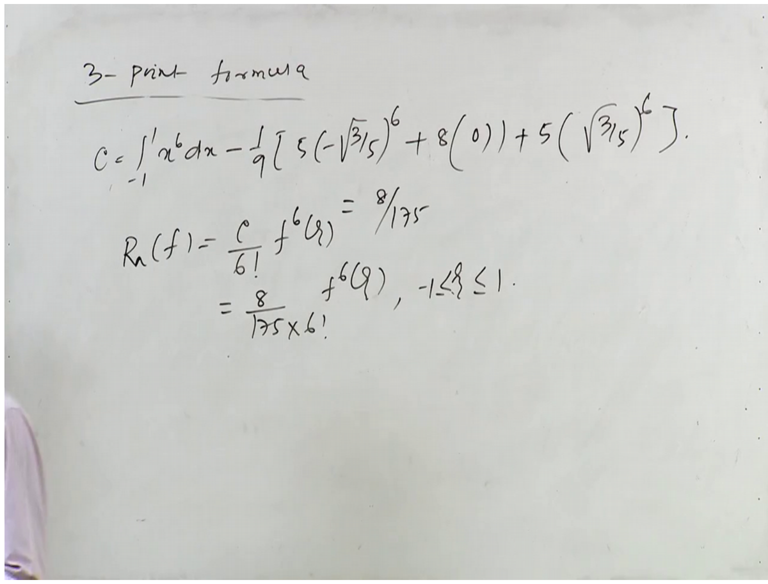
So if you just put all these values then we can just obtain these values are as like minus 1 to 1 for f of x equals to 1 this is dx this equals to lambda 0, lambda 1 plus lambda 2 this equals to 2 here. And for f of x equals to x this can be written as minus 1 to 1 x dx this equals to lambda 0 x 0, lambda 1 x 1, lambda 2 x 2 this equals to 0 here f of x equals to x square minus 1 to 1 x square dx this equals to lambda 0 x 0 square lambda 1 x 1 square lambda 2 x 2 square this equals to 2 by 3 here.

So similarly f of x equals to x cube it can be written as minus 1 to 1 x cube dx lambda 0 x 0 cube lambda 1 x 1 cube lambda 2 x 2 cube this equals to 0 here. And if you just consider f of x equals to x to the power 4 here then minus 1 to 1 x to the power 4 dx this equals to lambda 0 x 0 4 lambda 1 x 14 lambda 2 x 24 this equals to 2 by 5 here f of x equals to x to the power 5 if you just consider minus 1 to 1 x to the power 5 dx this can be written as lambda 0 x 0 to the power 5 lambda 1 x 1 to the power 5 lambda 2 x 2 to the power 5 this equals to 0 also here.

So if you just solve these 6 equations for 6 unknowns then we can just obtain these coefficients for like lambda 0, lambda 1, lambda 2 and x 0, x 1 and x 2 here. So in a complete form if you just write this formula that can be written in the form of like minus 1 to 1 f of dx this equals to 5 by 9 f of minus root 3 by 5 plus 8 by 9 f of 0 plus 5 by 9 f of root 3 by 5 this is the complete formula for this 3 point Gauss Legendre integration method.

So if you just go for this error computation so your error computation can be written in the form of like a C equals to the formula is just providing the exact result for a polynomial of degree 5 here then the error term can be obtained for polynomial of degree 6 here.

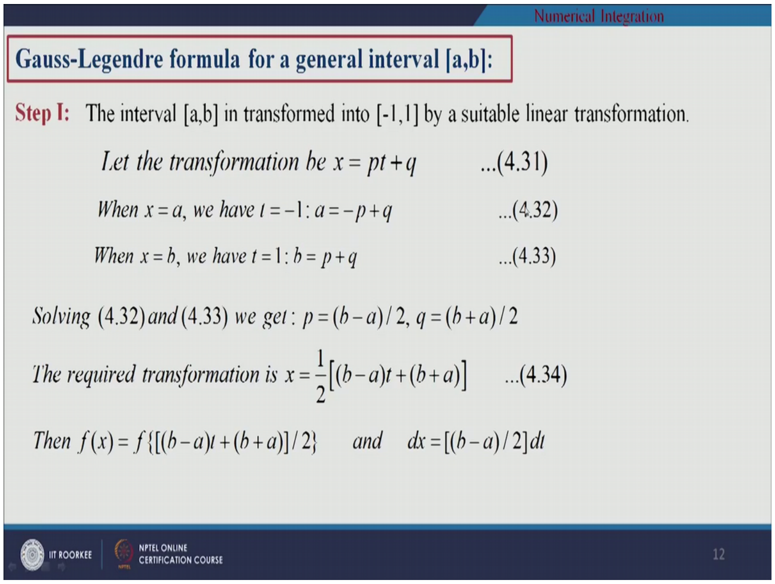
(Refer Slide Time: 21:03)

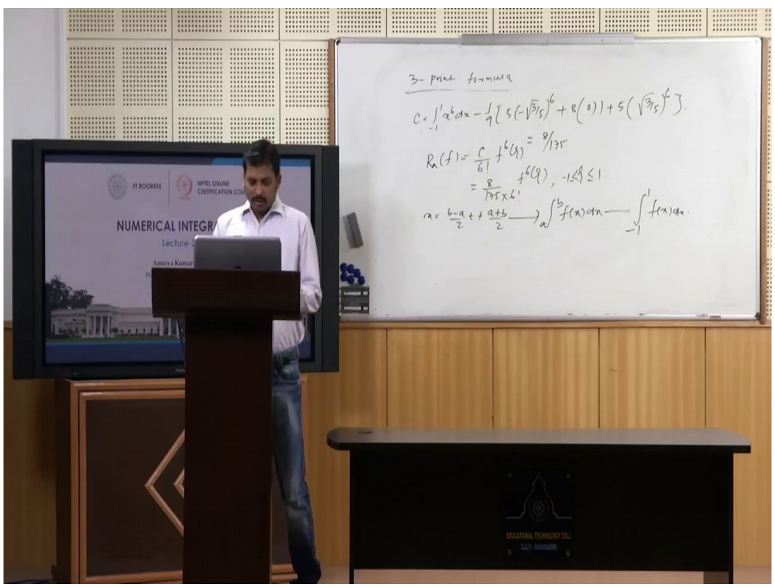


So that is why C can be written as like minus 1 to 1 x to the power 6 dx minus 1 by 9 if you just write so 5 minus root 3 by 5 whole to the power 6 plus 8 like f of 0 so that is why we can just write that one as 0 here plus 5 root 3 by 5 whole to the power 6 this one and the total value just it is just giving you like 8 by 175 here.

Hence the total error term that is you can just write R n of f this equals to C by 6 factorial f to the power 6 of zeta here that is 8 by 175 into 6 factorial f to the power 6 of zeta where zeta should be lies between minus 1 to here this is the error term for 3 point Gauss Legendre integration method.

(Refer Slide Time: 22:15)

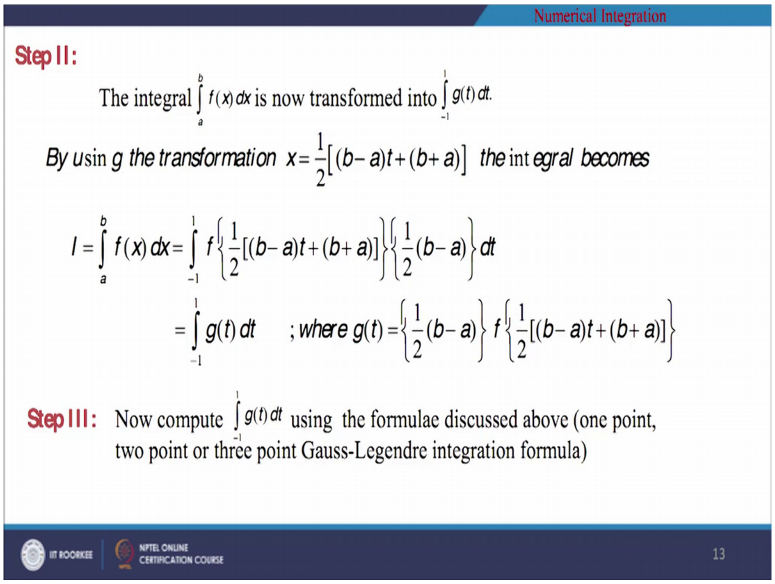


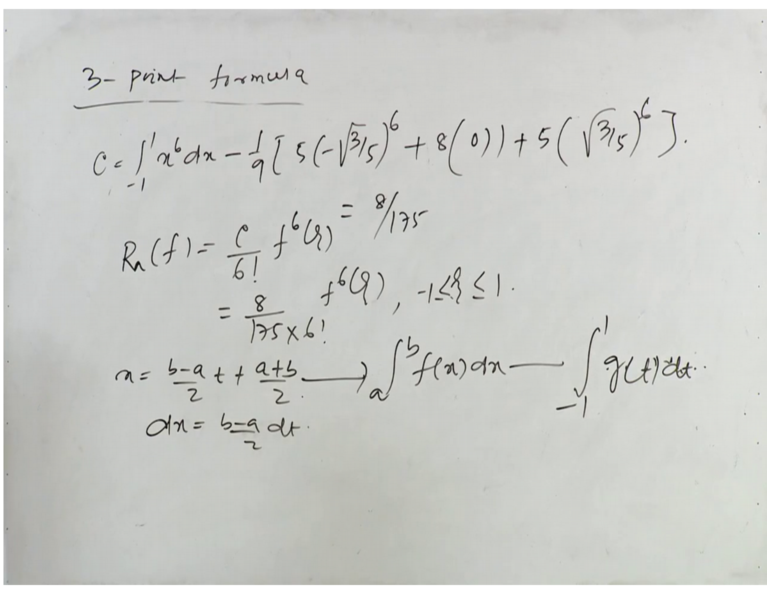


So the general transform method for like a interval from a2 b2 minus 1 to 1 can be done in a easy form that is if you just write this transformation x equals to pt plus q here then when x equals to a we have t equals to minus 1 and a equals to minus p plus q here when x equals to b we have t equals to 1 so b equals to p plus q here.

If you just combine both these equations here we can just obtain p equals to b minus a by 2 q equals to b plus a by 2. So the required transformation for x if we want to transform into a, b to minus 1 to 1 here then we can just write that one as b minus a by 2 into t the transformation basically it is just written as x equals to b minus a by 2 into t plus a plus b by 2 here. So automatically this range that is b to a can be transformed to minus 1 to 1 f of x dx here.

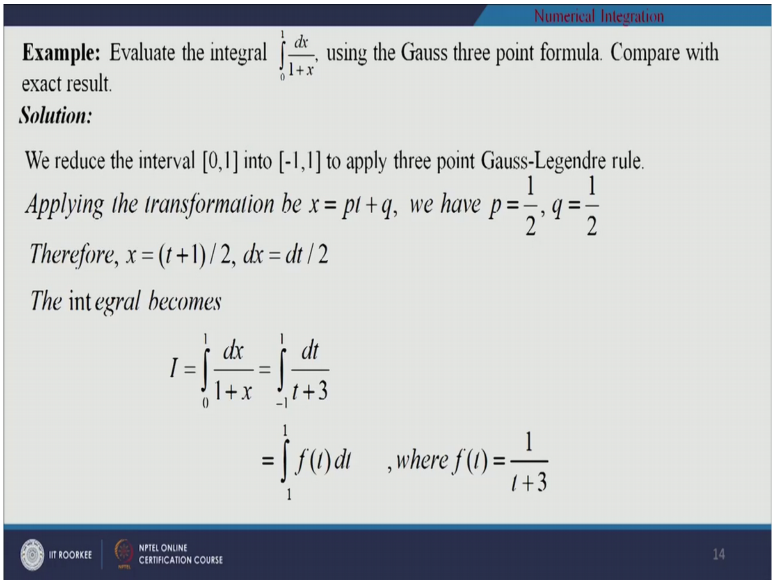
(Refer Slide Time: 23:38)





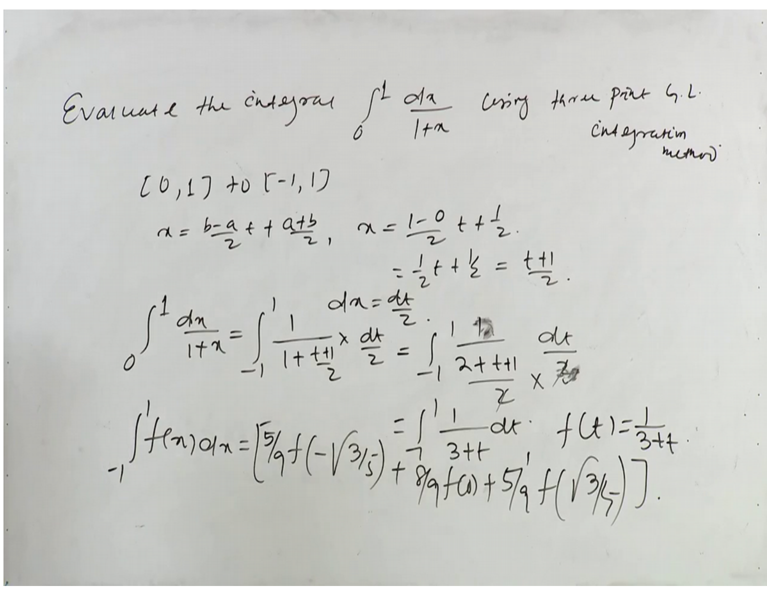
So obviously once you are just obtaining this transformation x in terms of t here then we can just write that dx in the form of dt so then you can just write this formulation that is in the form of if you want to find this transformation specially you can just write this part as g of t dt here. So t obviously if you just write here dx equals to b minus a by 2 into dt and then you can just put it over there and you can get this integration range there.

(Refer Slide Time: 24:15)



If the question is asked suppose evaluate this integral that is 0 to 1 dx by 1 plus x using Gauss Legendre 3 point formula compare with the exact result, then first we will just transform this range 0 to 1 to that is minus 1 to 1 here then we can just apply this 3 point formula there itself.

(Refer Slide Time: 24:38)

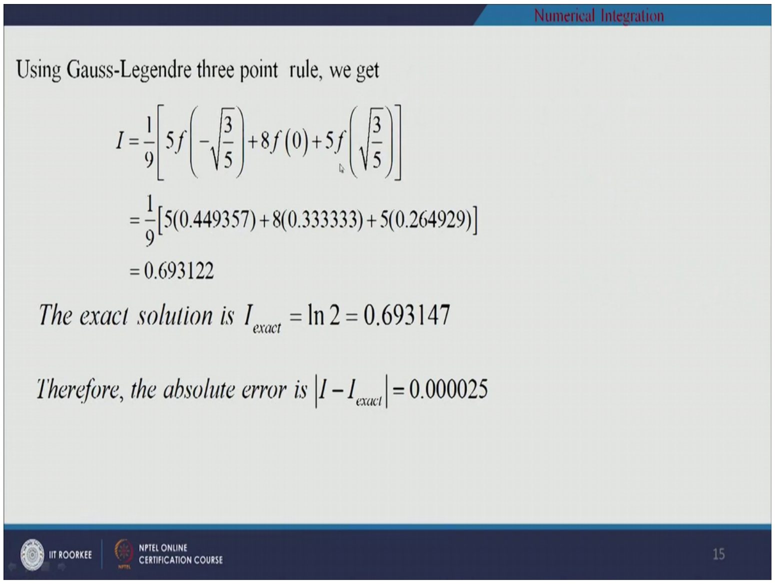


Suppose if you just write this integral evaluate the integral 0 to 1 dx by 1 plus x using 3 point Gauss Legendre integration method.

Then the first criteria is that we have to transform this integration range 0, 1 to minus 1 to 1 here, so for that we will just put here that is x equals to b minus a by 2 into t plus a plus b by 2 here if you just write in this form then we can just obtain here x equals to b as 1 here 1 minus 0 by 2 into t plus half here and this can be written as half of t plus half we can just write t plus 1 by 2 here. And dx can be written as like dt by 2 here and the complete integration range that is 0 to 1 dx by 1 plus x this can be transformed into minus 1 to 1, so first 1 by 1 plus x so that is why you can just write that one as 1 by 1 plus t plus 1 by 2 into dt by 2 here.

So this can be written in the form of like minus 1 to 1 so 1 by 2 plus t plus 1 if you just take out this will be 2 here, sorry by 2 here into dt by 2 here so 2, 2 can be cancel it out so it can be written as minus 1 to 1 1 by 3 plus t into dt here. So then if f of t equals to 1 by 3 plus t here we can now use this formula that as integration minus 1 to 1 f of x dx this as like 5 by n f of minus root 3 by 5 plus 8 by 9 f of 0 plus 5 by 9 f of root 3 by 5 here.

(Refer Slide Time: 27:46)



Since function is known to us and directly we can put these values and obtain these functional values there. For the exact solution if you just see here that is I exact equals to ln of 2 that is coming as 0.693147 but the computed value if you just see here that will just give you like 0.693122 and if you just take a for this error calculation the absolute error is like I minus I exact that is just giving you 0.000025 here. Thank you for listening this lecture in integration in the numerical methods for interpolation and the approximation of functions in various forms thank you for listening.